Annales Mathematicae Silesianae (2023),
DOI: 10.2478/amsil-2023-0022

BIDIMENSIONAL EXTENSIONS OF COBALANCING AND LUCAS-COBALANCING NUMBERS

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Abstract. A new bidimensional version of cobalancing numbers and Lucas-balancing numbers are introduced. Some properties and identities satisfied by these new bidimensional sequences are studied.

1. Introduction

The study of number sequences has been the subject of several studies published in recent decades. Algebraic properties, generating function, Binet's formula and some well-known identities have been studied in this research topic. There are studies of various number sequences, leaving the reader some suggestions about some of this variety (see, for example, the studies [1, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18]). Also the two-dimensional version of some sequences has been a topic of investigation for some researchers. This is the case with the work on the Narayana sequence [2], the Fibonacci sequence

Received: 25.03.2023. Accepted: 17.11.2023.

(2020) Mathematics Subject Classification: 11B39, 11B83.

Key words and phrases: cobalancing numbers, Lucas-cobalancing numbers, balancing numbers, Lucas-balancing numbers, bidimensional recurrence relations.

This research was partially funded by Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, under Projects UIDB/00013/2020, UIDP/00013/2020 and UIDB/00194/2020.

of Gaussian numbers [15], the Fibonacci sequence [16] and the Leonardo sequence [22], among others.

In this paper it is our purpose to introduce and study the bidimensional version of two numerical sequences: the sequence of cobalancing numbers and the sequence of Lucas-cobalancing numbers.

We will recall in this section some important results concerning these sequences in their unidimensional version, results that will be used later in establishing new results concerning the same sequences in their bidimensional form.

Panda and Ray in [18] introduced the sequence $\{c_n\}_{n\geq 1}$ of Lucas-cobalancing numbers that satisfy the following recurrence relation

$$c_{n+2} = 6c_{n+1} - c_n, \quad n \geq 1,$$

with the initial conditions $c_1 = 1$ and $c_2 = 7$. Binet's formula of the sequence $\{c_n\}_{n\geq 1}$ of Lucas-cobalancing numbers has the following form

$$c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2},$$

where

$$\alpha_1 = \sqrt{3 + 2\sqrt{2}} \tag{1.1}$$

and

$$\alpha_2 = \sqrt{3 - 2\sqrt{2}}. \tag{1.2}$$

Related to this sequence we have the sequence $\{Q_n\}_{n\geq 0}$ which is known as Pell-Lucas numbers and is defined by the recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2,$$

with the initial conditions $Q_0 = 1$ and $Q_1 = 1$. According to Theorem 4.2.3 in [19], one of the relationships that exists between these two sequences is

$$c_n = Q_{2n-1}.$$

Also, Panda and Ray in [18] introduced the sequence $\{b_n\}_{n\geq 1}$ of cobalancing numbers satisfying the recurrence relation

$$b_{n+2} = 6b_{n+1} - b_n + 2, \quad n \geq 1,$$

with the initial conditions $b_1 = 0$ and $b_2 = 2$. Binet's formula for the sequence $\{b_n\}_{n\geq 1}$ of cobalancing numbers has the following form

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, \tag{1.3}$$

where α_1, α_2 are defined by formulas (1.1) and (1.2), respectively.

Based on the Binet formula of $\{c_n\}_{n\geq 1}$,

$$c_{n+1} = \frac{\alpha_1^{2n+1} + \alpha_2^{2n+1}}{2},$$

hence

$$\begin{aligned} c_{n+1} &= \frac{\alpha_1^{2n-1} \cdot \alpha_1^2 + \alpha_2^{2n-1} \cdot \alpha_2^2}{2} \\ &= \frac{\alpha_1^{2n-1} \cdot \left(3 + 2\sqrt{2}\right) + \alpha_2^{2n-1} \cdot \left(3 - 2\sqrt{2}\right)}{2} \\ &= \left(\frac{3\alpha_1^{2n-1} + 3\alpha_2^{2n-1}}{2}\right) + \left(\frac{2\sqrt{2}\alpha_1^{2n-1} - 2\sqrt{2}\alpha_2^{2n-1}}{2}\right) \\ &= 3c_n + \sqrt{2}\left(\alpha_1^{2n-1} - \alpha_2^{2n-1}\right) \\ &= 3c_n + \sqrt{2}\left(\frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} \cdot 4\sqrt{2}\right) \\ &= 3c_n + 8\left(\frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} + \frac{1}{2}\right), \end{aligned}$$

therefore

$$c_{n+1} = 3c_n + 8b_n + 4,$$

which is the equivalent of saying that

$$b_n = \frac{1}{8}c_{n+1} - \frac{3}{8}c_n - \frac{1}{2}.$$

Behera and Panda in [1] introduced the sequence $\{B_n\}_{n\geq 0}$ of balancing numbers satisfying the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 1,$$

with the initial conditions $B_0 = 0$ and $B_1 = 1$. Binet's formula of the sequence $\{B_n\}_{n\geq 0}$ of balancing numbers is defined by

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

with $r_1 = \alpha_1^2$ and $r_2 = \alpha_2^2$. A generalization of this sequence was studied by Ray in [20].

There is another sequence of numbers called Lucas-balancing $\{C_n\}_{n\geq 0}$ in which its recurrence relation is the same as in the case of the sequence $\{B_n\}_{n\geq 0}$, differing only in the initial conditions which, in this case are, $C_0 = 1$ and $C_1 = 3$. In [19] Panda shows the relation existing between the terms of order n of the sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$, as being $C_n = \sqrt{8B_n^2 + 1}$.

The sequences that we have mentioned before, are in *The On-Line Encyclopedia of Integers Sequences*® (OEIS®) [21] and the Table 1 gives us the first elements of them. Note that for cobalancing and Lucas-cobalancing sequences, in Table 1, we consider as the first elements, the value of b_1 and c_1, respectively.

Table 1. Some first element of $\{B_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 1}$ and $\{c_n\}_{n\geq 1}$

n	0	1	2	3	4	5	6
B_n	0	1	6	35	204	1189	6930
C_n	1	3	17	99	577	3363	19601
b_n	-	0	2	14	84	492	2870
c_n	-	1	7	41	239	1393	8119

Chimpanzo et al. in [7] introduced the bidimensional balancing numbers sequence $\{B_{(n,m)}\}$ that satisfies the following bidimensional recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} B_{(n+1,m)} = 6B_{(n,m)} - B_{(n-1,m)}, \\ B_{(n,m+1)} = B_{(n,m)} - B_{(n,m-1)}, \end{cases}$$

with the initial conditions $B_{(0,0)} = 0$, $B_{(1,0)} = 1$, $B_{(0,1)} = i$, $B_{(1,1)} = 1 + i$ and $i^2 = -1$.

Below we present a series of results related to the sequences $\{B_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 1}$ that we use later in demonstrations of other results related to bidimensional versions of Lucas-cobalancing numbers and cobalancing numbers. Thus, we have the properties listed in the following proposition:

Proposition 1.1. *If B_n and c_n represent, respectively, the balancing numbers and the Lucas-cobalancing numbers of order n, then the following properties are true:*

1. $\sum_{l=1}^{n} c_l = \frac{-1-B_{n-1}+3B_n}{2}$;
2. $\sum_{l=1}^{n} c_{2l} = B_n^2 + B_n B_{n+1}$;
3. $\sum_{l=1}^{n} c_{2l-1} = \frac{-1-B_{2n-1}+3B_{2n}-2(B_n+B_{n+1})B_n}{2}$.

Proof. To prove item 1, we use Theorem 4.2.16 in [19] and Proposition 2.6, item 6 in [5]. Thus we have:

$$\begin{aligned}
\sum_{l=1}^{n} c_l &= \sum_{l=1}^{n} (B_{l-1} + B_l) = \sum_{l=1}^{n} B_{l-1} + \sum_{l=1}^{n} B_l \\
&= \frac{-1 - B_{n-1} + B_n}{4} + \sum_{l=1}^{n} B_{l-1} + B_n \\
&= \frac{-1 - B_{n-1} + B_n}{4} + \frac{-1 - B_{n-1} + B_n}{4} + B_n \\
&= \frac{-1 - B_{n-1} + B_n - 1 - B_{n-1} + B_n + 4B_n}{4} \\
&= \frac{-2 - 2B_{n-1} + 6B_n}{4} \\
&= \frac{-1 - B_{n-1} + 3B_n}{2}.
\end{aligned}$$

For item 2, the result follows, using, once again, in [19], Theorem 4.2.16 and also items (a) and (b) of Corollary 2.3.6.

Finally, to prove item 3, we use items 1 and 2 of this proposition. Thus we have:

$$\begin{aligned}
\sum_{l=1}^{n} c_{2l-1} &= \sum_{l=1}^{2n} c_l - \sum_{l=1}^{n} c_{2l} \\
&= \frac{-1 - B_{2n-1} + 3B_{2n}}{2} - \left(B_n^2 + B_n B_{n+1}\right) \\
&= \frac{-1 - B_{2n-1} + 3B_{2n}}{2} - \frac{2\left(B_n^2 + B_n B_{n+1}\right)}{2} \\
&= \frac{-1 - B_{2n-1} + 3B_{2n} - 2\left(B_n + B_{n+1}\right) B_n}{2}.
\end{aligned}$$ □

2. Bidimensional Lucas-cobalancing numbers

In this section we will present the bidimensional version of Lucas-cobalancing numbers, starting by introducing the following definition:

Definition 2.1. The *bidimensional Lucas-cobalancing numbers* $c_{(m,n)}$ are the numbers that satisfy the following bidimensional recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} c_{(n+1,m)} = 6c_{(n,m)} - c_{(n-1,m)}, \\ c_{(n,m+1)} = 6c_{(n,m)} - c_{(n,m-1)}, \end{cases}$$

with the initial conditions $c_{(0,0)} = 1$, $c_{(1,0)} = 7$, $c_{(0,1)} = 1 + i$, $c_{(1,1)} = 7 + i$ and $i^2 = -1$.

As noted in the previous section, some versions of numerical sequences have been the subject of various studies. In the following lemma we will use the sequence of bidimensional balancing numbers denoted by $B_{(n,m)}$ which were studied in [7]. In this lemma we present some properties involving not only the sequence of bidimensional balancing numbers, but especially the sequence of bidimensional Lucas-cobalancing numbers.

Lemma 2.2. *The following properties are valid for all non-negative integers m and n and for the numbers $c_{(n,m)}$:*
(a) $c_{(n,0)} = c_{n+1}$;
(b) $c_{(0,m)} = B_{(1,m)}$;
(c) $c_{(n,1)} = c_{n+1} + (B_n - B_{n-1})\, i$;
(d) $c_{(1,m)} = B_{(1,m)} + 6\,(B_m - B_{m-1})$,
where $B_{(n,m)}$ is a bidimensional balancing number, B_k is a balancing number of order k and c_k is a Lucas-cobalancing number of order k.

Proof. The proof of (a) will be done by induction on n:
For $n = 0$ we have that $c_{(0,0)} = 1 = c_1$ and the proposition is true.
For $n = 1$ we have that $c_{(1,0)} = 7 = c_2$ and the proposition is also true.
Suppose that the proposition is true for all values less than or equal to n. We are going to show that it remains true for $n + 1$:
By Definition 2.1 and the induction hypothesis, we obtain:

$$\begin{aligned} c_{(n+1,0)} &= 6c_{(n,0)} - c_{(n-1,0)} \\ &= 6c_{n+1} - c_n = c_{n+2}, \end{aligned}$$

which ends the proof of (a).

The proof of (b) will be done by induction on m:

For $m = 0$, since $B_{(1,0)} = 1$ (see [7]), we have that $c_{(0,0)} = 1 = B_{(1,0)}$ and the proposition is true.

For $m = 1$, since $B_{(1,1)} = 1+i$ (see [7]), we have that $c_{(0,1)} = 1+i = B_{(1,1)}$ and the proposition is also true.

Suppose that the proposition is true for all values less than or equal to m. We show that it remains true for $m + 1$:

By Definition 2.1, by the induction hypothesis and by the expression (3) or by Definition 3.1 in [7],

$$\begin{aligned} c_{(0,m+1)} &= 6c_{(0,m)} - c_{(0,m-1)} \\ &= 6B_{(1,m)} - B_{(1,m-1)} \\ &= B_{(1,m+1)}, \end{aligned}$$

which ends the proof of (b).

The proof of (c) will also be done by induction on n:

For $n = 0$, since $B_{-n} = -B_n$ (see [7]), we have that $c_{(0,1)} = c_1 + (B_0 - B_{-1})\, i = 1 + i$ and the proposition is true.

For $n = 1$ we have that $c_{(1,1)} = c_2 + (B_1 - B_0)\, i = 7+i$ and the proposition is also true, given the initial conditions of the sequence $\{B_n\}_{n\geq 0}$.

Suppose that the proposition is true for all values less than or equal to n. Let us show that it remains true for $n + 1$:

By Definition 2.1 and the recurrence relations of $\{B_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 1}$,

$$\begin{aligned} c_{(n+1,1)} &= 6c_{(n,1)} - c_{(n-1,1)} \\ &= 6\left(c_{n+1} + (B_n - B_{n-1})\, i\right) - \left(c_n + (B_{n-1} - B_{n-2})\, i\right) \\ &= 6c_{n+1} + 6\left(B_n - B_{n-1}\right) i - c_n - \left(B_{n-1} - B_{n-2}\right) i \\ &= 6c_{n+1} + 6B_n i - 6B_{n-1} i - c_n - B_{n-1} i + B_{n-2} i \\ &= (6c_{n+1} - c_n) + (6B_n i - B_{n-1} i) - (6B_{n-1} i - B_{n-2} i) \\ &= c_{n+2} + B_{n+1} i - B_n i \\ &= c_{n+2} + (B_{n+1} - B_n)\, i, \end{aligned}$$

which ends the proof of (c).

The proof of (d) will also be done by induction on m:

For $m = 0$ again, since $B_0 = 0$, $B_{-n} = -B_n$ and $B_{(1,0)} = 1$ (see [7]), we have that $c_{(1,0)} = B_{(1,0)} + 6\left(B_0 - B_{-1}\right) = 7$ and the proposition is true.

For $m = 1$, since $B_0 = 0$, $B_1 = 1$ and $B_{(1,1)} = 1 + i$ (see [7]), we have that $c_{(1,1)} = B_{(1,1)} + 6(B_1 - B_0) = 7 + i$ and the proposition is also true.

Suppose that the proposition is true for all values less than or equal to m. Let us show that it remains true for $m + 1$:

By Definition 2.1, by the recurrence relation of $\{B_n\}_{n\geq 0}$ and by property 4 of Lemma 3.2 in [7],

$$\begin{aligned}
c_{(1,m+1)} &= 6c_{(1,m)} - c_{(1,m-1)} \\
&= 6\left(B_{(1,m)} + 6(B_m - B_{m-1})\right) - \left(B_{(1,m-1)} + 6(B_{m-1} - B_{m-2})\right) \\
&= 6\left((B_m - B_{m-1}) + B_m i + 6(B_m - B_{m-1})\right) \\
&\quad - \left((B_{m-1} - B_{m-2}) + B_{m-1} i + 6(B_{m-1} - B_{m-2})\right) \\
&= 6(B_m - B_{m-1}) + 6B_m i + 36(B_m - B_{m-1}) - (B_{m-1} - B_{m-2}) \\
&\quad - B_{m-1} i - 6(B_{m-1} - B_{m-2}) \\
&= 42(B_m - B_{m-1}) - 7(B_{m-1} - B_{m-2}) + 6B_m i - B_{m-1} i \\
&= 42B_m - 42B_{m-1} - 7B_{m-1} + 7B_{m-2} + B_{m+1} i \\
&= (42B_m - 7B_{m-1}) - (42B_{m-1} - 7B_{m-2}) + B_{m+1} i \\
&= 7(6B_m - B_{m-1}) - 7(6B_{m-1} - B_{m-2}) + B_{m+1} i \\
&= 7B_{m+1} - 7B_m + B_{m+1} i \\
&= 7(B_{m+1} - B_m) + B_{m+1} i \\
&= (B_{m+1} - B_m) + B_{m+1} i + 6(B_{m+1} - B_m) \\
&= B_{(1,m+1)} + 6(B_{m+1} - B_m).
\end{aligned}$$

□

The following theorem gives us a way to determine the element $c_{(n,m)}$, for any n and m, non-negative integers, in terms of cobalancing and balancing numbers.

THEOREM 2.3. *For the non-negative integers m, n, the bidimensional Lucas-cobalancing numbers are described in the form:*

$$c_{(n,m)} = c_{n+1}(B_m - B_{m-1}) + (B_n - B_{n-1}) B_m i.$$

PROOF. We begin by performing the demonstration by induction on m, fixing n.

For $m = 0$ and given the initial conditions of the sequence $\{c_n\}_{n\geq 1}$ and again, using the fact that $B_{-m} = -B_m$ in [7], we have that $c_{(n,0)} = c_{n+1}(B_0 - B_{-1}) + (B_n - B_{n-1}) B_0 i = c_{n+1}$, which is true by item (a) of Lemma 2.2.

For $m = 1$ and again, given the initial conditions of the sequence $\{c_n\}_{n\geq 1}$ and $B_{-m} = -B_m$ in [7], we have $c_{(n,1)} = c_{n+1}(B_1 - B_0) + (B_n - B_{n-1}) B_1 i = c_{n+1} + (B_n - B_{n-1}) i$, which is found to be true by Lemma 2.2, item (c).

Suppose that the equality is true for non-negative n fixed and all values less than or equal to m. Let us prove that it is still true for $m+1$:

$$c_{(n,m+1)} = c_{n+1}(B_{m+1} - B_m) + (B_n - B_{n-1}) B_{m+1} i.$$

By Definition 2.1 and the recurrence relation of $\{B_n\}_{n\geq 0}$,

$$\begin{aligned}
c_{(n,m+1)} &= 6c_{(n,m)} - c_{(n,m-1)} \\
&= 6\left(c_{n+1}(B_m - B_{m-1}) + (B_n - B_{n-1}) B_m i\right) \\
&\quad - \left(c_{n+1}(B_{m-1} - B_{m-2}) + (B_n - B_{n-1}) B_{m-1} i\right) \\
&= 6c_{n+1}(B_m - B_{m-1}) + 6(B_n - B_{n-1}) B_m i \\
&\quad - c_{n+1}(B_{m-1} - B_{m-2}) - (B_n - B_{n-1}) B_{m-1} i \\
&= 6c_{n+1}B_m - 6c_{n+1}B_{m-1} + 6B_nB_m i - 6B_{n-1}B_m i \\
&\quad - c_{n+1}B_{m-1} + c_{n+1}B_{m-2} - B_nB_{m-1} i + B_{n-1}B_{m-1} i \\
&= c_{n+1}(6B_m - B_{m-1}) - c_{n+1}(6B_{m-1} - B_{m-2}) \\
&\quad + B_n(6B_m i - B_{m-1} i) - B_{n-1}(6B_m i - B_{m-1} i) \\
&= c_{n+1}B_{m+1} - c_{n+1}B_m + B_nB_{m+1} i - B_{n-1}B_{m+1} i \\
&= c_{n+1}(B_{m+1} - B_m) + (B_n - B_{n-1}) B_{m+1} i,
\end{aligned}$$

as we wanted to show.

Let us now fix m and, perform the induction on n.

For $n = 0$ and given the initial conditions of the sequence $\{c_n\}_{n\geq 1}$ and [7, Lemma 3.2, item 4], and again, using the fact that $B_{-n} = -B_n$, we have that $c_{(0,m)} = c_1(B_m - B_{m-1}) + (B_0 - B_{-1}) B_m i = (B_m - B_{m-1}) + B_m i = B_{(1,m)}$, which is true by Lemma 2.2, item (b).

For $n = 1$ and again, given the initial conditions of the sequence $\{c_n\}_{n\geq 1}$ and the fact that $B_{-n} = -B_n$ in [7], we have that $c_{(1,m)} = c_2(B_m - B_{m-1}) + (B_1 - B_0) B_m i = 7(B_m - B_{m-1}) + B_m i = B_{(1,m)} + 6(B_m - B_{m-1})$, which is found to be true by Lemma 2.2, item (d).

Suppose the theorem is true for all values less than or equal to n. We prove that it is still true for $n+1$:

$$c_{(n+1,m)} = c_{n+2}\left(B_m - B_{m-1}\right) + \left(B_{n+2} - B_{n+1}\right) B_m i.$$

By Definition 2.1 and the recurrence relations of $\{c_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 0}$,

$$\begin{aligned}
c_{(n+1,m)} &= 6c_{(n,m)} - c_{(n-1,m)} \\
&= 6\left(c_{n+1}\left(B_m - B_{m-1}\right) + \left(B_{n+1} - B_n\right) B_m i\right) \\
&\quad - \left(c_n\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right) B_m i\right) \\
&= 6c_{n+1}\left(B_m - B_{m-1}\right) + 6\left(B_{n+1} - B_n\right) B_m i \\
&\quad - c_n\left(B_m - B_{m-1}\right) - \left(B_n - B_{n-1}\right) B_m i \\
&= 6c_{n+1}B_m - 6c_{n+1}B_{m-1} + 6B_{n+1}B_m i - 6B_nB_m i \\
&\quad - c_nB_m + c_nB_{m-1} - B_nB_m i + B_{n-1}B_m i \\
&= B_m\left(6c_{n+1} - c_n\right) - B_{m-1}\left(6c_{n+1} - c_n\right) \\
&\quad + \left(6B_{n+1} - B_n\right) B_m i - \left(6B_n - B_{n-1}\right) B_m i \\
&= B_mc_{n+2} - B_{m-1}c_{n+2} + B_{n+2}B_m i - B_{n+1}B_m i \\
&= c_{n+2}\left(B_m - B_{m-1}\right) + \left(B_{n+2} - B_{n+1}\right) B_m i,
\end{aligned}$$

as we wanted to show. □

3. Some identities involving bidimensional Lucas-cobalancing numbers

From now on, we will explore some bidimensional identities of the bidimensional sequence of Lucas-cobalancing numbers, using somewhat inherent properties of the sequence.

IDENTITY 3.1. The sum of the numbers $c(p,m)$ of odd index p, can be described by:

$$\begin{aligned}
\sum_{l=1}^{n} c_{(2l-1,m)} &= \left(B_n^2 + B_nB_{n+1}\right)\left(B_m - B_{m-1}\right) \\
&\quad + \left(B_n^2 - B_{n-1}B_n + B_{2n}\right) B_m i.
\end{aligned}$$

PROOF. By Theorem 2.3, we have

$$\sum_{l=1}^{n} c_{(2l-1,m)} = \sum_{l=1}^{n} \left(c_{2l}\left(B_m - B_{m-1}\right) + \left(B_{2l-1} - B_{2l-2}\right) B_m i\right).$$

Thus,

$$\begin{aligned}\sum_{l=1}^{n} c_{(2l-1,m)} &= \left(c_2\left(B_m - B_{m-1}\right) + \left(B_1 - B_0\right) B_m i\right) \\ &\quad + \cdots + \left(c_{2n}\left(B_m - B_{m-1}\right) + \left(B_{2n-1} - B_{2n-2}\right) B_m i\right) \\ &= \left(c_2 + \cdots + c_{2n}\right)\left(B_m - B_{m-1}\right) \\ &\quad + \left(\left(B_1 - B_0\right) + \cdots + \left(B_{2n-1} - B_{2n-2}\right)\right) B_m i \\ &= \sum_{l=1}^{n} c_{2l}\left(B_m - B_{m-1}\right) + \Big(\sum_{l=1}^{n} B_{2l-1} - \sum_{l=1}^{n} B_{2l-2}\Big) B_m i.\end{aligned}$$

The result follows, using item 2 of Proposition 1.1 and items (a) and (b) of Corollary 2.3.6 in [19]. □

IDENTITY 3.2. The sum of the numbers $c(p,m)$ of even index p, can be described by:

$$\sum_{l=1}^{n} c_{(2l,m)} = \left(B_{n+1}^2 + B_{n+1}B_{n+2}\right)\left(B_m - B_{m-1}\right) + \left(B_n B_{n+1} - B_n^2\right) B_m i.$$

PROOF. By Theorem 2.3, we have

$$\sum_{l=1}^{n} c_{(2l,m)} = \sum_{l=1}^{n} \left(c_{2l+1}\left(B_m - B_{m-1}\right) + \left(B_{2l} - B_{2l-1}\right) B_m i\right).$$

Thus,

$$\begin{aligned}\sum_{l=1}^{n} c_{(2l,m)} &= \left(c_3\left(B_m - B_{m-1}\right) + \left(B_2 - B_1\right) B_m i\right) \\ &\quad + \cdots + \left(c_{2n+1}\left(B_m - B_{m-1}\right) + \left(B_{2n} - B_{2n-1}\right) B_m i\right) \\ &= \left(c_3 + \cdots + c_{2n+1}\right)\left(B_m - B_{m-1}\right) \\ &\quad + \left(\left(B_2 - B_1\right) + \cdots + \left(B_{2n} - B_{2n-1}\right)\right) B_m i\end{aligned}$$

$$= \sum_{l=1}^{n} c_{2l+1} \left(B_m - B_{m-1}\right) + \Big(\sum_{l=1}^{n} B_{2l} - \sum_{l=1}^{n} B_{2l-1}\Big) B_m i$$

$$= \sum_{l=2}^{n+1} c_{2l} \left(B_m - B_{m-1}\right) + \Big(\sum_{l=1}^{n} B_{2l} - \sum_{l=1}^{n} B_{2l-1}\Big) B_m i.$$

By item 2 of Proposition 1.1, by items (a) and (b) of Corollary 2.3.6 in [19], the result follows. □

IDENTITY 3.3. The sum of the first p numbers $c(p, m)$, can be described as follows:

$$\sum_{l=1}^{n} c_{(l,m)} = \frac{1}{2}\left(-3 + c_{n+1} - B_{n-1} + 3B_n\right)\left(B_m - B_{m-1}\right) + B_n B_m i.$$

PROOF. Using Theorem 2.3, we have

$$\sum_{l=1}^{n} c_{(l,m)} = \sum_{l=1}^{n} \left(c_{l+1}\left(B_m - B_{m-1}\right) + \left(B_l - B_{l-1}\right) B_m i\right).$$

Thus,

$$\begin{aligned}
\sum_{l=1}^{n} c_{(l,m)} &= \left(c_2\left(B_m - B_{m-1}\right) + \left(B_1 - B_0\right) B_m i\right) \\
&\quad + \cdots + \left(c_{n+1}\left(B_m - B_{m-1}\right) + \left(B_n - B_{n-1}\right) B_m i\right) \\
&= \left(c_2 + \cdots + c_{n+1}\right)\left(B_m - B_{m-1}\right) \\
&\quad + \left(\left(B_1 - B_0\right) + \cdots + \left(B_n - B_{n-1}\right)\right) B_m i \\
&= \left(-c_1 + c_1 + c_2 + \cdots + c_{n+1}\right)\left(B_m - B_{m-1}\right) \\
&\quad + \left(\left(B_1 - B_0\right) + \cdots + \left(B_n - B_{n-1}\right)\right) B_m i \\
&= \Big(-1 + \sum_{l=1}^{n} c_l + c_{n+1}\Big)\left(B_m - B_{m-1}\right) \\
&\quad + \Big(\sum_{l=1}^{n} B_l - B_0 + \sum_{l=1}^{n-1} B_l\Big) B_m i.
\end{aligned}$$

By item 1 of Proposition 1.1, the fact that $B_0 = 0$ and item 6 of Proposition 2.6 in [5], the result follows. □

IDENTITY 3.4. The sum of the numbers $c(n,p)$ of odd index p can be given by:

$$\sum_{j=1}^{m} c_{(n,2j-1)} = \left(B_m^2 - B_m B_{m+1}\right) c_{n+1} + \left(B_n - B_{n-1}\right) B_m^2 i.$$

PROOF. By Theorem 2.3, we have

$$\sum_{j=1}^{m} c_{(n,2j-1)} = \sum_{j=1}^{m} \left(c_{n+1}\left(B_{2j-1} - B_{2j-2}\right) + \left(B_n - B_{n-1}\right) B_{2j-1} i\right).$$

Thus,

$$\begin{aligned}\sum_{j=1}^{m} c_{(n,2j-1)} &= c_{n+1} \sum_{j=1}^{m} \left(B_{2j-1} - B_{2j-2}\right) + \left(B_n - B_{n-1}\right) \sum_{j=1}^{m} B_{2j-1} i \\ &= c_{n+1} \Big(\sum_{j=1}^{m} B_{2j-1} - \sum_{j=1}^{m} B_{2j-2} \Big) + \left(B_n - B_{n-1}\right) \sum_{j=1}^{m} B_{2j-1} i \\ &= \sum_{j=1}^{m} B_{2j-1} \left(c_{n+1} + \left(B_n - B_{n-1}\right) i\right) - c_{n+1} \sum_{j=1}^{m} B_{2j-2}\end{aligned}$$

The result follows, using items (a) and (b) of Corollary 2.3.6 in [19]. □

IDENTITY 3.5. The sum of the numbers $c(n,\ p)$ of even index p can be described by:

$$\sum_{j=1}^{m} c_{(n,2j)} = \left(B_m B_{m+1} - B_m^2\right) c_{n+1} + \left(B_n - B_{n-1}\right) B_m B_{m+1} i.$$

PROOF. Using Theorem 2.3, we have

$$\sum_{j=1}^{m} c_{(n,2j)} = \sum_{j=1}^{m} \left(c_{n+1}\left(B_{2j} - B_{2j-1}\right) + \left(B_n - B_{n-1}\right) B_{2j} i\right).$$

Thus,

$$\sum_{j=1}^{m} c_{(n,2j)} = c_{n+1} \sum_{j=1}^{m} \left(B_{2j} - B_{2j-1}\right) + \left(B_n - B_{n-1}\right) \sum_{j=1}^{m} B_{2j} i$$

$$= c_{n+1}\Big(\sum_{j=1}^{m} B_{2j} - \sum_{j=1}^{m} B_{2j-1}\Big) + (B_n - B_{n-1}) \sum_{j=1}^{m} B_{2j} i$$
$$= \sum_{j=1}^{m} B_{2j}\left(c_{n+1} + (B_n - B_{n-1})\, i\right) - c_{n+1} \sum_{j=1}^{m} B_{2j-1}.$$

By Corollary 2.3.6, items (b) and (a) in [19], the result follows. □

IDENTITY 3.6. The sum of the first p numbers $c(n,p)$, can be described by:

$$\sum_{j=1}^{m} c_{(n,j)} = \frac{1}{4}\left(B_{m-1} - 2B_m + B_{m+1}\right) c_{n+1} - \frac{1}{4}\left(1 + B_m - B_{m+1}\right)\left(B_n - B_{n-1}\right) i.$$

PROOF. By Theorem 2.3, we have

$$\sum_{j=1}^{m} c_{(n,j)} = \sum_{j=1}^{m} \left(c_{n+1}\left(B_j - B_{j-1}\right) + \left(B_n - B_{n-1}\right) B_j i\right).$$

Thus,

$$\sum_{j=1}^{m} c_{(n,j)} = c_{n+1} \sum_{j=1}^{m} (B_j - B_{j-1}) + (B_n - B_{n-1}) \sum_{j=1}^{m} B_j i$$
$$= c_{n+1}\Big(\sum_{j=1}^{m} B_j - \sum_{j=1}^{m} B_{j-1}\Big) + (B_n - B_{n-1}) \sum_{j=1}^{m} B_j i$$
$$= \sum_{j=1}^{m} B_j \left(c_{n+1} + (B_n - B_{n-1})\, i\right) - c_{n+1} \sum_{j=1}^{m} B_{j-1}$$
$$= \sum_{j=1}^{m} B_j \left(c_{n+1} + (B_n - B_{n-1})\, i\right) - c_{n+1}\Big(B_0 + \sum_{j=1}^{m-1} B_j\Big).$$

By [5, Proposition 2.6, item 6] and the fact that $B_0 = 0$, the result follows. □

4. Bidimensional cobalancing numbers

In this section we will present the bidimensional version of cobalancing numbers, starting with the following definition:

DEFINITION 4.1. The *bidimensional cobalancing numbers* $b_{(m,n)}$, $\forall n, m \in \mathbb{N}_0$ are defined by:

$$b_{(n,m)} = \frac{1}{8}c_{(n+1,m)} - \frac{3}{8}c_{(n,m)} - \frac{1}{2},$$

where $c_{(i,j)}$ represents the bidimensional Lucas-cobalancing numbers studied in the previous sections.

In particular, for $m = n = 0$, we get

$$b_{(0,0)} = \frac{1}{8}c_{(1,0)} - \frac{3}{8}c_{(0,0)} - \frac{1}{2} = \frac{1}{8} \cdot 7 - \frac{3}{8} \cdot 1 - \frac{1}{2} = 0.$$

For $n = 0$ and $m = 1$, we obtain that

$$\begin{aligned} b_{(0,1)} &= \frac{1}{8}c_{(1,1)} - \frac{3}{8}c_{(0,1)} - \frac{1}{2} \\ &= \frac{1}{8}(7+i) - \frac{3}{8}(1+i) - \frac{1}{2} \\ &= \frac{7}{8} + \frac{1}{8}i - \frac{3}{8} - \frac{3}{8}i - \frac{1}{2} = -\frac{2}{8}i = -\frac{1}{4}i. \end{aligned}$$

For $n = 1$ and $m = 0$, we obtain that

$$b_{(1,0)} = \frac{1}{8}c_{(2,0)} - \frac{3}{8}c_{(1,0)} - \frac{1}{2} = 2.$$

For $m = n = 1$, it comes that

$$b_{(1,1)} = \frac{1}{8}c_{(2,1)} - \frac{3}{8}c_{(1,1)} - \frac{1}{2}.$$

By Lemma 2.2, item (c) we have that $c_{(2,1)} = 41 + 5i$. Then,

$$b_{(1,1)} = \frac{1}{8}(41+5i) - \frac{3}{8}(7+i) - \frac{1}{2} = 2 + \frac{1}{4}i.$$

The following results present identities satisfied by bidimensional cobalancing numbers in whose expressions some numbers $c_{(m,n)}$, $B_{(m,n)}$ and also balancing numbers are involved.

LEMMA 4.2. *The following properties are valid for all non-negative integers m and n and for the numbers $b_{(m,n)}$:*
(a) $b_{(n,0)} = b_{n+1}$;
(b) $b_{(0,m)} = -\frac{1}{4}B_{(1,m)} + \frac{3}{4}\left(B_m - B_{m-1}\right) - \frac{1}{2}$;
(c) $b_{(n,1)} = b_{n+1} + \frac{1}{2}\left(\frac{1}{4}B_{n+1} - B_n + \frac{3}{4}B_{n-1}\right)i$;
(d) $b_{(1,m)} = \frac{1}{8}\left(c_{(2,m)} - 3B_{(1,m)}\right) - \frac{9}{4}\left(B_m - B_{m-1}\right) - \frac{1}{2}$.

PROOF. The proof of (a) will be done by induction on n:
For $n = 0$ we have that $b_{(0,0)} = 0 = b_1$ and the proposition is true.
For $n = 1$ we have that $b_{(1,0)} = 2 = b_2$ and the proposition is also true.
Suppose that the proposition is true for values less than or equal to n. Let us show that it remains true for $n+1$:
By Definition 4.1 and by Lemma 2.2, item (a),

$$\begin{aligned}
b_{(n+1,0)} &= \frac{1}{8}c_{(n+2,0)} - \frac{3}{8}c_{(n+1,0)} - \frac{1}{2} \\
&= \frac{1}{8}c_{n+3} - \frac{3}{8}c_{n+2} - \frac{1}{2} \\
&= \frac{1}{8}c_{n+3} - \frac{3}{8}\left(3c_{n+1} + 8b_{n+1} + 4\right) - \frac{1}{2} \\
&= \frac{1}{8}c_{n+3} - \frac{9}{8}c_{n+1} - 3b_{n+1} - 2 \\
&= \frac{1}{8}\left(3c_{n+2} + 8b_{n+2} + 4\right) - \frac{9}{8}c_{n+1} - 3b_{n+1} - 2 \\
&= 3\left(\frac{1}{8}c_{n+2} - \frac{3}{8}c_{n+1} - \frac{1}{2}\right) + 2 + b_{n+2} - 3b_{n+1} - 2 \\
&= 3b_{n+1} + 2 + b_{n+2} - 3b_{n+1} - 2 = b_{n+2},
\end{aligned}$$

which ends the proof of (a).
The proof of (b) will be done by induction on m:
For $m = 0$, since $B_{(1,0)} = 1$ and $B_{-n} = -B_n$ (see [7]), we have that $b_{(0,0)} = -\frac{1}{4}B_{(1,0)} + \frac{3}{4}\left(B_0 - B_{-1}\right) - \frac{1}{2} = -\frac{1}{4}(1) + \frac{3}{4}(1) - \frac{1}{2} = 0$ and the proposition is true.
For $m = 1$, since $B_{(1,1)} = 1 + i$ (see [7]), we have that $b_{(0,1)} = -\frac{1}{4}B_{(1,1)} + \frac{3}{4}\left(B_1 - B_0\right) - \frac{1}{2} = -\frac{1}{4}(1+i) + \frac{3}{4}(1) - \frac{1}{2} = -\frac{1}{4}i$ and the proposition is also true.

Suppose that the proposition is true for all values less than or equal to m. Let us show that it remains true for $m+1$:

By Definition 4.1 and by Lemma 2.2, items (d) and (b),

$$\begin{aligned}
b_{(0,m+1)} &= \frac{1}{8}c_{(1,m+1)} - \frac{3}{8}c_{(0,m+1)} - \frac{1}{2} \\
&= \frac{1}{8}\left(B_{(1,m+1)} + 6\left(B_{m+1} - B_m\right)\right) - \frac{3}{8}B_{(1,m+1)} - \frac{1}{2} \\
&= \frac{1}{8}B_{(1,m+1)} + \frac{3}{4}\left(B_{m+1} - B_m\right) - \frac{3}{8}B_{(1,m+1)} - \frac{1}{2} \\
&= -\frac{1}{4}B_{(1,m+1)} + \frac{3}{4}\left(B_{m+1} - B_m\right) - \frac{1}{2},
\end{aligned}$$

which ends the proof of (b).

The proof of (c) will also be done by induction on n:

For $n = 0$ again, since $B_{-n} = -B_n$ (see [7]), we have that $b_{(0,1)} = b_0 + \frac{1}{2}\left(\frac{1}{4}B_1 - B_0 + \frac{3}{4}B_{-1}\right)i = 0 + \frac{1}{2}\left(\frac{1}{4}\cdot 1 - 0 + \frac{3}{4}(-1)\right)i = -\frac{1}{4}i$ and the proposition is true.

For $n = 1$ we have that

$$b_{(1,1)} = b_1 + \frac{1}{2}\Big(\frac{1}{4}B_2 - B_1 + \frac{3}{4}B_0\Big)i = 2 + \frac{1}{2}\Big(\frac{1}{4}\cdot 6 - 1 + \frac{3}{4}\cdot 0\Big)i = 2 + \frac{1}{4}i$$

and the proposition is also true.

Suppose that the proposition is true for all values less than or equal to n. Let us show that it remains true for $n+1$:

By Definition 4.1 and Lemma 2.2, item (c),

$$\begin{aligned}
b_{(n+1,1)} &= \frac{1}{8}c_{(n+2,1)} - \frac{3}{8}c_{(n+1,1)} - \frac{1}{2} \\
&= \frac{1}{8}\left(c_{n+3} + \left(B_{n+2} - B_{n+1}\right)i\right) - \frac{3}{8}\left(c_{n+2} + \left(B_{n+1} - B_n\right)i\right) - \frac{1}{2} \\
&= \frac{1}{8}c_{n+3} + \frac{1}{8}\left(B_{n+2} - B_{n+1}\right)i - \frac{3}{8}c_{n+2} - \frac{3}{8}\left(B_{n+1} - B_n\right)i - \frac{1}{2} \\
&= \frac{1}{8}c_{n+3} - \frac{3}{8}c_{n+2} - \frac{1}{2} + \frac{1}{8}B_{n+2}i - \frac{1}{8}B_{n+1}i - \frac{3}{8}B_{n+1}i + \frac{3}{8}B_n i \\
&= b_{n+2} + \frac{1}{8}B_{n+2}i - \frac{1}{2}B_{n+1}i + \frac{3}{8}B_n i \\
&= b_{n+2} + \frac{1}{2}\Big(\frac{1}{4}B_{n+2} - B_{n+1} + \frac{3}{4}B_n\Big)i,
\end{aligned}$$

which ends the proof of (c).

The proof of (d) will also be done by induction on m:

For $m = 0$, given that in [7], $B_{-n} = -B_n$, we have that $b_{(1,0)} = \frac{1}{8}\left(c_{(2,0)} - 3B_{(1,0)}\right) - \frac{9}{4}\left(B_0 - B_{-1}\right) - \frac{1}{2} = 2$ and the proposition is true.

For $m = 1$, given Lemma 2.2, item (c) and again, in [7], the fact that $B_{(1,1)} = 1 + i$, we have that $b_{(1,1)} = \frac{1}{8}\left(c_{(2,1)} - 3B_{(1,1)}\right) - \frac{9}{4}\left(B_1 - B_0\right) - \frac{1}{2} = \frac{1}{8}\left((41 + 5i) - 3\left(1 + i\right)\right) - \frac{9}{4}\left(1\right) - \frac{1}{2} = 2 + \frac{1}{4}i$ and the proposition is also true.

Suppose that the proposition is true for values less than or equal to m. Let us show that it remains true for $m + 1$:

By Definition 4.1 and by Lemma 2.2, item (d),

$$\begin{aligned} b_{(1,m+1)} &= \frac{1}{8}c_{(2,m+1)} - \frac{3}{8}c_{(1,m+1)} - \frac{1}{2} \\ &= \frac{1}{8}c_{(2,m+1)} - \frac{3}{8}\left(B_{(1,m+1)} + 6\left(B_{m+1} - B_m\right)\right) - \frac{1}{2} \\ &= \frac{1}{8}c_{(2,m+1)} - \frac{3}{8}B_{(1,m+1)} - \frac{9}{4}\left(B_{m+1} - B_m\right) - \frac{1}{2} \\ &= \frac{1}{8}\left(c_{(2,m+1)} - 3B_{(1,m+1)}\right) - \frac{9}{4}\left(B_{m+1} - B_m\right) - \frac{1}{2}, \end{aligned}$$

which ends the proof of (d). □

5. Some identities involving bidimensional cobalancing numbers

From now on, we will explore some bidimensional identities of bidimensional cobalancing sequence, using certain properties attached to it.

Identity 5.1. The sum of the numbers $b(p, m)$ of odd index p, can be described by:

$$\begin{aligned} \sum_{l=1}^{n} b_{(2l-1,m)} &= \frac{1}{8}\left(\left(C_n + B_{n+2}\right)B_{n+1} - 3B_n^2\right)\left(B_m - B_{m-1}\right) \\ &\quad + \frac{1}{4}\left(3B_{n-1} - 2B_n - B_{n+1}\right)B_nB_mi - \frac{1}{2}n. \end{aligned}$$

Proof. By Definition 4.1, we have

$$\sum_{l=1}^{n} b_{(2l-1,m)} = \sum_{l=1}^{n}\Big(\frac{1}{8}c_{(2l,m)} - \frac{3}{8}c_{(2l-1,m)} - \frac{1}{2}\Big).$$

Thus,

$$\sum_{l=1}^{n} b_{(2l-1,m)} = \Big(\sum_{l=1}^{n} \frac{1}{8} c_{(2l,m)}\Big) - \Big(\sum_{l=1}^{n} \frac{3}{8} c_{(2l-1,m)}\Big) - \Big(\sum_{l=1}^{n} \frac{1}{2}\Big)$$
$$= \frac{1}{8}\sum_{l=1}^{n} c_{(2l,m)} - \frac{3}{8}\sum_{l=1}^{n} c_{(2l-1,m)} - \frac{1}{2}\sum_{l=1}^{n} 1.$$

The result follows, by Identities 3.2 and 3.1, by Theorem 2.4.3 in [19] and by Corollary 2.3.5, item (b) also in [19]. □

IDENTITY 5.2. The sum of numbers $b(p,m)$ of even index p, can be described by:

$$\sum_{l=1}^{n} b_{(2l,m)} = \frac{1}{8}\left(\left(C_{n+1} + B_{n+3}\right) B_{n+2} - 3B_{n+1}^{2}\right)\left(B_m - B_{m-1}\right)$$
$$+ \frac{1}{8}\left(\left(B_{n+2} - 4B_n\right) B_{n+1} + 3B_n^2\right) B_m i - \frac{1}{2} n.$$

PROOF. By Definition 4.1, we have

$$\sum_{l=1}^{n} b_{(2l,m)} = \sum_{l=1}^{n} \Big(\frac{1}{8} c_{(2l+1,m)} - \frac{3}{8} c_{(2l,m)} - \frac{1}{2}\Big).$$

Thus,

$$\sum_{l=1}^{n} b_{(2l,m)} = \Big(\sum_{l=1}^{n} \frac{1}{8} c_{(2l+1,m)}\Big) - \Big(\sum_{l=1}^{n} \frac{3}{8} c_{(2l,m)}\Big) - \Big(\sum_{l=1}^{n} \frac{1}{2}\Big)$$
$$= \frac{1}{8}\sum_{l=1}^{n} c_{(2l+1,m)} - \frac{3}{8}\sum_{l=1}^{n} c_{(2l,m)} - \frac{1}{2}\sum_{l=1}^{n} 1.$$

By Theorem 2.3, by Identity 3.2, by the fact that $\sum_{i=m}^{n} x_i = \sum_{i=m+p}^{n+p} x_{i-p}$, by Proposition 1.1, item 2 and also item 3, by Corollary 2.3.6, item (b) in [19] and by Theorem 2.4.3 also in [19], the result follows. □

Since the idea behind the proof is similar to the ones we have just presented, we will omit the corresponding proofs in the results that follow.

Identity 5.3. The sum of the first p numbers $b(p,\ m)$, can be described by:

$$\sum_{l=1}^{n} b_{(l,m)} = \frac{1}{16}\left(10 + 8b_n + 10B_n + 3\left(B_{n+1} - B_{n-1}\right)\right)\left(B_m - B_{m-1}\right) - \frac{1}{8}C_n B_m i - \frac{1}{2}n.$$

Identity 5.4. The sum of the numbers $b(n,p)$ of odd index, can be described by:

$$\sum_{j=1}^{m} b_{(n,2j-1)} = \frac{1}{8}\left(c_{n+2} - 3c_{n+1}\right)\left(B_m^2 - B_m B_{m+1}\right) + \frac{1}{8}\left(B_{n+1} - 4B_n + 3B_{n-1}\right) B_m^2 i - \frac{1}{2}m.$$

Identity 5.5. The sum of the numbers $b(n,p)$ of even index, can be given by:

$$\sum_{j=1}^{m} b_{(n,2j)} = \frac{1}{8}\left(c_{n+2} - 3c_{n+1}\right)\left(B_m B_{m+1} - B_m^2\right) + \frac{1}{8}\left(B_{n+1} - 4B_n + 3B_{n-1}\right) B_m B_{m+1} i - \frac{1}{2}m.$$

Identity 5.6. The sum of p numbers $b(n,p)$, can be given by:

$$\sum_{j=1}^{m} b_{(n,j)} = \frac{1}{32}\left(c_{n+2} - 3c_{n+1}\right)\left(B_{m-1} - 2B_m + B_{m+1}\right) - \frac{1}{8}\left(B_{n+1} - 4B_n + 3B_{n-1}\right)\left(1 + B_m - B_{m+1}\right) i - \frac{1}{2}m.$$

6. Conclusion

This research paper presents new sequences of numbers in their bidimensional versions. Some identities involved in these sequences are stated and we think that this work makes a contribution to the mathematical field and gives an opportunity to researchers interested in this topic of numerical sequences.

Acknowledgements. The first author is grateful to the Portuguese Funds through the Calouste Gulbenkian Foundation. The second author is part of the Mathematics Research Group of the University of Santiago de Compostela – USC GI 2136 and also of the Project Homology, Homotopy and Categorical Invariants in Groups and Non-Associative Algebras – PID2020 115155GB I00, within the goal Dynamical Systems, Genetic Algebras and Non-Stochastic Quadratic Operators. The other authors are members of the CMAT-UTAD Research Center and the last author is also a collaborator member of the CIDTFF Research Center.

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