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SOME GENERAL THEOREMS ABOUT A CLASS OF SETS OF NUMBERS

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Abstract. We prove a theorem which unifies some formulas, for example the counting function, of some sets of numbers including all positive integers, h-free numbers, h-full numbers, etc. We also establish a conjecture and give some examples where the conjecture holds.

1. Introduction

Let $h \geq 1$ be an arbitrary but fixed positive integer. A number is h-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to h, that is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is h-full if $s_i \geq h$ $(i = 1, \ldots, r)$ $(r \geq 1)$. If $h = 1$ we obtain all the positive integers. If $h = 2$ the numbers are called square-full or powerful.

Let $h \geq 2$ be an arbitrary but fixed positive integer. A number is h-free if all the distinct primes in its prime factorization have multiplicity (or exponent) less than or equal to $h-1$, that is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is h-free if $s_i \leq h-1$ $(i = 1, \ldots, r)$ $(r \geq 1)$. If $h = 2$ we obtain all the square-free numbers. If $h = 3$ the numbers are called cube-free.

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Let $Q_h(x)$ be the number of h-free numbers not exceeding x. It is well-known ([1]) that

$$Q_h(x) = \frac{1}{\zeta(h)} x + o(x). \tag{1.1}$$

A number is k-free h-full ($k > h$) if it is simultaneously a k-free number and h-full number, that is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is k-free h-full if $h \leq s_i \leq k-1$ $(i = 1, \ldots, r)$ $(r \geq 1)$.

These three special cases of numbers are very well studied and they are particular cases of the following sets of numbers.

Definition 1.1. We shall associate each prime p with a finite or infinite set E_p of possible exponents, namely $E_p = \{k, k_{1,p}, k_{2,p}, \ldots\}$, where $1 \leq k < k_{1,p} < k_{2,p} < \cdots$. Therefore all sets E_p have the same least element k. Let us consider the positive integers n whose prime factorization is of the form $n = q_1^{r_1} q_2^{r_2} \cdots q_s^{r_s}$, where q_i $(i = 1, \ldots, s)$ are distinct primes and $r_i \in E_{q_i}$ $(i = 1, \ldots, s)$ are the exponents. The set of these positive integers n will be denoted by A.

Consequently, h-free, h-full and k-free h-full numbers ($k > h$) are particular sets of A.

In this article we prove some general theorems about sets A which unifies some apparently unconnected formulas. For example, let $A(x)$ be the number of positive integers n in the set A not exceeding x, that is, the counting function of the set A. We shall prove that

$$A(x) = c x^{\frac{1}{k}} + o\left(x^{\frac{1}{k}}\right),$$

where the positive constant c is

$$c = \frac{6}{\pi^2} \left(\prod_p \left(1 + \frac{p}{p+1} \left(\frac{1}{p^{\frac{k_{1,p}}{k}}} + \frac{1}{p^{\frac{k_{2,p}}{k}}} + \cdots \right) \right) \right).$$

We also obtain in these theorems some general results about partitions of a set of positive integers into infinite disjoint subsets.

Consider, as motivation, the following example.

Example 1.2. We can divide the set of all positive integers, whose positive density is 1, in the following infinite disjoint subsets. The numbers whose greatest exponent in their prime factorization is 1, that is, the 2-free numbers or square-free numbers and consequently they have positive density (see (1.1)) $\frac{1}{\zeta(2)}$. The numbers whose greatest exponent in their prime factorization is 2, they have positive density $\frac{1}{\zeta(3)} - \frac{1}{\zeta(2)}$. The numbers whose greatest exponent in their prime factorization is 3, they have positive density $\frac{1}{\zeta(4)} - \frac{1}{\zeta(3)}$, etc. Now, we shall prove that the sum of the infinite positive densities is the density of the union, namely 1. The proof is trivial, since

$$\frac{1}{\zeta(2)} + \sum_{k=3}^{\infty}\Big(\frac{1}{\zeta(k)} - \frac{1}{\zeta(k-1)}\Big) = \lim_{m\to\infty}\bigg(\frac{1}{\zeta(2)} + \sum_{k=3}^{m}\Big(\frac{1}{\zeta(k)} - \frac{1}{\zeta(k-1)}\Big)\bigg)$$
$$= \lim_{m\to\infty}\frac{1}{\zeta(m)} = 1,$$

for it is well-known that $\zeta(m) \to 1$ as $m \to \infty$.

The author knows many examples where the following conjecture is true but cannot prove it. Example 1.2 is a particular case of this conjecture.

Conjecture 1.3. *Let s be an arbitrary fixed positive integer. Suppose that we have disjoint infinite sets of numbers S_i $(i \geq 1)$ whose union is the set S, that is, a partition of the set S. Let $S_i(x)$ be the number of numbers in the set S_i not exceeding x and let $S(x)$ be the number of numbers in the set S not exceeding x. Suppose that*

$$S_i(x) = \rho_i x^{\frac{1}{s}} + o\big(x^{\frac{1}{s}}\big),$$

where $\rho_i > 0$ and suppose that $S(x) \leq Hx^{\frac{1}{s}}$, where $H > 0$. Then

$$S(x) = \sigma x^{\frac{1}{s}} + o\big(x^{\frac{1}{s}}\big),$$

where

$$\sigma = \sum_{i=1}^{\infty} \rho_i.$$

Note that the sum $\sum_{i=1}^{n} \rho_i$ of positive terms ρ_i is bounded by H for all n. Therefore the series $\sum_{i=1}^{\infty} \rho_i$ has a certain positive sum σ.

We suppose that $\rho_i > 0$ for $i \geq 1$ since in the contrary case the conjecture can be false. For example, the number of square-free with k prime factors $(k \geq 1)$ is $o(x)$ by Landau's Theorem ([1]) and the set of all square-free has positive density $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$ (see equation (1.1)).

Perhaps, it is necessary add to the conjecture some additional conditions.

In the following theorem, we give a sufficient condition such that Conjecture 1.3 is true. Before, note that the equation (see Conjecture 1.3)

$$S_i(x) = \rho_i x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right) \tag{1.2}$$

implies that there exists a positive number c_i such that

$$S_i(x) \leq c_i \rho_i x^{\frac{1}{s}} \quad (x \geq 1).$$

Theorem 1.4. *Suppose that there exists a positive number C such that $c_i \leq C$ for all $i \geq 1$. Then Conjecture 1.3 holds.*

Proof. Given $\epsilon > 0$, there exists M depending on ϵ such that

$$\sum_{i>M} \rho_i < \epsilon. \tag{1.3}$$

We have (see (1.2))

$$\begin{aligned} S(x) &= \Big(\sum_{1 \leq i \leq M} \rho_i \Big) x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right) + F(x) \\ &= \sigma x^{\frac{1}{s}} - \Big(\sum_{i>M} \rho_i \Big) x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right) + F(x), \end{aligned}$$

where $F(x)$ is the contribution to $S(x)$ of the rest of the numbers not exceeding x. Therefore (see (1.3))

$$0 \leq F(x) \leq \sum_{i>M} C\rho_i x^{\frac{1}{s}} = Cx^{\frac{1}{s}} \sum_{i>M} \rho_i < C\epsilon x^{\frac{1}{s}}.$$

By combining these equations, we obtain

$$\left| \frac{S(x)}{x^{\frac{1}{s}}} - \sigma \right| \leq \epsilon + \epsilon + C\epsilon \quad (x \geq x_\epsilon),$$

that is,

$$S(x) = \sigma x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right),$$

since $\epsilon > 0$ can be arbitrarily small. □

Similar proofs, as the proof of Theorem 1.4, will be used in the proofs of other theorems in this article.

2. Lemmas

Let $h \geq 1$ and let $A_h(x)$ be the number of h-full numbers not exceeding x. It was proved by Ivić and Shiu (see either [2, Chapter 14] or [3]) that

$$A_h(x) = \gamma_{0,h} x^{\frac{1}{h}} + \gamma_{1,h} x^{\frac{1}{h+1}} + \cdots + \gamma_{h-1,h} x^{\frac{1}{2h-1}} + \Delta_h(x), \tag{2.1}$$

where $\Delta_h(x) = O\left(x^{\rho}\right)$ for ρ small.

We need the weaker lemma.

Lemma 2.1. *The following asymptotic formula holds*

$$A_h(x) = \gamma_{0,h} x^{\frac{1}{h}} + o\big(x^{\frac{1}{h}}\big), \tag{2.2}$$

where

$$\gamma_{0,h} = \frac{6}{\pi^2} C_h = \frac{6}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{h}} - 1)}\right) = \prod_p \left(1 + \frac{p - p^{\frac{1}{h}}}{p^2\big(p^{\frac{1}{h}} - 1\big)}\right). \tag{2.3}$$

Note that if $h = 1$ then we obtain the trivial formula $A_1(x) = x + o(x)$.

Proof. Equation (2.2) is a weak consequence of (2.1). For equation (2.3) see the reference [4]. □

Lemma 2.2. *Let $h \geq 1$ be an arbitrary but fixed integer. The following series converges*

$$\sum_Q \frac{1}{Q^{\frac{1}{h}}},$$

where the sum runs over all $(h+1)$-full numbers Q.

Proof. Let a_n be the n-th $(h+1)$-full number and let $A_{h+1}(x)$ be the number of $(h+1)$-full numbers not exceeding x. By Lemma 2.1, we have $A_{h+1}(x) \sim \gamma_{0,h} \sqrt[h+1]{x}$. Therefore if $x = a_n$ we obtain $n = A_{h+1}(a_n) \sim \gamma_{0,h} \sqrt[h+1]{a_n}$, that is, $a_n \sim \frac{n^{h+1}}{\gamma_{0,h}^{h+1}}$. Now, the lemma follows by the Comparison Criterion, since the series $\sum \frac{1}{n^{\frac{h+1}{h}}}$ converges. □

Lemma 2.3. *Let $h \geq 2$ be an arbitrary but fixed integer. Let $r \geq 1$ be an arbitrary but fixed integer. Let us consider r distinct primes $q_1, \ldots, q_r$. Let $B_{q_1,\ldots,q_r}(x)$ be the number of h-free numbers not exceeding x relatively prime to $q_1 \cdots q_r$. The following asymptotic formula holds*

$$B_{q_1,\ldots,q_r}(x) = \frac{1}{\zeta(h)} \prod_{i=1}^{r} \frac{1 - \frac{1}{q_i}}{1 - \frac{1}{q_i^h}} x + o(x).$$

Proof. See [5]. $\square$

We have the following general theorem.

Theorem 2.4. *Let $f(i)$ be a sequence such that $0 < f(i) < 1$ $(i \geq 1)$. Then*

$$\sum_{h=1}^{n} \left(\left(\prod_{i=1}^{h} f(i) \right) \left(\frac{1}{f(h)} - 1 \right) \right) = 1 - \left(\prod_{i=1}^{n} f(i) \right).$$

Proof. Use mathematical induction. $\square$

Corollary 2.5. *If, in addition, $\prod_{i=1}^{n} f(i) \to L$ as $i \to \infty$, then*

$$\sum_{h=1}^{\infty} \left(\left(\prod_{i=1}^{h} f(i) \right) \left(\frac{1}{f(h)} - 1 \right) \right) = 1 - L.$$

3. Main results

Our main results are some general theorems and corollaries about sets A and some examples where Conjecture 1.3 holds.

Theorem 3.1. *Let $A(x)$ be the number of positive integers n in the set A not exceeding x. Then*

(3.1) $$A(x) = c x^{\frac{1}{k}} + o\left(x^{\frac{1}{k}}\right),$$

where the positive constant c is

$$c = \frac{6}{\pi^2} \left(\prod_{p} \left(1 + \frac{p}{p+1} \left(\frac{1}{p^{\frac{k_{1,p}}{k}}} + \frac{1}{p^{\frac{k_{2,p}}{k}}} + \cdots \right) \right) \right).$$

PROOF. The proof is similar to the proof of Theorem 1.4. Let us consider the numbers n in the set A of the form q^k, where q is square-free. The inequality $q^k \le x$ is equivalent to the inequality $q \le x^{\frac{1}{k}}$. Therefore, by equation (1.1) with $h = 2$, the number of $q^k \le x$ is

$$\frac{6}{\pi^2} x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big). \tag{3.2}$$

The rest of the numbers n in the set A are of the form $q^k Q$, where $\gcd(q, Q) = 1$, q is square-free and Q is $(k+1)$-full number. The prime factorization of the $(k+1)$-full number Q is $Q = \prod_{i=1}^{s} s_i^{r_i}$, where s_i are different primes and $r_i \in E_{s_i} \setminus \{k\}$. By Lemma 2.3 with $h = 2$ the number of these numbers n in the set A not exceeding x, that is $q^k Q \le x$, where Q is fixed, is

$$\frac{6}{\pi^2} a(Q) \frac{x^{\frac{1}{k}}}{Q^{\frac{1}{k}}} + o\big(x^{\frac{1}{k}}\big), \tag{3.3}$$

where, for simplicity, we put $a(Q) = \prod_{i=1}^{s} \frac{s_i}{s_i+1}$.

Given $\epsilon > 0$, there exists M, depending on ϵ, such that (Lemma 2.2)

$$\sum_{Q > M} \frac{1}{Q^{\frac{1}{k}}} < \epsilon. \tag{3.4}$$

Equations (3.2), (3.3) and (3.4) give

$$\begin{aligned} A(x) &= \frac{6}{\pi^2} \left(1 + \sum_{Q \le M} a(Q) \frac{1}{Q^{\frac{1}{k}}} \right) x^{\frac{1}{k}} + o\left(x^{\frac{1}{k}} \right) + F(x) \\ &= c x^{\frac{1}{k}} - \frac{6}{\pi^2} \left(\sum_{Q > M} a(Q) \frac{1}{Q^{\frac{1}{k}}} \right) x^{\frac{1}{k}} + o\left(x^{\frac{1}{k}} \right) + F(x), \end{aligned}$$

where

$$c = \frac{6}{\pi^2} \left(1 + \sum_{Q} a(Q) \frac{1}{Q^{\frac{1}{k}}} \right) = \frac{6}{\pi^2} \left(\prod_{p} \left(1 + \frac{p}{p+1} \left(\frac{1}{p^{\frac{k_{1,p}}{k}}} + \frac{1}{p^{\frac{k_{2,p}}{k}}} + \cdots \right) \right) \right)$$

and (see (3.4))

$$0 \le F(x) \le \sum_{M < Q \le x} \left\lfloor \frac{x^{\frac{1}{k}}}{Q^{\frac{1}{k}}} \right\rfloor < \epsilon x^{\frac{1}{k}}.$$

Note that the number of solutions q to the equation $q^k Q \le x$ does not exceed $\left\lfloor x^{\frac{1}{k}} / Q^{\frac{1}{k}} \right\rfloor$.

Therefore, by combining these equations, we obtain equation (3.1), since $\epsilon > 0$ can be arbitrarily small. The theorem is proved. □

COROLLARY 3.2. *Let us consider positive integers n in the set A relatively prime to the square-free $q_1 \cdots q_s$, where q_i $(i = 1, \ldots, s)$ are distinct primes. The set of these positive integers n will be denoted $A_{q_1,\ldots,q_s} \subset A$. The number of positive integers n in the set $A_{q_1,\ldots,q_s}$ not exceeding x will be denoted $A_{q_1,\ldots,q_s}(x)$. We have*

$$A_{q_1,\ldots,q_s}(x) = c' c x^{\frac{1}{k}} + o\left(x^{\frac{1}{k}}\right), \tag{3.5}$$

where the positive constant c' is

$$c' = \frac{1}{\prod_{i=1}^{s}\left(1 + \frac{1}{q_i} + \frac{1}{q_i^{\frac{k_{1,q_i}}{k}}} + \frac{1}{q_i^{\frac{k_{2,q_i}}{k}}} + \cdots\right)}. \tag{3.6}$$

PROOF. The proof is similar to the proof of Theorem 1.4. Let us consider the numbers $n = q^k$ in the set $A_{q_1,\ldots,q_s}$, where q is square-free. The inequality $q^k \leq x$ is equivalent to the inequality $q \leq x^{\frac{1}{k}}$. By Lemma 2.3 with $h = 2$, the number of $q^k \leq x$ is

$$\frac{6}{\pi^2}\left(\prod_{i=1}^{s} \frac{q_i}{q_i + 1}\right) x^{\frac{1}{k}} + o\left(x^{\frac{1}{k}}\right). \tag{3.7}$$

The rest of the numbers n in the set $A_{q_1,\ldots,q_s}$ are of the form $q^k Q$, where $\gcd(q, Q) = 1$, q is square-free and Q is $(k+1)$-full number. The prime factorization of the $(k+1)$-full number Q is $Q = \prod_{i=1}^{s} s_i^{r_i}$, where s_i are different primes, $r_i \in E_{s_i} \setminus \{k\}$ and $\gcd(Q, q_1 \cdots q_s) = 1$. By Lemma 2.3 with $h = 2$, the number of these numbers n in the set $A_{q_1,\ldots,q_s}$ not exceeding x, that is $q^k Q \leq x$, where Q is fixed, is

$$\frac{6}{\pi^2}\left(\prod_{i=1}^{s} \frac{q_i}{q_i + 1}\right) a(Q) \frac{x^{\frac{1}{k}}}{Q^{\frac{1}{k}}} + o\left(x^{\frac{1}{k}}\right), \tag{3.8}$$

where, for simplicity, we put $a(Q) = \prod_{i=1}^{s} \frac{s_i}{s_i+1}$.

Given $\epsilon > 0$, there exists M, depending on ϵ, such that (Lemma 2.2)

$$\sum_{Q > M} \frac{1}{Q^{\frac{1}{k}}} < \epsilon. \tag{3.9}$$

Equations (3.7), (3.8) and (3.9) give

$$\begin{aligned}A_{q_1,\ldots,q_s}(x) &= \frac{6}{\pi^2}\Big(\prod_{i=1}^{s}\frac{q_i}{q_i+1}\Big)\Big(1+\sum_{Q\le M}a(Q)\frac{1}{Q^{\frac{1}{k}}}\Big)x^{\frac{1}{k}}+o\big(x^{\frac{1}{k}}\big)+F(x)\\ &= c''x^{\frac{1}{k}}-\frac{6}{\pi^2}\Big(\prod_{i=1}^{s}\frac{q_i}{q_i+1}\Big)\Big(\sum_{Q>M}a(Q)\frac{1}{Q^{\frac{1}{k}}}\Big)x^{\frac{1}{k}}+o\big(x^{\frac{1}{k}}\big)+F(x),\end{aligned}$$

where

$$\begin{aligned}c'' &= \frac{6}{\pi^2}\Big(\prod_{i=1}^{s}\frac{q_i}{q_i+1}\Big)\Big(1+\sum_{Q}a(Q)\frac{1}{Q^{\frac{1}{k}}}\Big)\\ &= \frac{6}{\pi^2}\Big(\prod_{i=1}^{s}\frac{q_i}{q_i+1}\Big)\Big(\prod_{p\ne q_i}\Big(1+\frac{p}{p+1}\Big(\frac{1}{p^{\frac{k_{1,p}}{k}}}+\frac{1}{p^{\frac{k_{2,p}}{k}}}+\cdots\Big)\Big)\Big)=cc'\end{aligned}$$

and

$$0\le F(x)\le\sum_{M<Q\le x}\left\lfloor\frac{x^{\frac{1}{k}}}{Q^{\frac{1}{k}}}\right\rfloor<\epsilon x^{\frac{1}{k}}.$$

Therefore, by combining these equations, we obtain equation (3.5), since $\epsilon>0$ can be arbitrarily small. The corollary is proved. □

COROLLARY 3.3. *Let p_h be the h-th prime number and let B_{p_h} be the set of positive integers n in the set A such that p_h is their least prime factor. Then the infinite sets B_{p_h} are a partition of the set A, since if $h_1\ne h_2$ then $B_{p_{h_1}}\cap B_{p_{h_2}}$ is the empty set. Let $B_{p_h}(x)$ be the number of positive integers n in the set B_{p_h} not exceeding x. Then*

$$B_{p_h}(x)=c_{p_h}cx^{\frac{1}{k}}+o\big(x^{\frac{1}{k}}\big), \tag{3.10}$$

where the positive constant c_{p_h} is

$$c_{p_h}=\frac{\frac{1}{p_h}+\frac{1}{p_h^{\frac{k_{1,p_h}}{k}}}+\frac{1}{p_h^{\frac{k_{2,p_h}}{k}}}+\cdots}{\prod_{i=1}^{h}\Big(1+\frac{1}{p_i}+\frac{1}{p_i^{\frac{k_{1,p_i}}{k}}}+\frac{1}{p_i^{\frac{k_{2,p_i}}{k}}}+\cdots\Big)} \tag{3.11}$$

and

$$\sum_{h=1}^{\infty}c_{p_h}=1. \tag{3.12}$$

Hence

$$\sum_{h=1}^{\infty} c_{p_h} c = c$$

and consequently Conjecture 1.3 *holds for this partition of the set A (see equation* (3.1)*).*

Proof. The proof is similar to the proof of Theorem 1.4. If $q_1, \ldots, q_s$ is the set of the first h primes, that is $p_1, \ldots, p_h$, then equations (3.5) and (3.6) become

$$A_{p_1,\ldots,p_h}(x) = c'cx^{\frac{1}{k}} + o\left(x^{\frac{1}{k}}\right),$$

where the positive constant c' is

$$c' = \frac{1}{\prod_{i=1}^{h}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^{\frac{k_{1,p_i}}{k}}} + \frac{1}{p_i^{\frac{k_{2,p_i}}{k}}} + \cdots\right)}.$$

Furthermore, there exists a positive constant b such that

$$A_{p_1,\ldots,p_h}(x) < bx^{\frac{1}{k}} \quad (x \geq 1). \tag{3.13}$$

Let us consider the numbers with least prime factor p_h not exceeding x such that p_h has exponent m, that is $p_h^m a \leq x$, where $m \in E_{p_h} = \{k, k_{1,p_h}, k_{2,p_h}, \ldots\}$, with $1 \leq k < k_{1,p_h} < k_{2,p_h} < \cdots$ and $a \in A_{p_1,\ldots,p_h}$. The number of these numbers is

$$c'c\frac{x^{\frac{1}{k}}}{p_h^{\frac{m}{k}}} + o\big(x^{\frac{1}{k}}\big). \tag{3.14}$$

Note that the following series

$$\sum_{m\in E_{p_h}} \frac{1}{p_h^{\frac{m}{k}}}$$

is convergent. Therefore if $\epsilon > 0$, then there exists M such that

$$\sum_{\substack{m\in E_{p_h}\\ m>M}} \frac{1}{p_h^{\frac{m}{k}}} < \epsilon. \tag{3.15}$$

Equations (3.14) and (3.15) give

$$\begin{aligned}B_{p_h}(x) &= c'cx^{\frac{1}{k}}\Bigg(\sum_{\substack{m\in E_{p_h}\\ m\le M}}\frac{1}{p_h^{\frac{m}{k}}}\Bigg)+o\big(x^{\frac{1}{k}}\big)+F(x)\\ &= c_{p_h}cx^{\frac{1}{k}}-c'cx^{\frac{1}{k}}\Bigg(\sum_{\substack{m\in E_{p_h}\\ m> M}}\frac{1}{p_h^{\frac{m}{k}}}\Bigg)+o\big(x^{\frac{1}{k}}\big)+F(x),\end{aligned}$$

where (see (3.13) and (3.15))

$$0\le F(x)\le \sum_{\substack{m\in E_{p_h}\\ m> M}} b\frac{x^{\frac{1}{k}}}{p_h^{\frac{m}{k}}}<b\epsilon x^{\frac{1}{k}}.$$

Finally, by combining these equations, we obtain equation (3.10), since $\epsilon>0$ can be arbitrarily small.

Equation (3.12) is an immediate consequence of equation (3.11), Theorem 2.4 and Corollary 2.5. Note that in this case $L=0$, since $\prod_{i=1}^{\infty}\left(1+\frac{1}{p_i}\right)=\infty$. The corollary is proved. □

Corollary 3.4. *Let $B_{p_h\ge p_s}$ be the set of positive integers n in the set A such that $p_h\ge p_s$ is their least prime factor. Let $B_{p_h\ge p_s}(x)$ be the number of positive integers n in the set $B_{p_h\ge p_s}$ not exceeding x. Then*

$$B_{p_h\ge p_s}(x)=d_s cx^{\frac{1}{k}}+o\big(x^{\frac{1}{k}}\big), \tag{3.16}$$

where the positive constant d_s is

$$d_s=\sum_{h=s}^{\infty}c_{p_h}=\frac{1}{\prod_{i=1}^{s-1}\left(1+\frac{1}{p_i}+\frac{1}{p_i^{\frac{k_{1,p_i}}{k}}}+\frac{1}{p_i^{\frac{k_{2,p_i}}{k}}}+\cdots\right)}.$$

Therefore

$$\lim_{s\to\infty} d_s=0. \tag{3.17}$$

Proof. By Corollary 3.2, we have

$$B_{p_h \geq p_s}(x) = A_{p_1,\ldots,p_{s-1}}(x) = c' c x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big) = d_s c x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big),$$

where

$$d_s = c' = \frac{1}{\prod_{i=1}^{s-1}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^{\frac{k_{1,p_i}}{k}}} + \frac{1}{p_i^{\frac{k_{2,p_i}}{k}}} + \cdots\right)}.$$

On the other hand, by equation (3.12), we have

$$\sum_{h=1}^{s-1} c_{p_h} + \sum_{h=s}^{\infty} c_{p_h} = 1.$$

Therefore

$$\sum_{h=s}^{\infty} c_{p_h} = 1 - \sum_{h=1}^{s-1} c_{p_h}$$

and by Theorem 2.4 (see (3.11))

$$1 - \sum_{h=1}^{s-1} c_{p_h} = c'.$$

The corollary is proved. □

Let $H(n)$ be the greatest exponent in the prime factorization of n. Niven ([6]) proved the following equality

$$\lim_{x \to \infty} \frac{\sum_{n \leq x} H(n)}{x} = 1 + \sum_{k=2}^{\infty}\left(1 - \frac{1}{\zeta(k)}\right). \tag{3.18}$$

In the following theorem we generalize this limit to sets A.

Theorem 3.5. *Let us consider the set A. Then*

$$\sum_{\substack{n\in A\\ n\le x}} H(n) = c_A x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big), \tag{3.19}$$

where the constant c_A is

$$c_A = \frac{6}{\pi^2}\left(k + \sum_{Q\in A} a(Q)\frac{H(Q)}{Q^{\frac{1}{k}}}\right). \tag{3.20}$$

The sum runs over all $(k+1)$-full number Q in the set A, that is, $Q = \prod_{i=1}^{s} s_i^{r_i}$, where s_i are different primes, $r_i \in E_{s_i} \setminus \{k\}$ and $a(Q) = \prod_{i=1}^{s} \frac{s_i}{s_i+1}$.

If $A = N$ is the set of all positive integers then equations (3.19) *and* (3.20) *become*

$$\sum_{n\le x} H(n) = c_N x + o(x),$$

where the constant c_N is the Niven's constant (see (3.18)*)*

$$c_N = 1 + \sum_{k=2}^{\infty}\left(1 - \frac{1}{\zeta(k)}\right). \tag{3.21}$$

Proof. The proof is similar to the proof of Theorem 1.4. Let us consider the numbers n in the set A of the form q^k, where q is square-free. Therefore $H(q^k) = k$. The inequality $q^k \le x$ is equivalent to the inequality $q \le x^{\frac{1}{k}}$. Therefore, by equation (1.1) with $h = 2$, the contribution of the numbers $q^k \le x$ to the sum $\sum_{\substack{n\in A\\ n\le x}} H(n)$ is

$$k\frac{6}{\pi^2} x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big). \tag{3.22}$$

The rest of the numbers n in the set A are of the form $q^k Q$, where $\gcd(q, Q) = 1$, q is square-free and Q is $(k+1)$-full number. The prime factorization of the $(k+1)$-full number Q is $Q = \prod_{i=1}^{s} s_i^{r_i}$, where s_i are different primes and $r_i \in E_{s_i} \setminus \{k\}$. Therefore $H(q^k Q) = H(Q)$. By Lemma 2.3 with $h = 2$, the contribution of these numbers n in the set A not exceeding x to the sum $\sum_{\substack{n\in A\\ n\le x}} H(n)$, that is $q^k Q \le x$, where Q is fixed, is

$$H(Q)\frac{6}{\pi^2} a(Q)\frac{x^{\frac{1}{k}}}{Q^{\frac{1}{k}}} + o\big(x^{\frac{1}{k}}\big), \tag{3.23}$$

where, for simplicity, we put $a(Q) = \prod_{i=1}^{s} \frac{s_i}{s_i+1}$.

Let $d(n)$ be the number of divisors of n. It is well known ([1]) that $d(n) = o(n^\sigma)$ for all $\sigma > 0$ and consequently, since $H(n) < d(n)$, we obtain that the series (see the proof of Lemma 2.2)

$$\sum_{\substack{n\in A\\ n\leq x}} \frac{H(n)}{Q^{\frac{1}{k}}}$$

converges.

Consequently, given $\epsilon > 0$, there exists M, depending on ϵ, such that

$$\sum_{Q>M} \frac{H(Q)}{Q^{\frac{1}{k}}} < \epsilon. \tag{3.24}$$

Equations (3.22), (3.23) and (3.24) give

$$\begin{aligned}\sum_{\substack{n\in A\\ n\leq x}} H(n) &= \frac{6}{\pi^2}\left(k + \sum_{\substack{Q\leq M\\ Q\in A}} a(Q)\frac{H(Q)}{Q^{\frac{1}{k}}}\right)x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big) + F(x)\\ &= c_A x^{\frac{1}{k}} - \frac{6}{\pi^2}\left(\sum_{\substack{Q>M\\ Q\in A}} a(Q)\frac{H(Q)}{Q^{\frac{1}{k}}}\right)x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big) + F(x),\end{aligned}$$

where

$$c_A = \frac{6}{\pi^2}\left(k + \sum_{Q\in A} a(Q)\frac{H(Q)}{Q^{\frac{1}{k}}}\right)$$

and (see (3.24))

$$0 \leq F(x) \leq \sum_{\substack{M<Q\leq x\\ Q\in A}} H(Q)\frac{x^{\frac{1}{k}}}{Q^{\frac{1}{k}}} < \epsilon x^{\frac{1}{k}}.$$

Therefore, by combining these equations, we obtain equation (3.19), since $\epsilon > 0$ can be arbitrarily small.

Now, if A is the set of all positive integers and consequently $k = 1$, we can write (3.20) in the form (3.21). Let us consider the square-full Q with the same greatest exponent m. We have

$$\begin{aligned}&\frac{6}{\pi^2}\sum_{Q} a(Q)\frac{m}{Q} = \frac{6m}{\pi^2}\prod_{p}\left(1 + \frac{p}{p+1}\left(\frac{1}{p^2} + \cdots + \frac{1}{p^m}\right)\right)\\ &\quad - \frac{6m}{\pi^2}\prod_{p}\left(1 + \frac{p}{p+1}\left(\frac{1}{p^2} + \cdots + \frac{1}{p^{m-1}}\right)\right) = m\left(\frac{1}{\zeta(m+1)} - \frac{1}{\zeta(m)}\right),\end{aligned}$$

where Q runs over all square-full Q with the same greatest exponent m. Therefore, we have

$$\frac{6}{\pi^2}\left(1+\sum_Q a(Q)\frac{H(Q)}{Q}\right) = \frac{1}{\zeta(2)} + \sum_{m=2}^{\infty} m\left(\frac{1}{\zeta(m+1)} - \frac{1}{\zeta(m)}\right)$$
$$= 1 + \sum_{k=2}^{\infty}\left(1 - \frac{1}{\zeta(k)}\right).$$

The theorem is proved. □

4. Formulas when a small number of primes is removed from the set *A*

In this section we obtain formulas when a small number of primes is removed from the set A.

We need two lemmas.

The set of square-free numbers has positive density $\frac{6}{\pi^2}$. Let P be the set of all positive prime numbers. In the following lemma we study the counting function of square-free numbers such that their prime factors are in the set $P \setminus B$ where B is a small set of infinite primes.

Lemma 4.1. *Suppose that B is a set of infinite primes p such that the series $\sum_{p\in B}\frac{1}{p}$ converges. Let $C(x)$ be the number of square-free not exceeding x such that their prime factors are in the set $P \setminus B$. Then*

$$C(x) = \frac{6}{\pi^2}\frac{1}{c_B}x + o(x), \tag{4.1}$$

where

$$c_B = \prod_{p\in B}\left(1+\frac{1}{p}\right).$$

Proof. We have

$$\prod_{p\in B}\left(1+\frac{1}{p}\right) = c_B,$$

where c_B is a positive number, since the series $\sum_{p\in B}\frac{1}{p}$ converges.

Therefore, given $\epsilon > 0$, there exists M, depending on ϵ, such that

$$\prod_{\substack{p>M \\ p\in B}} \frac{1}{(1+\frac{1}{p})} > 1-\epsilon$$

and

$$\sum_{\substack{p>M \\ p\in B}} \frac{1}{p} < \epsilon.$$

Lemma 2.3 with $h = 2$ gives

$$\begin{aligned} C(x) = x\frac{6}{\pi^2}\prod_{\substack{p\leq M \\ p\in B}} \frac{1}{(1+\frac{1}{p})} + o(x) - F(x) = \frac{6}{\pi^2}\frac{1}{c_B}x + o(x) \\ + \frac{6}{\pi^2}\prod_{\substack{p\leq M \\ p\in B}} \frac{1}{(1+\frac{1}{p})}\left(1 - \prod_{\substack{p>M \\ p\in B}} \frac{1}{(1+\frac{1}{p})}\right) - F(x), \end{aligned}$$

where

$$0 \leq F(x) \leq \sum_{\substack{p>M \\ p\in B}} \frac{x}{p} < \epsilon x.$$

By combining these equations we obtain equation (4.1), since $\epsilon > 0$ can be arbitrarily small. $\square$

In the following lemma we study the square-free numbers relatively prime to a certain number and such that the prime factors of these square-free are in the set $P \setminus B$ where B is a small set of infinite primes

LEMMA 4.2. *Suppose that B is a set of infinite primes p such that the series $\sum_{p\in A} \frac{1}{p}$ converges. Let $C(x)$ be the number of square-free not exceeding x such that their prime factors are in the set $P - B$ and $C_{q_1,\ldots,q_s}(x)$ the number of these square-free not exceeding x relatively prime to the square-free $q_1 \cdots q_s$, where $q_i \in P \setminus B$. Then*

$$C_{q_1,\ldots,q_s}(x) = \frac{6}{\pi^2}\left(\prod_{i=1}^{s} \frac{1}{1-\frac{1}{q_i}}\right)\frac{1}{c_B}x + o(x).$$

PROOF. The proof is the same as in Lemma 4.1. $\square$

Now, we can prove the main theorem of this section.

THEOREM 4.3. *We shall associate each prime $p \in P \setminus B$, where the set B was defined in Lemma 4.1, with a finite or infinite set E_p of possible exponents, namely $E_p = \{k, k_{1,p}, k_{2,p}, \dots\}$, where $1 \leq k < k_{1,p} < k_{2,p} < \cdots$. Therefore all sets E_p have the same least element k. Let us consider the positive integers n whose prime factorization is of the form $n = q_1^{r_1} q_2^{r_2} \cdots q_s^{r_s}$, where $q_i \in P \setminus B$ $(i = 1, \dots, s)$ are distinct primes and $r_i \in E_{q_i}$ $(i = 1, \dots, s)$ are the exponents. The set of these positive integers n will be denoted by A'.*

Let $A'(x)$ be the number of positive integers n in the set A' not exceeding x. We have

$$A'(x) = \frac{1}{c_B} c x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big),$$

where the positive constant c is

$$c = \frac{6}{\pi^2} \Bigg(\prod_{p \in P - B} \bigg(1 + \frac{p}{p+1} \bigg(\frac{1}{p^{\frac{k_{1,p}}{k}}} + \frac{1}{p^{\frac{k_{2,p}}{k}}} + \cdots \bigg) \bigg) \Bigg).$$

Let us consider the positive integers n in the set A' relatively prime to the square-free $q_1 \cdots q_s$, where $q_i \in P \setminus B$ $(i = 1, \dots, s)$ are distinct primes. The set of these positive integers n will be denoted $A'_{q_1,\dots,q_s} \subset A'$. The number of positive integers n in the set $A'_{q_1,\dots,q_s}$ not exceeding x will be denoted $A'_{q_1,\dots,q_s}(x)$. We have

$$A'_{q_1,\dots,q_s}(x) = \frac{1}{c_B} c' c x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big),$$

where the positive constant c' is

$$c' = \frac{1}{\prod_{i=1}^{s} \bigg(1 + \frac{1}{q_i} + \frac{1}{q_i^{\frac{k_{1,q_i}}{k}}} + \frac{1}{q_i^{\frac{k_{2,q_i}}{k}}} + \cdots \bigg)}.$$

PROOF. The proof is the same as the proof of Theorem 3.1 and Corollary 3.2. In this case we use Lemma 4.1 and Lemma 4.2. □

5. Applications and examples

EXAMPLE 5.1. All positive integers, h-free numbers ($h \geq 2$), h-full numbers and h-full k-free numbers ($k > h$) are particular cases of sets A. Therefore the theorems and corollaries proved in Section 3 are true for these special sets of numbers.

1) All positive integers.

In this case equation (3.1) becomes the trivial equation $A(x) = x + o(x)$.

Equation (3.5) becomes

$$A_{q_1,\ldots,q_s}(x) = \left(\prod_{i=1}^{s}\left(1 - \frac{1}{q_i}\right)\right) x + o(x)$$

and equation (3.10) becomes

$$B_{p_h}(x) = \frac{1}{p_h}\prod_{i=1}^{h-1}\left(1 - \frac{1}{p_i}\right) x + o(x).$$

2) h-free numbers.

For example, in this case equation (3.1) becomes equation (1.1) and equation (3.5) becomes Lemma 2.3.

3) h-full numbers.

For example, in this case equation (3.1) becomes Lemma 2.1 and equation (3.5) becomes

$$A_{q_1,\cdots,q_s}(x) = \left(\prod_{i=1}^{s}\frac{q_i\left(q_i^{\frac{1}{h}} - 1\right)}{q_i\left(q_i^{\frac{1}{h}} - 1\right) + q_i^{\frac{1}{h}}}\right)\gamma_{0,h} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

where $\gamma_{0,h}$ is defined by equation (2.3).

4) h-full k-free numbers ($k > h$).

For example, equation (3.1) becomes

$$A(x) = \frac{6}{\pi^2}\prod_{p}\left(\frac{\frac{1}{p^{1+\frac{1}{h}}} - \frac{1}{p^{\frac{k}{h}}}}{\left(1 + \frac{1}{p}\right)\left(1 - \frac{1}{p^{\frac{1}{h}}}\right)}\right) x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right) \tag{5.1}$$

and equation (3.5) becomes

$$A_{q_1,\ldots,q_s}(x) = \prod_{i=1}^{s}\left(1 + \frac{\frac{1}{q_i} - \frac{1}{q_i^{\frac{k}{h}}}}{1 - \frac{1}{q_i^{\frac{1}{h}}}}\right)^{-1}\frac{6}{\pi^2}\prod_{p}\left(\frac{\frac{1}{p^{1+\frac{1}{h}}} - \frac{1}{p^{\frac{k}{h}}}}{\left(1 + \frac{1}{p}\right)\left(1 - \frac{1}{p^{\frac{1}{h}}}\right)}\right) x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right).$$

Now, we give an example where Conjecture 1.3 holds.

EXAMPLE 5.2. Let $h \geq 2$ be an arbitrary but fixed integer. Equation (5.1) can be written, after some calculations, in the form

$$\begin{aligned} A(x) &= \gamma_{0,h} \prod_p \left(1 - \frac{1}{p^{\frac{k}{h}}} \frac{p^{1+\frac{1}{h}}}{(p+1)\big(p^{\frac{1}{h}} - 1\big) + 1}\right) x^{\frac{1}{h}} + o\big(x^{\frac{1}{h}}\big) \\ &= \rho_k x^{\frac{1}{h}} + o\big(x^{\frac{1}{h}}\big). \end{aligned}$$

We can divide the set of h-full numbers, whose positive density is $\gamma_{0,h}$, in the following infinite disjoint subsets. The numbers whose greatest exponent in their prime factorization is h, and consequently they have positive density ρ_{h+1}. The numbers whose greatest exponent in their prime factorization is $h+1$, they have positive density $\rho_{h+2} - \rho_{h+1}$. The numbers whose greatest exponent in their prime factorization is $h+2$, they have positive density $\rho_{h+3} - \rho_{h+2}$, etc. The sum of the infinite positive densities is the density of the union, namely $\gamma_{0,h}$, that is

$$\rho_{h+1} + \sum_{k=h+2}^{\infty} (\rho_k - \rho_{k-1}) = \gamma_{0,h}. \tag{5.2}$$

Therefore Conjecture 1.3 holds.

The proof of equation (5.2) is as the proof in Example 1.2, since

$$\lim_{k \to \infty} \prod_p \left(1 - \frac{1}{p^{\frac{k}{h}}} \frac{p^{1+\frac{1}{h}}}{(p+1)\big(p^{\frac{1}{h}} - 1\big) + 1}\right) = 1.$$

The proof of this limit is as follows. Note that

$$\prod_{p_i} \left(1 - \frac{1}{p_i^{\frac{k}{h}}} \frac{p_i^{1+\frac{1}{h}}}{(p_i+1)\big(p_i^{\frac{1}{h}} - 1\big) + 1}\right) = \prod_{p_i} \left(1 - \frac{c(p_i)}{p_i^{\frac{k}{h}}}\right),$$

where $\lim_{i \to \infty} c(p_i) = 1$. Therefore, given $\epsilon > 0$, there exists m such that $c(p_i) < 1 + \epsilon$ if $i \geq m$ and consequently $\frac{c(p_i)}{p_i^{\frac{k}{h}}} < \frac{1+\epsilon}{2}$ if $i \geq m$ and $\frac{1}{1 - \frac{c(p_i)}{p_i^{\frac{k}{h}}}} < \frac{1}{1 - \frac{1+\epsilon}{2}} = \frac{2}{1-\epsilon}$ if $i \geq m$. Now, we also have (logarithmic power series and geometric power series)

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \leq x + x^2 + x^3 + \cdots = \frac{x}{1-x},$$

where $0 < x < 1$. Consequently

$$\begin{aligned}
&\lim_{k\to\infty}\prod_{p_i}\left(1-\frac{1}{p_i^{\frac{k}{h}}}\frac{p_i^{1+\frac{1}{h}}}{(p_i+1)\big(p_i^{\frac{1}{h}}-1\big)+1}\right)\\
&=\lim_{k\to\infty}\prod_{i=1}^{m-1}\prod_{p_i}\left(1-\frac{1}{p_i^{\frac{k}{h}}}\frac{p_i^{1+\frac{1}{h}}}{(p_i+1)\big(p_i^{\frac{1}{h}}-1\big)+1}\right)\lim_{k\to\infty}\prod_{i=m}^{\infty}\left(1-\frac{c(p_i)}{p_i^{\frac{k}{h}}}\right)\\
&=\lim_{k\to\infty}\prod_{i=m}^{\infty}\left(1-\frac{c(p_i)}{p_i^{\frac{k}{h}}}\right)=\lim_{k\to\infty}\exp\left(-\sum_{i=m}^{\infty}-\log\left(1-\frac{c(p_i)}{p_i^{\frac{k}{h}}}\right)\right)\\
&=e^0=1,
\end{aligned}$$

since

$$\begin{aligned}
0 &\le \sum_{i=m}^{\infty}-\log\left(1-\frac{c(p_i)}{p_i^{\frac{k}{h}}}\right)\le\sum_{i=m}^{\infty}\frac{c(p_i)}{p_i^{\frac{k}{h}}}\frac{1}{1-\frac{c(p_i)}{p_i^{\frac{k}{h}}}}\\
&\le\frac{2(1+\epsilon)}{(1-\epsilon)}\sum_{i=m}^{\infty}\frac{1}{p_i^{\frac{k}{h}}}\to 0
\end{aligned}$$

as $k\to\infty$, and $(\zeta(s)-1)\to 0$ as $s\to\infty$. The proof is complete.

In the following theorem we study a particular case of Conjecture 1.3.

THEOREM 5.3. *Consider the set A. By Theorem 3.1, we have*

$$A(x)=cx^{\frac{1}{k}}+o\big(x^{\frac{1}{k}}\big). \tag{5.3}$$

Let p_h be the h-th prime number and let B_{p_h} be the set of positive integers n in the set A such that p_h is their least prime factor. Then, by Corollary 3.3, the infinite subsets B_{p_h} are a partition of the set A. Let $B_{p_h}(x)$ be the number of positive integers n in the subset B_{p_h} not exceeding x. Then, by Corollary 3.3,

$$B_{p_h}(x)=c_{p_h}cx^{\frac{1}{k}}+o\big(x^{\frac{1}{k}}\big).$$

Let us consider a set S included in A ($S\subseteq A$) such that $S(x)$ is the number of numbers in the set S not exceeding x. Now, consider the partition of the *set S in the infinite subsets $S_{p_h} = S \cap B_{p_h}$ and suppose that the number of numbers in the subset S_{p_h} not exceeding x is*

$$S_{p_h}(x) = s_{p_h} x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big), \tag{5.4}$$

where s_{p_h} is a positive constant depending of p_h. Then

$$S(x) = s x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big),$$

where

$$s = \sum_{h=1}^{\infty} s_{p_h}$$

and consequently Conjecture 1.3 holds for the set S.

Proof. Note that the series $\sum_{h=1}^{\infty} s_{p_h}$ has increasing partial sums bounded by c (see equation (5.3)) therefore it has a positive sum s.

Given $\epsilon > 0$ there exists a prime p_s, depending of ϵ, such that

$$\sum_{p_h > p_s} s_{p_h} < \epsilon.$$

By equation (5.4), we have

$$\begin{aligned} S(x) &= \Bigg(\sum_{p_h \le p_s} s_{p_h} \Bigg) x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big) + F(x) \\ &= s x^{\frac{1}{k}} - \Bigg(\sum_{p_h > p_s} s_{p_h} \Bigg) x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big) + F(x), \end{aligned}$$

where (see (3.16) and (3.17))

$$0 \le F(x) \le B_{p_h \ge p_s}(x) \le \epsilon x^{\frac{1}{k}}.$$

By combining these equations we obtain

$$S(x) = s x^{\frac{1}{k}} + o\big(x^{\frac{1}{k}}\big),$$

since $\epsilon > 0$ can be arbitrarily small. $\square$

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