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ON ALMOST EVERYWHERE K-ADDITIVE SET-VALUED MAPS

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Dedicated to Professor Kazimierz Nikodem on his 70^{th} birthday

Abstract. Let X be an Abelian group, Y be a commutative monoid, $K \subset Y$ be a submonoid and $F\colon X \to 2^Y \setminus \{\emptyset\}$ be a set-valued map. Under some additional assumptions on ideals $\mathcal{I}_1$ in X and $\mathcal{I}_2$ in X^2, we prove that if F is $\mathcal{I}_2$-almost everywhere K-additive, then there exists a unique up to K K-additive set-valued map $G\colon X \to 2^Y \setminus \{\emptyset\}$ such that $F = G$ $\mathcal{I}_1$-almost everywhere in X. Our considerations refers to the well known de Bruijn's result [1].

1. Introduction and preliminaries

1.1. K-additive set-valued maps

In the paper [9] the following notions of K-subadditive and K-superadditive set-valued maps (shortly called s.v. maps) have been introduced.

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Definition 1. Let $(X,+)$, $(Y,+)$ be commutative monoids[1] (with two various operations) and $K \subset Y$ be a submonoid. Denote by $n(Y) := 2^Y \setminus \{\emptyset\}$. An s.v. map $F\colon X \to n(Y)$ is called:

– *K-subadditive*, if

$$F(x) + F(y) \subset F(x+y) + K, \quad x, y \in X,$$

– *K-superadditive*, if

$$F(x+y) \subset F(x) + F(y) + K, \quad x, y \in X.$$

These notions generalize the well known notions of subadditivity and superadditivity of real functions. Since additive real functions can be characterized as functions which are simultaneously subadditive and superadditive, the following natural definition of K-additivity was introduced in [8].

Definition 2. Let $(X,+),(Y,+)$ be commutative monoids and $K \subset Y$ be a submonoid. An s.v. map $F\colon X \to n(Y)$ is called *K-additive*, if it is simultaneously K-subadditive and K-superadditive.

In the case when $K = \{0\}$ the notion of K-additivity coincides with the definition of additivity of s.v. maps introduced by Nikodem in [12]. If $K = [0,\infty)$, $Y = \mathbb{R}$ and F is additionally single-valued, K-additivity of F means classical additivity of the real function.

For a monoid $(Y,+)$ and a submonoid $K \subset Y$ we introduce the following relation $=_K$:

$$A =_K B \iff (A \subset B + K \ \wedge \ B \subset A + K)$$

for every $A, B \in n(Y)$. It is easy to show that $=_K$ is an equivalence relation in $n(Y)$ satisfying condition

$$(A =_K B \ \wedge \ C =_K D) \implies A + C =_K B + D$$

for every $A, B, C, D \in n(Y)$.

Consequently, for monoids $(X,+)$, $(Y,+)$ and a submonoid $K \subset Y$ we can write that an s.v. map $F\colon X \to n(Y)$ is K-additive, if

$$F(x+y) =_K F(x) + F(y), \quad x, y \in X.$$

We will use this notion in the whole paper.

[1] A monoid is a semigroup with a neutral element.

1.2. Almost everywhere K-additive set-valued maps

Let us recall some known notions.

Definition 3. Let X be an Abelian group. A nonempty family $\mathcal{I} \subset 2^X$ is called a *proper linearly invariant ideal*, if it satisfies the following conditions:

(i) $X \notin \mathcal{I}$,
(ii) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$,
(iii) $A \cup B \in \mathcal{I}$ for every $A, B \in \mathcal{I}$,
(iv) $x - A \in \mathcal{I}$ for every $x \in X$ and $A \in \mathcal{I}$.

Clearly, condition (iv) implies that

$$-A, x + A \in \mathcal{I} \quad \text{for every } x \in X \text{ and } A \in \mathcal{I},$$

because X contains the neutral element and $A = -(-A)$.

For a nonempty set X and a family $\mathcal{I} \subset 2^X$ we say that a *condition is satisfied $\mathcal{I}$-almost everywhere in X* (shortly written $\mathcal{I}$-a.e. in X), if there exists a set $A \in \mathcal{I}$ such that this condition is satisfied for all $x \in X \setminus A$.

Definition 4. Let X be a nonempty set. The ideals $\mathcal{I}_1$ in X and $\mathcal{I}_2$ in X^2 are called *conjugate*, if for every set $A \in \mathcal{I}_2$ the set

$$A[x] := \{y \in X : (x, y) \in A\} \in \mathcal{I}_1, \quad \mathcal{I}_1\text{-a.e. in } X.$$

Remark 1. The most important examples of pairs of proper linearly invariant ideals which are conjugate seem to be:

– the ideal $\mathcal{N}_{\mathbb{R}^n}$ of null sets in $\mathbb{R}^n$ and the ideal $\mathcal{N}_{\mathbb{R}^{2n}}$ of null sets in $\mathbb{R}^{2n}$ (in view of the Fubini theorem),
– the ideal $\mathcal{M}_X$ of meager subsets of a second countable Baire space X and the ideal $\mathcal{M}_{X^2}$ of meager subets of X^2 (according to the Kuratowski-Ulam theorem).

Further examples can be found in [11, Section 17.5].

In 1966 de Bruijn proved the following result.

Theorem 1 ([1, Theorem 1]). *Let $(X, +)$, $(Y, +)$ be Abelian groups and $\mathcal{I}_1, \mathcal{I}_2$ be proper linearly invariant ideals in X and X^2, respectively, which are conjugate. If $f\colon X \to Y$ is additive $\mathcal{I}_2$-a.e. in X^2, then there is a unique homomorphism $g\colon X \to Y$ such that $f = g$ $\mathcal{I}_1$-a.e. in X.*

This theorem answers the question of Erdös [5] from 1960. Independently, problem of Erdös was solved by Jurkat [10] and Denny [3, 4]; a generalization for non-commutative groups was proved by Ger [6, 7] (see also [11, Section 17.6]).

The aim of the paper is proving a generalization of Theorem 1 for s.v. maps which are $\mathcal{I}$-a.e. K-additive.

2. The main result

To prove the main result we need stronger assumptions on ideals than those used by de Bruijn [1] and Ger [6, 7]. Here, a concept of S-conjugate ideals introduced by Chmieliński and Rätz in [2] seems to be much more useful. According to their definition, for a subset S of a field $\mathbb{K}$ and a linear space X over $\mathbb{K}$, ideals $\mathcal{I}_1$ in X and $\mathcal{I}_2$ in X^2 are S-conjugate, if for every set $A \in \mathcal{I}_2$ and $\mathcal{I}_1$-almost all $x \in X$

$$A[x,k] := \{y \in X \colon (x+ky, y) \in A\} \in \mathcal{I}_1 \quad \text{for every } k \in S.$$

We just adapt their notion to the case of a group X.

Definition 5. Let X be an Abelian group. For $x \in X$ let us denote (as usual) elements $0x := 0$ and $-1x := -x$. We say that ideals $\mathcal{I}_1$ in X and $\mathcal{I}_2$ in X^2 are $\{0,-1\}$-*conjugate*, if for every set $A \in \mathcal{I}_2$ there is $V \in \mathcal{I}_1$ such that for every $x \in X \setminus V$ and $\varepsilon \in \{0,-1\}$

$$A[x,\varepsilon] := \{y \in X \colon (x+\varepsilon y, y) \in A\} \in \mathcal{I}_1.$$

Clearly, $A[x,0] = A[x]$ for every $x \in X$. This means that ideals $\mathcal{I}_1$ in an Abelian group X and $\mathcal{I}_2$ in X^2 which are conjugate and satisfy

$$\{y \in X \colon (x-y, y) \in A\} \in \mathcal{I}_1, \quad \mathcal{I}_1\text{-a.e. in } X$$

for every set $A \in \mathcal{I}_2$, have to be $\{0,-1\}$-conjugate (see [2, Remark 1]).

Remark 2. Pairs of ideals given in Remark 1 are $\{0,-1\}$-conjugate (see [2, Examples 3 and 4]).

Now, we are ready to present the main result.

THEOREM 2. *Let $(X,+)$ be an Abelian group, Y be a commutative monoid, $K \subset Y$ be a submonoid, and $\mathcal{I}_1, \mathcal{I}_2$ be proper linearly invariant ideals in X and X^2, respectively, which are $\{0,-1\}$-conjugate. If $F\colon X \to n(Y)$ is an s.v. map which is K-additive $\mathcal{I}_2$-a.e. in X^2, then there exists a K-additive s.v. map $G\colon X \to n(Y)$ such that $F =_K G$ $\mathcal{I}_1$-a.e. in X, which is unique up to K, i.e. if $G_1, G_2\colon X \to n(Y)$ are K-additive s.v. maps such that $F =_K G_1$ and $F =_K G_2$ $\mathcal{I}_1$-a.e. in X, then $G_1 =_K G_2$ on X.*

PROOF. In the proof we use de Bruijn's idea from [1] (see also [11, Theorem 17.6.1]).

Since

$$F(x+y) =_K F(x) + F(y), \quad (x,y) \in X^2 \setminus N,$$

with some $N \in \mathcal{I}_2$, and $\mathcal{I}_1, \mathcal{I}_2$ are $\{0,-1\}$-conjugate, there exists a set $M \in \mathcal{I}_1$ such that

$$N[x], N[x,-1] \in \mathcal{I}_1 \quad \text{for every } x \in X \setminus M.$$

Hence

$$F(x+y) =_K F(x) + F(y) \quad \text{for every } x \in X \setminus M,\ y \in X \setminus N[x] \tag{1}$$

and

$$F(x) =_K F(x-z) + F(z) \quad \text{for every } x \in X \setminus M,\ z \in X \setminus N[x,-1]. \tag{2}$$

Fix arbitrary $x \in X$. Since $M, x - M \in \mathcal{I}_1$, so $M \cup (x - M) \neq X$, and hence there exists $w_x \in X$ such that $w_x \notin M$ and $x - w_x \notin M$. Moreover,

$$L_x := (w_x - N[x - w_x]) \cup N[w_x, -1] \in \mathcal{I}_1.$$

Thus, for $y \in X \setminus L_x \neq \emptyset$ we obtain that $w_x - y \in X \setminus N[x - w_x]$, and then, according to (1)–(2),

$$\begin{aligned} F(x-y) + F(y) &= F(x - w_x + w_x - y) + F(y) \\ &=_K F(x - w_x) + F(w_x - y) + F(y) =_K F(x - w_x) + F(w_x). \end{aligned}$$

Now, define $G\colon X \to n(Y)$ by

$$G(x) = F(x - w_x) + F(w_x), \quad x \in X.$$

Then,

$$G(x) =_K F(x-y) + F(y), \quad x \in X,\ y \in X \setminus L_x. \tag{3}$$

Further, for every $x \in X \setminus M$, $K_x := N[x, -1] \cup L_x \in \mathcal{I}_1$, so, in view of (2),

$$G(x) =_K F(x-t) + F(t) =_K F(x), \quad x \in X \setminus M,\ t \in X \setminus K_x.$$

In this way we proved that $F =_K G$ $\mathcal{I}_1$-a.e. in X.

To prove K-additivity of G on X fix $a, b \in X$. By properties of the ideal $\mathcal{I}_1$, we can choose

$$w \in X \setminus [L_a \cup M \cup (a - M)] \neq \emptyset,$$

$$z \in X \setminus [L_b \cup (L_{a+b} - w) \cup N[w] \cup (b - N[a-w])] \neq \emptyset.$$

Then, $w \in X \setminus L_a$, $z \in X \setminus L_b$ and $w + z \in X \setminus L_{a+b}$, so, by (3), we get

$$F(a-w) + F(w) =_K G(a),$$

$$F(b-z) + F(z) =_K G(b),$$

$$F(a+b-w-z) + F(w+z) =_K G(a+b).$$

Moreover, since $w \in X \setminus M$, $z \in X \setminus N[w]$, $a-w \in X \setminus M$, $b-z \in X \setminus N[a-w]$, according to (1),

$$F(w+z) =_K F(w) + F(z),$$

$$F(a+b-w-z) =_K F(a-w) + F(b-z).$$

Hence, we get

$$\begin{aligned} G(a+b) &=_K F(a+b-w-z) + F(w+z) \\ &=_K F(a-w) + F(b-z) + F(w) + F(z) =_K G(a) + G(b), \end{aligned}$$

which proves K-additivity of G on X.

Finally, assume that there exist two K-additive s.v. maps $G_1, G_2 \colon X \to n(Y)$ such that

$$G_1(x) =_K F(x), \quad x \in X \setminus V_1,$$

$$G_2(x) =_K F(x), \quad x \in X \setminus V_2,$$

for some $V_1, V_2 \in \mathcal{I}_1$. Fix $x \in X$. Since $V_1 \cup V_2 \cup (x - V_1) \cup (x - V_2) \in \mathcal{I}_1$, we can choose $y \in X \setminus [V_1 \cup V_2 \cup (x - V_1) \cup (x - V_2)] \neq \emptyset$. Then, we get

$$\begin{aligned} G_1(x) &=_K G_1(x-y) + G_1(y) =_K F(x-y) + F(y) \\ &=_K G_2(x-y) + G_2(y) =_K G_2(x), \end{aligned}$$

which ends the proof. □

In view of Remark 2, we obtain two important corollaries.

Corollary 3. *Let Y be a commutative monoid and K be a submonoid of Y. If $F\colon \mathbb{R}^n \to n(Y)$ is an s.v. map which is K-additive $\mathcal{N}_{\mathbb{R}^{2n}}$-a.e. in $\mathbb{R}^{2n}$, then there exists a unique (up to K) K-additive s.v. map $G\colon \mathbb{R}^n \to n(Y)$ such that $F =_K G$ $\mathcal{N}_{\mathbb{R}^n}$-a.e. in $\mathbb{R}^n$.*

Corollary 4. *Let X be a second countable Baire topological Abelian group, Y be a commutative monoid and K be a submonoid of Y. If $F\colon X \to n(Y)$ is an s.v. map which is K-additive $\mathcal{M}_{X^2}$-a.e. in X^2, then there exists a unique (up to K) K-additive s.v. map $G\colon X \to n(Y)$ such that $F =_K G$ $\mathcal{M}_X$-a.e. in X.*

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