Annales Mathematicae Silesianae (2023),
DOI: 10.2478/amsil-2023-0027

DETERMINANTS OF TOEPLITZ–HESSENBERG MATRICES WITH GENERALIZED LEONARDO NUMBER ENTRIES

Taras Goy, Mark Shattuck

Abstract. Let $u_n = u_n(k)$ denote the *generalized Leonardo* number defined recursively by $u_n = u_{n-1} + u_{n-2} + k$ for $n \geq 2$, where $u_0 = u_1 = 1$. Terms of the sequence $u_n(1)$ are referred to simply as *Leonardo* numbers. In this paper, we find expressions for the determinants of several Toeplitz–Hessenberg matrices having generalized Leonardo number entries. These results are obtained as special cases of more general formulas for the generating function of the corresponding sequence of determinants. Special attention is paid to the cases $1 \leq k \leq 7$, where several connections are made to entries in the *On-Line Encyclopedia of Integer Sequences*. By Trudi's formula, one obtains equivalent multi-sum identities involving sums of products of generalized Leonardo numbers. Finally, in the case $k = 1$, we also provide combinatorial proofs of the determinant formulas, where we make extensive use of sign-changing involutions on the related structures.

1. Introduction

Given a variable k, let $u_n = u_n(k)$ denote the n-th generalized Leonardo number defined by the recursion $u_n = u_{n-1} + u_{n-2} + k$ for $n \geq 2$, with $u_0 = u_1 = 1$. The u_n were apparently first introduced by Bicknell-Johnson and Bergum in [5] and studied further in [4, 17] and [23] from the algebraic and combinatorial standpoints, respectively. The case $k = 1$ of $u_n(k)$ gives

Received: 22.08.2023. Accepted: 11.12.2023.
(2020) Mathematics Subject Classification: 11C20, 15B05, 05A19.
Key words and phrases: Leonardo number, Fibonacci number, Toeplitz–Hessenberg matrix, Trudi's formula, combinatorial proof, generating function.

what are known as the (classical) Leonardo numbers, which were introduced by Dijkstra in conjunction with his smoothsort algorithm [8] (see also [9]). We will denote here terms of the sequence $u_n(1)$ by ℓ_n for $n \geq 0$. For further information on ℓ_n, we refer the reader to entry A001595 in the OEIS [24], and for a complete list of identities, see, e.g., [2]. The u_n have generating function formula

$$\sum_{n\geq 0} u_n x^n = \frac{1-x+kx^2}{1-2x+x^3}, \tag{1.1}$$

from which it is seen that u_n is also given recursively by $u_n = 2u_{n-1} - u_{n-3}$ for $n \geq 3$, with initial values $u_0 = u_1 = 1$ and $u_2 = k+2$.

The Leonardo numbers have been an object of ongoing research and several generalizations and variants have been recently considered. For example, incomplete Leonardo numbers were introduced in [7] and a p-version of ℓ_n was studied in [27] in analogy with Fibonacci and Lucas numbers. For other extensions of ℓ_n, see, e.g., [18, 21, 22, 25]. Complex Leonardo numbers were considered in [1, 15, 16], where various properties including recurrences and explicit formulas were shown. Further extensions in terms of hybrid numbers with Leonardo [1] or complex Leonardo [15] coefficients or in terms of the quaternions [14] or octonians [28] have subsequently been studied. Here, we consider some new combinatorial aspects of the generalized Leonardo numbers u_n as it pertains to their occurrence in certain Toeplitz–Hessenberg matrices. This extends to u_n some recent determinant formulas found for Toeplitz–Hessenberg matrices whose nonzero entries were derived from such sequences as the Catalan [10], generalized Fibonacci [11], Motzkin [12] and Schröder [13] numbers.

The organization of this paper is as follows. In the next section, we provide some general generating function formulas for determinants of Toeplitz–Hessenberg matrices whose nonzero entries are given by arbitrary translates of the sequence u_n or u_{2n}. From this, one can obtain explicit formulas for these determinants in terms of Fibonacci polynomials. In the third section, we consider the case $k = 1$ of $u_n(k)$ and obtain, as special cases of the results from the second, some simple explicit formulas of determinants of Toeplitz–Hessenberg matrices having Leonardo number entries. We consider in the fourth section further special cases of the results from the second and find determinants of matrices whose nonzero entries come from $u_n(k)$ for $2 \leq k \leq 7$. Note that this leads to new expressions in terms of determinants of several sequences appearing in [24]. In the final section, we provide combinatorial proofs of our formulas in the case $k = 1$. To do so, we make use of the definition of the determinant as a signed sum over the symmetric group and employ a variety of counting techniques, including direct enumeration, bijections between related structures and, perhaps most notably, sign-changing involutions.

Let us now recall some well-known sequences. Let F_n, L_n and Q_n denote, respectively, the Fibonacci, Lucas and Pell–Lucas numbers satisfying $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$, with initial values $F_0 = 0, F_1 = 1$, $L_0 = 2, L_1 = 1$ and $Q_0 = Q_1 = 2$. See, respectively, entries A000045, A000032 and A002203 in [24] for further information on these numbers. Recall $u_n = (k+1)F_{n+1} - k$ for all $n \geq 0$ (see, e.g., [17, Theorem 1]). Given a parameter z, let $f_n(z)$ denote the n-th Fibonacci polynomial defined by $f_n(z) = zf_{n-1}(z) + f_{n-2}(z)$ for $n \geq 2$, with $f_0(z) = 0$ and $f_1(z) = z$. Note that $f_n(1) = F_n$ for all $n \geq 0$.

2. Some general formulas

Let A_n denote the Toeplitz–Hessenberg matrix (see, e.g., [20]) given by

$$(2.1) \quad A_n := A_n(a_0; a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix},$$

where $a_0 \neq 0$. The following result, known as *Trudi's formula* [19, Theorem 1], expresses $\det(A_n)$ in terms of a multinomial sum involving products of the a_i.

LEMMA 2.1. *If $n \geq 1$, then*

$$(2.2) \qquad \det(A_n) = \sum_{\widetilde{s}=n} (-a_0)^{n-|s|} \binom{|s|}{s_1, \ldots, s_n} a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n},$$

where $\binom{|s|}{s_1,\ldots,s_n} = \frac{|s|!}{s_1! s_2! \cdots s_n!}$, $\widetilde{s} = s_1 + 2s_2 + \cdots + ns_n$, $|s| = s_1 + s_2 + \cdots + s_n$ *and* $s_i \geq 0$ *for all* i.

REMARK. The case $a_0 = 1$ of (2.2) is known as *Brioschi's formula* [20]. Note that the sum in (2.2) may be regarded as being over the set of partitions of the positive integer n.

Let $d(x) = \sum_{n\geq 1} \det(A_n)x^n$ and $g(x) = \sum_{i \geq 1} a_i x^i$. Using (2.2), together with the fact $\frac{y}{1-y} = y + y^2 + \cdots$, one can establish the following relation between the generating functions f and g.

LEMMA 2.2. *We have*

$$d(x) = \frac{-\frac{1}{a_0} g(-a_0 x)}{1 + \frac{1}{a_0} g(-a_0 x)}. \tag{2.3}$$

We have the following generating function formulas for $\det(A_n)$ in the case when a_i is given by an arbitrary translate of the sequence u_i, where we assume here $u_i = 0$ if $i < 0$.

THEOREM 2.1. *Let $d_m(x) = d_m(x; a)$ be given by*

$$d_m(x) = \sum_{n \geq 1} \det(A_n(a; u_{m+1}, u_{m+2}, \ldots, u_{m+n}))x^n,$$

where m is an integer and a is arbitrary. If $m \geq 0$, then

$$d_m(x) = \begin{cases} \frac{u_{m+1}x + u_{m-1}ax^2 - u_m a^2 x^3}{1-(u_{m+1}-2a)x - u_{m-1}ax^2 + (u_m - a)a^2 x^3}, & m \geq 1; \\ \frac{x - kax^2 - a^2x^3}{1+(2a-1)x + kax^2 - (a-1)a^2x^3}, & m = 0. \end{cases} \tag{2.4}$$

If $m < 0$, then

$$d_m(x) = -\frac{(-ax)^{-m}(1 + ax + ka^2x^2)}{a(1 + 2ax - a^3x^3) + (-ax)^{-m}(1 + ax + ka^2x^2)}. \tag{2.5}$$

PROOF. First assume $m \geq 3$. Then, by (1.1), we have

$$\begin{aligned} \sum_{n \geq 1} u_{n+m}x^n &= \frac{1}{x^m} \sum_{n \geq m+1} u_n x^n = \frac{1}{x^m}\left(\frac{x + kx^2 - x^3}{1 - 2x + x^3} - \sum_{n=1}^{m} u_n x^m\right) \\ &= \frac{x + kx^2 - x^3 - (1 - 2x + x^3)\sum_{n=1}^{m} u_n x^n}{x^m(1 - 2x + x^3)}. \end{aligned}$$

Note that $m \geq 3$ implies

$$\begin{aligned} (1 - 2x + x^3)\sum_{n=1}^{m} u_n x^n &= x + kx^2 - x^3 + 0x^4 + \cdots + 0x^m \\ &\quad + (u_{m-2} - 2u_m)x^{m+1} + u_{m-1}x^{m+2} + u_m x^{m+3} \\ &= x + kx^2 - x^3 - u_{m+1}x^{m+1} + u_{m-1}x^{m+2} + u_m x^{m+3}, \end{aligned}$$

upon making use of the recurrence $u_n = 2u_{n-1} - u_{n-3}$. Thus, we have

$$g_m(x) := \sum_{n \geq 1} u_{n+m}x^n = \frac{u_{m+1}x - u_{m-1}x^2 - u_m x^3}{1 - 2x + x^3}.$$

Hence, by (2.3),

$$d_m(x) = \frac{-\frac{1}{a}g_m(-ax)}{1+\frac{1}{a}g_m(-ax)} = \frac{\frac{u_{m+1}x+u_{m-1}ax^2-u_m a^2x^3}{1+2ax-a^3x^3}}{1-\frac{u_{m+1}x+u_{m-1}ax^2-u_m a^2x^3}{1+2ax-a^3x^3}}$$

$$= \frac{u_{m+1}x+u_{m-1}ax^2-u_m a^2x^3}{1-(u_{m+1}-2a)x-u_{m-1}ax^2+(u_m-a)a^2x^3},$$

which yields the first formula in (2.4) for $m \geq 3$.

A similar argument applies in the $m=1$ and $m=2$ cases and yields

$$d_1(x) = \frac{(k+2)x+ax^2-a^2x^3}{1-(k-2a+2)x-ax^2-(a-1)a^2x^3}$$

and

$$d_2(x) = \frac{(2k+3)x+ax^2-(k+2)a^2x^3}{1-(2k-2a+3)x-ax^2+(k-a+2)a^2x^3},$$

from which the first formula in (2.4) is seen to hold in the $m=1$ and $m=2$ cases as well. On the other hand, if $m=0$, then

$$d_0(x) = \frac{-\frac{1}{a}g_0(-ax)}{1+\frac{1}{a}g_0(-ax)} = \frac{\frac{x-kax^2-a^2x^3}{1+2ax-a^3x^3}}{1-\frac{x-kax^2-a^2x^3}{1+2ax-a^3x^3}}$$

$$= \frac{x-kax^2-a^2x^3}{1+(2a-1)x+kax^2-(a-1)a^2x^3},$$

which finishes the proof of (2.4). Finally, if $m<0$, then

$$g_m(x) = \sum_{n\geq 1} u_{n+m}x^n = \sum_{n\geq m+1} u_n x^{n-m} = x^{-m}\sum_{n\geq 0} u_n x^n$$

$$= \frac{x^{-m}(1-x+kx^2)}{1-2x+x^3},$$

and hence

$$d_m(x) = \frac{-\frac{(-ax)^{-m}(1+ax+ka^2x^2)}{a(1+2ax-a^3x^3)}}{1+\frac{(-ax)^{-m}(1+ax+ka^2x^2)}{a(1+2ax-a^3x^3)}}$$

$$= -\frac{(-ax)^{-m}(1+ax+ka^2x^2)}{a(1+2ax-a^3x^3)+(-ax)^{-m}(1+ax+ka^2x^2)},$$

which gives (2.5) and completes the proof. $\square$

Comparable formulas can be found for the generating function of $\det(A_n)$ in the case when a_i is given by an arbitrary translate of u_{2n}.

THEOREM 2.2. *Let $h_m(x) = h_m(x;a)$ be given by*

$$h_m(x) = \sum_{n\geq 1} \det(A_n(a; u_{m+2}, u_{m+4}, \ldots, u_{m+2n}))x^n,$$

where m is an integer and a is arbitrary. If $m \geq 0$, then

$$h_m(x) = \begin{cases} \frac{u_{m+2}x+(4u_m-u_{m-2})ax^2+u_ma^2x^3}{1-(u_{m+2}-4a)x-(4u_m-u_{m-2}-4a)ax^2-(u_m-a)a^2x^3}, & m \geq 2; \\ \frac{(2k+3)x+(k+4)ax^2+a^2x^3}{1-(2k-4a+3)x-(k-4a+4)ax^2+(a-1)a^2x^3}, & m = 1; \\ \frac{(k+2)x+3ax^2+a^2x^3}{1-(k-4a+2)x+(4a-3)ax^2+(a-1)a^2x^3}, & m = 0. \end{cases} \tag{2.6}$$

If $m < 0$, then

$$h_m(x) = \begin{cases} -\frac{(-ax)^t(1-(k-2)ax+a^2x^2)}{a(1+4ax+4a^2x^2+a^3x^3)+(-ax)^t(1-(k-2)ax+a^2x^2)}, & m = -2t; \\ \frac{(-ax)^{t-1}(x-(2k-1)ax^2-ka^2x^3)}{1+4ax+4a^2x^2+a^3x^3-(-ax)^{t-1}(x-(2k-1)ax^2-ka^2x^3)}, & m = 1-2t, \end{cases} \tag{2.7}$$

where t denotes a positive integer.

PROOF. First note

$$\begin{aligned} \sum_{n\geq 1} u_{2n}x^{2n} &= \frac{1}{2}\left(\sum_{n\geq 1} u_nx^n + \sum_{n\geq 1} u_n(-x)^n\right) \\ &= \frac{1}{2}\left(\frac{x+kx^2-x^3}{1-2x+x^3} + \frac{-x+kx^2+x^3}{1+2x-x^3}\right) = \frac{(k+2)x^2-3x^4+x^6}{1-4x^2+4x^4-x^6} \end{aligned}$$

and

$$\sum_{n\geq 1} u_{2n-1}x^{2n-1} = \frac{1}{2}\left(\sum_{n\geq 1} u_nx^n - \sum_{n\geq 1} u_n(-x)^n\right) = \frac{x+(2k-1)x^3-kx^5}{1-4x^2+4x^4-x^6},$$

which implies

$$\sum_{n\geq 1} u_{2n}x^n = \frac{(k+2)x-3x^2+x^3}{1-4x+4x^2-x^3}, \quad \sum_{n\geq 1} u_{2n-1}x^n = \frac{x+(2k-1)x^2-kx^3}{1-4x+4x^2-x^3}.$$

To show (2.6), first assume $m = 2s$, where s is a non-negative integer. If $s \geq 3$, then

$$\begin{aligned} j_m(x) :&= \sum_{n\geq 1} u_{2n+m}x^n = \sum_{n\geq 1} u_{2n+2s}x^n = \sum_{n\geq s+1} u_{2n}x^{n-s} \\ &= \frac{1}{x^s}\left(\frac{(k+2)x-3x^2+x^3}{1-4x+4x^2-x^3} - \sum_{n=1}^{s} u_{2n}x^n\right). \end{aligned}$$

From the generating function for u_{2n}, we have the recurrence $u_{2n} = 4u_{2n-2} - 4u_{2n-4} + u_{2n-6}$ for $n \geq 3$, which implies

$$\begin{aligned} &(1-4x+4x^2-x^3)\sum_{n=1}^{s} u_{2n}x^n \\ &\quad = u_2x + (u_4-4u_2)x^2 + (u_6-4u_4+4u_2)x^3 + 0x^4 + \cdots + 0x^s \\ &\qquad - (4u_{2s}-4u_{2s-2}+u_{2s-4})x^{s+1} + (4u_{2s}-u_{2s-2})x^{s+2} - u_{2s}x^{s+3} \\ &\quad = (k+2)x - 3x^2 + x^3 - u_{2s+2}x^{s+1} + (4u_{2s}-u_{2s-2})x^{s+2} - u_{2s}x^{s+3}, \end{aligned}$$

and hence

$$\begin{aligned} j_m(x) &= \frac{u_{2s+2}x - (4u_{2s}-u_{2s-2})x^2 + u_{2s}x^3}{1-4x+4x^2-x^3} \\ &= \frac{u_{m+2}x - (4u_m-u_{m-2})x^2 + u_mx^3}{1-4x+4x^2-x^3}. \end{aligned}$$

Thus, by (2.3), we have

$$\begin{aligned} h_m(x) &= \frac{-\frac{1}{a}j_m(-ax)}{1+\frac{1}{a}j_m(-ax)} \\ &= \frac{u_{m+2}x + (4u_m-u_{m-2})ax^2 + u_ma^2x^3}{1-(u_{m+2}-4a)x-(4u_m-u_{m-2}-4a)ax^2-(u_m-a)a^2x^3}, \end{aligned}$$

which establishes the first formula in (2.6) for $m \geq 6$ even.

A similar argument shows that this formula also holds in the $m = 2$ and $m = 4$ cases. On the other hand, if $m = 0$, then

$$j_0(x) = \frac{(k+2)x-3x^2+x^3}{1-4x+4x^2-x^3},$$

which gives

$$h_0(x) = \frac{-\frac{1}{a}j_0(-ax)}{1+\frac{1}{a}j_0(-ax)} = \frac{(k+2)x+3ax^2+a^2x^3}{1-(k-4a+2)x+(4a-3)ax^2+(a-1)a^2x^3}.$$

This finishes the proof of (2.6) for m even. A comparable proof applies in the odd case of m, which we leave to the reader, upon letting $m = 2s-1$ for some $s > 0$ and using the formula above for $\sum_{n\geq 1} u_{2n-1}x^n$. Finally, assume $m < 0$. If $m = -2t$ for some $t > 0$, then

$$\begin{aligned} j_m(x) &= \sum_{n\geq 1} u_{2n+m}x^n = \sum_{n\geq 1} u_{2n-2t}x^n = \sum_{n\geq 1-t} u_{2n}x^{n+t} = x^t \sum_{n\geq 0} u_{2n}x^n \\ &= \frac{x^t(1+(k-2)x+x^2)}{1-4x+4x^2-x^3}, \end{aligned}$$

which leads to the first formula in (2.7). A similar argument applies if $m = 1-2t$, which completes the proof. □

From the expressions found for the generating functions in the prior two theorems, it is possible to obtain the following general explicit formulas in terms of Fibonacci polynomials. For convenience of notation, we will often denote $\det(A_n(\pm 1; a_1, \ldots, a_n))$ simply by $D_\pm(a_1, \ldots, a_n)$.

Corollary 2.1. *We have*

$$D_+(u_1, u_2, \ldots, u_n) = -k(k+1)(i\sqrt{k})^{n-3} f_{n-3}(i/\sqrt{k}), \qquad n \geq 3, \tag{2.8}$$

and

$$D_+(u_2, u_3, \ldots, u_{n+1}) = \frac{k+1}{k}\left(f_n(k) + f_{n-1}(k)\right), \qquad n \geq 2, \tag{2.9}$$

where $k \neq 0$ is arbitrary and i is the imaginary unit.

Proof. By the well-known generating function formula $\sum_{n\geq 0} f_n(z)x^n = \frac{zx}{1-zx-x^2}$, we have

$$\begin{aligned} \sum_{n\geq 3}(i\sqrt{k})^{n-3} f_{n-3}(i/\sqrt{k})x^n &= x^3 \sum_{n\geq 0} f_n(i/\sqrt{k})(i\sqrt{k}x)^n \\ &= \frac{\frac{i}{\sqrt{k}}(i\sqrt{k}x)x^3}{1-\frac{i}{\sqrt{k}}(i\sqrt{k}x)-(i\sqrt{k}x)^2} = \frac{-x^4}{1+x+kx^2}. \end{aligned}$$

Thus,

$$\begin{aligned} &x-(k+1)x^2-k(k+1)\sum_{n\geq 3}(i\sqrt{k})^{n-3} f_{n-3}(i/\sqrt{k})x^n \\ &\qquad = x-(k+1)x^2+\frac{k(k+1)x^4}{1+x+kx^2} = \frac{x-kx^2-x^3}{1+x+kx^2} = d_0(x;1), \end{aligned}$$

by the $m=0$ case of (2.4), which implies (2.8). Similarly, we have

$$(k+2)x+\frac{k+1}{k}\sum_{n\geq 2}(f_n(k)+f_{n-1}(k))x^n=\frac{(k+2)x+x^2-x^3}{1-kx-x^2}=d_1(x;1),$$

which implies (2.9) and completes the proof. $\square$

Corollary 2.2. *We have*

$$D_+(u_2,u_4,\ldots,u_{2n})=\frac{(k+1)i^n}{k-2}(f_n(\alpha)-if_{n-1}(\alpha)),\quad n\geq 2, \tag{2.10}$$

and

$$D_+(u_3,u_5,\ldots,u_{2n+1})=\frac{(k+1)k^{(n-2)/2}}{2k-1}\Big((2k+1)f_n(\beta) + \left(k^{1/2}+k^{-1/2}\right)f_{n-1}(\beta)\Big),\quad n\geq 2, \tag{2.11}$$

excluding division by zero, where $\alpha=-i(k-2)$ *and* $\beta=\frac{2k-1}{k^{1/2}}$.

Proof. For (2.10), first note

$$\sum_{n\geq 2}(f_n(\alpha)-if_{n-1}(\alpha))(ix)^n=\frac{\alpha ix(1+x)}{1-\alpha ix+x^2}-\alpha ix$$
$$=\frac{\alpha x^2(i-\alpha-ix)}{1-\alpha ix+x^2}=\frac{(k-2)x^2(k-1-x)}{1-(k-2)x+x^2}.$$

Hence, we have

$$(k+2)x+\frac{k+1}{k-2}\sum_{n\geq 2}(f_n(\alpha)-if_{n-1}(\alpha))(ix)^n$$
$$=(k+2)x+\frac{(k+1)x^2(k-1-x)}{1-(k-2)x+x^2}=\frac{(k+2)x+3x^2+x^3}{1-(k-2)x+x^2}=h_0(x;1),$$

by the $m=0$ case of (2.6), which implies (2.10). Now observe

$$\sum_{n\geq 2}\left(k^{(n-2)/2}(2k+1)f_n(\beta)+k^{(n-2)/2}\left(k^{1/2}+k^{-1/2}\right)f_{n-1}(\beta)\right)x^n$$
$$=\frac{2k+1}{k}\left(\frac{(2k-1)x}{1-(2k-1)x-kx^2}-(2k-1)x\right)+\frac{k+1}{k}\cdot\frac{(2k-1)x^2}{1-(2k-1)x-kx^2}$$

$$= \frac{(4k^2-1)x+(k+1)(2k-1)x^2}{k(1-(2k-1)x-kx^2)} - \frac{(4k^2-1)x}{k}$$
$$= \frac{(4k+1)(2k-1)x^2+(4k^2-1)x^3}{1-(2k-1)x-kx^2}$$

and

$$(2k+3)x + \frac{k+1}{2k-1}\cdot\frac{(4k+1)(2k-1)x^2+(4k^2-1)x^3}{1-(2k-1)x-kx^2}$$
$$= (2k+3)x + \frac{(k+1)(4k+1)x^2+(k+1)(2k+1)x^3}{1-(2k-1)x-kx^2}$$
$$= \frac{(2k+3)x+(k+4)x^2+x^3}{1-(2k-1)x-kx^2} = h_1(x;1),$$

by the $m = 1$ case of (2.6), which implies (2.11) and completes the proof. $\square$

Remark. Recall that the Fibonacci polynomial $f_n(z)$ is given explicitly by

$$f_n(z) = \frac{z}{\sqrt{z^2+4}}\left(\left(\frac{z+\sqrt{z^2+4}}{2}\right)^n - \left(\frac{z-\sqrt{z^2+4}}{2}\right)^n\right), \quad n \geq 0,$$

as well as by the binomial expansion

$$f_n(z) = \sum_{j=0}^{n-1} z^{n-2j}\binom{n-1-j}{j}, \quad n \geq 0.$$

Combining these formulas with (2.8), for example, yields

$$D_+(u_1,u_2,\ldots,u_n)$$
$$= \frac{k(k+1)i}{\sqrt{4k-1}}\left(\left(\frac{-1-i\sqrt{4k-1}}{2}\right)^{n-3} - \left(\frac{-1+i\sqrt{4k-1}}{2}\right)^{n-3}\right), \quad n \geq 2,$$

and

$$D_+(u_1,u_2,\ldots,u_n) = (-1)^n k(k+1)\sum_{j=0}^{n-4}(-k)^j\binom{n-4-j}{j}, \qquad n \geq 3.$$

Comparable formulas may be given for $D_+(u_2,\ldots,u_{n+1})$, $D_+(u_2,\ldots,u_{2n})$ and $D_+(u_3,\ldots,u_{2n+1})$ using (2.9), (2.10) and (2.11).

3. Leonardo number determinant formulas

We have the following simple formulas for determinants involving the classical Leonardo numbers.

THEOREM 3.1. *If $n \geq 2$, then*

$$D_-(\ell_0, \ell_1, \ldots, \ell_{n-1}) = F_{2n-1} + F_{2n-4}, \tag{3.1}$$

$$D_+(\ell_1, \ell_2, \ldots, \ell_n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3; \\ 2, & \text{if } n \equiv 1 \pmod 3; \\ -2, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{3.2}$$

$$D_+(\ell_2, \ell_3, \ldots, \ell_{n+1}) = 2F_{n+1}, \tag{3.3}$$

$$D_-(\ell_1, \ell_3, \ldots, \ell_{2n-1}) = 2 \cdot A010903[n-2], \tag{3.4}$$

$$D_+(\ell_2, \ell_4, \ldots, \ell_{2n}) = \begin{cases} -2, & \text{if } n \equiv 0 \pmod 3; \\ 2, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{3.5}$$

$$D_+(\ell_3, \ell_5, \ldots, \ell_{2n+1}) = 2F_{n+3}. \tag{3.6}$$

PROOF. We obtain these expressions as special cases of Theorems 2.1 and 2.2 when $k = 1$. First recall the generating functions

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2}, \quad \sum_{n\geq 0} F_{2n} x^n = \frac{x}{1-3x+x^2},$$

$$\sum_{n\geq 1} F_{2n-1} x^n = \frac{x(1-x)}{1-3x+x^2}.$$

By the $m = a = -1$ case of (2.5), we have

$$d_{-1}(x; -1) = \frac{x(1-x)}{1-3x+x^2} + \frac{x^3}{1-3x+x^2} = \sum_{n\geq 1} F_{2n-1} x^n + \sum_{n\geq 2} F_{2n-4} x^n,$$

which implies (3.1), where $d_{-1}(x; -1)$ (and other such subsequent functions) is understood here to be evaluated at $k = 1$. By the $m = 0$, $a = 1$ case of (2.4), we have

$$d_0(x;1) = \frac{x(1-x-x^2)}{1+x+x^2} = \frac{x(1-2x+x^3)}{1-x^3} = x + \frac{2x^4}{1-x^3} - \frac{2x^2}{1-x^3},$$

which implies (3.2). By the $m = a = 1$ case of (2.4), we have

$$\begin{aligned} d_1(x;1) &= \frac{x(3+x-x^2)}{1-x-x^2} = 3x + \frac{4x^2+2x^3}{1-x-x^2} \\ &= 3x + 2\left(\frac{1}{1-x-x^2} - 1 - x\right) = 3x + 2\sum_{n\geq 2} F_{n+1}x^n, \end{aligned}$$

which implies (3.3).

By the $m = a = -1$ case of (2.7), we have

$$h_{-1}(x;-1) = \frac{x(1+x-x^2)}{1-5x+3x^2} = x + \frac{6x^2-4x^3}{1-5x+3x^2} = x + 2\sum_{n\geq 2} A010903[n-2]x^n,$$

which implies (3.4), where we have made use of the fact $\sum_{n\geq 0} A010903[n]x^n = \frac{3-2x}{1-5x+3x^2}$ in the last equality. By the $m = 0$, $a = 1$ case of (2.6), we have

$$h_0(x;1) = \frac{x(3+3x+x^2)}{1+x+x^2} = \frac{x(3-2x^2-x^3)}{1-x^3} = 3x - \frac{2x^3}{1-x^3} + \frac{2x^4}{1-x^3},$$

which implies (3.5). Finally, by the $m = a = 1$ case of (2.6), we have

$$\begin{aligned} h_1(x;1) &= \frac{x(5+5x+x^2)}{1-x-x^2} = 5x + \frac{10x^2+6x^3}{1-x-x^2} \\ &= 5x + \frac{2}{x^2}\left(\frac{1}{1-x-x^2} - 1 - x - 2x^2 - 3x^3\right) \\ &= 5x + \frac{2}{x^2}\sum_{n\geq 4} F_{n+1}x^n = 5x + 2\sum_{n\geq 2} F_{n+3}x^n, \end{aligned}$$

which implies (3.6) and completes the proof. $\square$

By Lemma 2.1, the determinant formulas from the preceding theorem yield the following identities involving Leonardo numbers and multinomial coefficients.

Corollary 3.1. *If $n \geq 2$, then*

$$\sum_{\widetilde{s}=n} \binom{|s|}{s_1, \ldots, s_n} \ell_0^{s_1} \ell_1^{s_2} \cdots \ell_{n-1}^{s_n} = F_{2n-1} + F_{2n-4},$$

$$\sum_{\widetilde{s}=n} (-1)^{n-|s|} \binom{|s|}{s_1, \ldots, s_n} \ell_1^{s_1} \ell_2^{s_2} \cdots \ell_n^{s_n} = \begin{cases} 0, & \text{if } n \equiv 0 \ (mod\ 3); \\ 2, & \text{if } n \equiv 1 \ (mod\ 3); \\ -2, & \text{if } n \equiv 2 \ (mod\ 3), \end{cases}$$

$$\sum_{\widetilde{s}=n} (-1)^{n-|s|} \binom{|s|}{s_1, \ldots, s_n} \ell_2^{s_1} \ell_3^{s_2} \cdots \ell_{n+1}^{s_n} = 2F_{n+1},$$

$$\sum_{\widetilde{s}=n} \binom{|s|}{s_1, \ldots, s_n} \ell_1^{s_1} \ell_3^{s_2} \cdots \ell_{2n-1}^{s_n} = 2 \cdot A010903[n-2],$$

$$\sum_{\widetilde{s}=n} (-1)^{n-|s|} \binom{|s|}{s_1, \ldots, s_n} \ell_2^{s_1} \ell_4^{s_2} \cdots \ell_{2n}^{s_n} = \begin{cases} -2, & \text{if } n \equiv 0 \ (mod\ 3); \\ 2, & \text{if } n \equiv 1 \ (mod\ 3); \\ 0, & \text{if } n \equiv 2 \ (mod\ 3), \end{cases}$$

$$\sum_{\widetilde{s}=n} (-1)^{n-|s|} \binom{|s|}{s_1, \ldots, s_n} \ell_3^{s_1} \ell_5^{s_2} \cdots \ell_{2n+1}^{s_n} = 2F_{n+3}.$$

4. Generalized Leonardo determinants for $2 \leq k \leq 7$

In this section, we give determinant formulas of some Toeplitz–Hessenberg matrices whose nonzero entries are derived from the sequence $u_n(k)$ for $2 \leq k \leq 7$. The first several terms of the sequence $u_n(k)$ for $2 \leq k \leq 7$, starting with the $n = 0$ term, are as follows:

$k = 2$: 1, 1, 4, 7, 13, 22, 37, 61, 100, 163, 265, 430, 697, 1129, 1828, 2959, ...
$k = 3$: 1, 1, 5, 9, 17, 29, 49, 81, 133, 217, 353, 573, 929, 1505, 2437, 3945, ...
$k = 4$: 1, 1, 6, 11, 21, 36, 61, 101, 166, 271, 441, 716, 1161, 1881, 3046, 4931, ...
$k = 5$: 1, 1, 7, 13, 25, 43, 73, 121, 199, 325, 529, 859, 1393, 2257, 3655, 5917, ...
$k = 6$: 1, 1, 8, 15, 29, 50, 85, 141, 232, 379, 617, 1002, 1625, 2633, 4264, 6903, ...
$k = 7$: 1, 1, 9, 17, 33, 57, 97, 161, 265, 433, 705, 1145, 1857, 3009, 4873, 7889, ...

Note that the sequences $u_n(2)$ and $u_n(5)$ correspond respectively to entries A111314 and A111721 in [24].

The results below involving particular determinants with entries from $u_n(k)$ for $2 \leq k \leq 7$ may be obtained by applying the formulas from the second section above. In each case, the recurrence stated for the corresponding sequence of determinants, denoted by a_n, is valid for all $n \geq 4$. All other formulas hold for $n \geq 2$, unless stated otherwise. We also give in each case at least the first ten terms of the sequence a_n, starting with a_1. Note that by Trudi's formula, one obtains analogues of the identities in Corollary 3.1 involving $u_n(k)$ for each k.

Case $k = 2$:

2.1 $$D_+(u_1, u_2, \dots, u_n) = \frac{6i}{\sqrt{7}}\left(\left(\frac{-1-i\sqrt{7}}{2}\right)^{n-3} - \left(\frac{-1+i\sqrt{7}}{2}\right)^{n-3}\right),$$
$$= 6(-1)^n \sum_{j=0}^{n-4} (-2)^j \binom{n-4-j}{j}, \quad n \geq 3,$$
$$= 6 \cdot A001607[n-3], \quad n \geq 3.$$

Sequence: 1, −3, 0, 6, −6, −6, 18, −6, −30, 42, 18, −102, ...
Recurrence: $a_n = -a_{n-1} - 2a_{n-2}$.

2.2 $$D_+(u_2, u_3, \dots, u_{n+1}) = \frac{3}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big) = \frac{3}{2}Q_{n-1},$$
$$= \frac{3}{2}\big(f_{n-1}(2) + f_n(2)\big) = 3 \cdot A001333[n].$$

Sequence: 4, 9, 21, 51, 123, 297, 717, 1731, 4179, 10089, ...
Recurrence: $a_n = 2a_{n-1} + a_{n-2}$.

2.3 $$D_+(u_0, u_2, \dots, u_{2n-2}) = \frac{3}{2}\sum_{j=0}^{n-3} (-2)^{n-1-j} \binom{n-3+j}{2j}, \quad n \geq 3,$$
$$= 6 \cdot A087168[n-3], \quad n \geq 3.$$

Sequence: 1, −3, 6, −6, −6, 42, −102, 138, −6, −534, 1626, −2742, ...
Recurrence: $a_n = -3a_{n-1} - 4a_{n-2}$.

2.4 $$D_+(u_2, u_4, \dots, u_{2n}) = 3 \cdot (-1)^{\lfloor \frac{n-1}{2} \rfloor}.$$

Sequence: 4, 3, −3, −3, 3, 3, −3, −3, 3, 3, −3, −3, 3, ...

Case $k = 3$:

3.1 $$D_+(u_1, u_2, \dots, u_n) = \frac{12i}{\sqrt{11}}\left(\left(\frac{-1-i\sqrt{11}}{2}\right)^{n-3} - \left(\frac{-1+i\sqrt{11}}{2}\right)^{n-3}\right),$$
$$= 12(-1)^n \sum_{j=0}^{n-4} (-3)^j \binom{n-4-j}{j}, \quad n \geq 3,$$
$$= -4 \cdot A110523[n-2].$$

Sequence: 1, −4, 0, 12, −12, −24, 60, 12, −192, 156, 420, ...
Recurrence: $a_n = -a_{n-1} - 3a_{n-2}$.

3.2 $D_+(u_2, u_3, \dots, u_{n+1})$

$$= \frac{4}{\sqrt{13}}\left((7+2\sqrt{13})\left(\frac{3+\sqrt{13}}{2}\right)^{n-2} - (7-2\sqrt{13})\left(\frac{3-\sqrt{13}}{2}\right)^{n-2}\right),$$
$$= \frac{4}{3}\big(f_{n-1}(3) + f_n(3)\big) = 4 \cdot A003688[n].$$

Sequence: 5, 16, 52, 172, 568, 1876, 6196, 20464, 67588, 223228, ...
Recurrence: $a_n = 3a_{n-1} + a_{n-2}$.

3.3
$$D_+(u_2, u_4, \dots, u_{2n}) = (-1)^n \cdot \begin{cases} -4, & \text{if } n \equiv 0 \pmod 3; \\ -4, & \text{if } n \equiv 1 \pmod 3; \\ 8, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Sequence: 5, 8, 4, −4, −8, −4, 4, 8, 4, −4, −8, −4, 4, ...
Recurrence: $a_n = a_{n-1} - a_{n-2}$.

Case $k = 4$:

4.1 $$D_+(u_1, u_2, \dots, u_n) = \frac{20i}{\sqrt{15}}\left(\left(\frac{-1-i\sqrt{15}}{2}\right)^{n-3} - \left(\frac{-1+i\sqrt{15}}{2}\right)^{n-3}\right),$$
$$= 20(-1)^n \sum_{j=0}^{n-4} (-4)^j \binom{n-4-j}{j}, \quad n \geq 3,$$
$$= 20(-1)^n A106853[n-4], \quad n \geq 4.$$

Sequence: 1, −5, 0, 20, −20, −60, 140, 100, −660, 260, 2380, ...
Recurrence: $a_n = -a_{n-1} - 4a_{n-2}$.

4.2 $$D_+(u_2,u_3,\ldots,u_{n+1}) = 5F_{3n-1} = \frac{5}{4}\big(f_{n-1}(4)+f_n(4)\big).$$

Sequence: 6, 25, 105, 445, 1885, 7985, 33825, 143285, 606965, 2571145, ...
Recurrence: $a_n = 4a_{n-1} + a_{n-2}$.

4.3 $$D_+(u_0,u_2,\ldots,u_{2n-2}) = 4i\sqrt{15}\left(\left(\frac{-3+i\sqrt{15}}{2}\right)^{n-4} - \left(\frac{-3-i\sqrt{15}}{2}\right)^{n-4}\right),$$
$$= 60(-1)^n A190960[n-4], \quad n \geq 4.$$

Sequence: 1, −5, 10, 0, −60, 180, −180, −540, 2700, −4860, −1620, ...
Recurrence: $a_n = -3a_{n-1} - 6a_{n-2}$.

4.4 $$D_+(u_2,u_4,\ldots,u_{2n}) = 10n-5.$$

Sequence: 6, 15, 25, 35, 45, 55, 65, 75, 85, 95, ...

Case $k=5$:

5.1 $$D_+(u_1,u_2,\ldots,u_n) = \frac{30i}{\sqrt{19}}\left(\left(\frac{-1-i\sqrt{19}}{2}\right)^{n-3} - \left(\frac{-1+i\sqrt{19}}{2}\right)^{n-3}\right),$$
$$= 30(-1)^n \sum_{j=0}^{n-4}(-5)^j \binom{n-4-j}{j}, \quad n \geq 3,$$
$$= 30(-1)^n A106854[n-4], \quad n \geq 4.$$

Sequence: 1, −6, 0, 30, −30, −120, 270, 330, −1680, 30, 8370, ...
Recurrence: $a_n = -a_{n-1} - 5a_{n-2}$.

5.2 $D_+(u_2,u_3,\ldots,u_{n+1})$
$$= \frac{6}{\sqrt{29}}\left((16+3\sqrt{29})\left(\frac{5+\sqrt{29}}{2}\right)^{n-2} - (16-3\sqrt{29})\left(\frac{5-\sqrt{29}}{2}\right)^{n-2}\right),$$
$$= \frac{6}{5}\big(f_{n-1}(5)+f_n(5)\big) = 6\cdot A015449[n].$$

Sequence: 7, 36, 186, 966, 5016, 26046, 135246, 702276, 3646626, 18935406, ...
Recurrence: $a_n = 5a_{n-1} + a_{n-2}$.

5.3 $$D_+(u_2,u_4,\ldots,u_{2n}) = 6L_{2n-1}.$$

Sequence: 7, 24, 66, 174, 456, 1194, 3126, 8184, 21426, 56094, ...
Recurrence: $a_n = 3a_{n-1} - a_{n-2}$.

Case $k=6$:

6.1 $$D_+(u_1,u_2,\ldots,u_n)=\frac{42i}{\sqrt{23}}\left(\Big(\frac{-1-i\sqrt{23}}{2}\Big)^{n-3}-\Big(\frac{-1+i\sqrt{23}}{2}\Big)^{n-3}\right),$$
$$=42(-1)^n\sum_{j=0}^{n-4}(-6)^j\binom{n-4-j}{j},\quad n\geq 3,$$
$$=42(-1)^n A145934[n-4],\quad n\geq 4.$$

Sequence: 1, −7, 0, 42, −42, −210, 462, 798, −3570, −1218, 22638, ...
Recurrence: $a_n=-a_{n-1}-6a_{n-2}$.

6.2 $$D_+(u_2,u_3,\ldots,u_{n+1})$$
$$=\frac{7}{10}\left((35-11\sqrt{10})(3-\sqrt{10})^{n-2}+(35+11\sqrt{10})(3+\sqrt{10})^{n-2}\right),$$
$$=\frac{7}{6}\big(f_{n-1}(6)+f_n(6)\big)=7\cdot A015451[n].$$

Sequence: 8, 49, 301, 1855, 11431, 70441, 434077, 2674903, 16483495, 87065034, ...
Recurrence: $a_n=6a_{n-1}+a_{n-2}$.

6.3 $$D_+(u_2,u_4,\ldots,u_{2n})=\frac{7}{2}\Big((5+3\sqrt{3})(2+\sqrt{3})^{n-2}$$
$$+(5-3\sqrt{3})(2-\sqrt{3})^{n-2}\Big),$$
$$=7\cdot A001834[n-1].$$

Sequence: 8, 35, 133, 497, 1855, 6923, 25837, 96425, 359863, 1343027, ...
Recurrence: $a_n=4a_{n-1}-a_{n-2}$.

Case $k=7$:

7.1 $$D_+(u_1,u_2,\ldots,u_n)=\frac{56i}{\sqrt{27}}\left(\Big(\frac{-1-i\sqrt{27}}{2}\Big)^{n-3}-\Big(\frac{-1+i\sqrt{27}}{2}\Big)^{n-3}\right),$$
$$=56(-1)^n\sum_{j=0}^{n-4}(-7)^j\binom{n-4-j}{j},\quad n\geq 3,$$
$$=56(-1)^n A145976[n-4],\quad n\geq 4.$$

Sequence: 1, −8, 0, 56, −56, −336, 728, 1624, −6720, −4648, 51688, ...
Recurrence: $a_n=-a_{n-1}-7a_{n-2}$.

7.2 $D_+(u_2, u_3, \dots, u_{n+1})$

$$= \frac{8}{53}\left((212-29\sqrt{53})\Big(\frac{7-\sqrt{53}}{2}\Big)^{n-2} + (212+29\sqrt{53})\Big(\frac{7+\sqrt{53}}{2}\Big)^{n-2}\right),$$
$$= \frac{8}{7}\big(f_{n-1}(7)+f_n(7)\big) = 8\cdot A015453[n].$$

Sequence: 9, 64, 456, 3256, 23248, 165992, 1185192, 8462336, 60421544, 431413144, ...
Recurrence: $a_n = 7a_{n-1} + a_{n-2}$.

7.3 $D_+(u_2, u_4, \dots, u_{2n})$

$$= \frac{8}{\sqrt{21}}\left((14+3\sqrt{21})\Big(\frac{5+\sqrt{21}}{2}\Big)^{n-2} - (14-3\sqrt{21})\Big(\frac{5-\sqrt{21}}{2}\Big)^{n-2}\right),$$
$$= 8\cdot A030221[n-1].$$

Sequence: 9, 48, 232, 1112, 5328, 25528, 122312, 586032, 2807848, 13453208, ...
Recurrence: $a_n = 5a_{n-1} - a_{n-2}$.

5. Combinatorial proofs

In this section, we provide combinatorial proofs of formulas (3.1)–(3.6) above involving the Leonardo numbers. Before doing so, let us recall combinatorial interpretations of the Fibonacci and Leonardo number sequences. By a (linear) *tiling* of length n, we mean a covering of the numbers $1, 2, \dots, n$, written in a row, by rectangular $1 \times m$ pieces for some $m \geq 1$, called *tiles*, that are capable of covering m consecutive numbers. Various restrictions are usually placed as to the lengths of the individual tiles which are otherwise indistinguishable. A rectangular piece covering a single or two adjacent numbers is referred to as a *square* or *domino* and is denoted by s or d, respectively. The length of a tiling λ is denoted by $|\lambda|$.

A well-known combinatorial interpretation of the Fibonacci number F_{n+1} is that it enumerates the set $\mathcal{F}_n$ of tilings in $\{s, d\}$ of length n. This interpretation of F_{n+1} has been used in providing combinatorial proofs of a large number of Fibonacci identities; see, e.g., [3] and references contained therein. In [23], a combinatorial interpretation for ℓ_n was given in terms of tilings as follows. Consider tilings of length n using three types of tiles: squares, dominos and a special kind of tile of variable length, which we will denote by d_ℓ.

We require that a d_ℓ tile be first, if it occurs, but can have any length $\ell \geq 2$, with this length being specified by its subscript. Let $\mathcal{L}_n$ denote the set of such tilings of length n; note that $|\mathcal{L}_n| = \ell_n$ for all $n \geq 0$. For example, when $n = 4$, we have

$$\mathcal{L}_4 = \{s^4,\, s^2d,\, sds,\, ds^2,\, d^2,\, d_2s^2,\, d_2d,\, d_3s,\, d_4\},$$

where a sequence of length m of consecutive copies of a tile x is denoted by x^m. We will refer to members of $\mathcal{L}_n$ as *Leonardo* tilings and those in $\mathcal{F}_n$ as *square-and-domino* tilings. Note that $\lambda \in \mathcal{L}_n$ for $n \geq 2$ implies it must either belong to $\mathcal{F}_n$ or have the form $\lambda = d_\ell\lambda'$, where $2 \leq \ell \leq n$ and $\lambda' \in \mathcal{F}_{n-\ell}$. From this, one obtains immediately the well-known relation

$$\ell_n = F_{n+1} + \sum_{i=0}^{n-2} F_{i+1} = 2F_{n+1} - 1, \qquad n \geq 2,$$

which is seen to hold also for $n = 0, 1$. See [23] for a more general interpretation of u_n that might be useful in explaining some of the determinant formulas in the fourth section involving u_n where $k > 1$, which we leave for the interested reader to explore.

Recall that the determinant of an $n \times n$ matrix A is given explicitly by

$$\det(A) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\operatorname{sgn}(\sigma)} a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}, \tag{5.1}$$

where $A = (a_{i,j})$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. If A is Toeplitz–Hessenberg, then one need only consider terms corresponding to those permutations σ in (5.1) in which every cycle consists of a set of consecutive integers in increasing order (assuming that the smallest element is first in each cycle), as all other terms must contain at least one $a_{i,j}$ factor that equals zero. Assume that the cycles of such a permutation σ are arranged in increasing order of their smallest elements. Then such permutations are synonymous with the compositions of n, upon regarding the various cycle lengths going from left to right as a sequence of parts.

Thus, one may regard the sum in (5.1) when $A = A_n$ has the form (2.1) as being over the set of compositions of n weighted as follows. Each part of size i receives weight a_i and the weight of a composition is the product of the weights of its parts. We will restrict our attention to cases when $a_0 = \pm 1$. If $a_0 = -1$, then the $(-1)^{\operatorname{sgn}(\sigma)}$ factor in each term in (5.1) is equal to the product of the superdiagonal -1 factors, and hence $\det(A)$ in this case gives the sum of the weights of all the compositions of n. On the other hand, if $a_0 = 1$, then one has that $\det(A)$ is a signed weighted sum over the compositions of n, where the weight is defined as before and the sign is given by $(-1)^{n-m}$ with

m denoting the number of parts in a composition (i.e., the number of cycles in the corresponding permutation σ in (5.1)).

Suppose now that a_i enumerates some set Ω_i of tilings for each $i \geq 1$. In this case, consider overlaying each part of size i of a composition with a member of Ω_i. That is, if $\sigma = (\sigma_1, \ldots, \sigma_m)$ with $\sum_{i=1}^{m} \sigma_i = n$ and $\sigma_i \geq 1$, then we overlay σ_i with some $\lambda_i \in \Omega_{\sigma_i}$ for each i. Let $\lambda = \lambda_1 \cdots \lambda_m$ denote the concatenation of the tilings λ_i, where we mark the final tile of each λ_i. Let $\Upsilon_{n,m}$ denote the set of all (marked) tilings that arise in this way. Define the sign of $\lambda \in \Upsilon_{n,m}$ by $(-1)^{n-m}$ and let $\Upsilon_n = \cup_{m=1}^{n} \Upsilon_{n,m}$. In certain cases (see proofs of (3.3) and (3.6) below), it will be convenient to view members of Υ_n as vectors whose components are certain kinds of linear tilings such that the sum of the functional values of some function of the lengths of the components is n.

Let $\sigma(S)$ denote the sum of the signs of all members of a signed set S. If $a_0 = 1$ in (2.1), then it is seen that $\det(A) = \sigma(\Upsilon_n)$, as every part σ_i within a composition in the sum (5.1) is weighted by $a_{\sigma_i} = |\Omega_{\sigma_i}|$, with the sign of each $\lambda \in \Upsilon_n$ the same as the corresponding σ from which it arose. On the other hand, if $a_0 = -1$, then all terms are non-negative in $\det(A)$ as described above and we get $\det(A) = |\Upsilon_n|$. Below, we consider some cases when Ω_i corresponds to a subset of $\mathcal{L}_{ai+b}$ for various constants a and b and thus the corresponding Υ_n consists of a certain set of marked tilings in $\{s, d, d_\ell\}$ such that d_ℓ can only occur at the very beginning or directly following a marked tile. The task at hand then is to determine $\sigma(\Upsilon_n)$ or $|\Upsilon_n|$ in these cases depending on if $a_0 = 1$ or -1. If $a_0 = 1$, it will be useful to define a pairing of the members of Υ_n (i.e., an involution on Υ_n with no fixed points) such that each member is paired with another whose number of marked tiles is of opposite parity. See [12, 13], where a comparable strategy involving lattice paths instead of linear tilings has been employed in establishing formulas for $\det(A)$ by combinatorial arguments.

Before proceeding, let us define a couple of further classes of tilings. By a *generalized* tiling, we mean one whose tiles consist of s, d or d_ℓ for $\ell \geq 2$, where there are no restrictions on the number of d_ℓ tiles or their positions. Let $\mathcal{J}_n$ denote the set of generalized tilings of length n. Then $\mathcal{L}_n$ enumerated by the Leonardo number corresponds the subset of $\mathcal{J}_n$ whose members contain at most one d_ℓ piece where $\ell \geq 2$, with d_ℓ first if it occurs. Let $\mathcal{M}_n$ denote the set of "marked" generalized tilings of length n derived from the members of $\mathcal{J}_n$ by marking some subset of the tiles, such that the final tile as well as each tile directly preceding a d_ℓ is always marked. To establish the formulas in Theorem 3.1 combinatorially, we consider the problem of finding the cardinality or sum of signs of a certain subset of $\mathcal{M}_n$ in several cases.

5.1. Proof of (3.1)

Let $\mathcal{A}_n$ denote the subset of $\mathcal{M}_n$ in which only squares may be marked. Note that members of $\mathcal{A}_n$ must then end in a marked s and that a d_ℓ tile can only occur directly after a marked s or at the very beginning. Since a sequence x of tiles lying (strictly) between two consecutive marked squares, or prior to the first marked square, within a member of $\mathcal{A}_n$ corresponds to a member of $\mathcal{L}_p$ for some $p \geq 0$ (where p denotes the sum of the lengths of the tiles in x), it is seen that $D_-(\ell_0, \ldots, \ell_{n-1}) = |\mathcal{A}_n|$ for all $n \geq 1$.

Let $\mathcal{T}_n$ denote the set of square-and-domino tilings of length $2n$ that end in s, but not sds. By subtraction, we have

$$|\mathcal{T}_n| = F_{2n} - F_{2n-3} = F_{2n-1} + F_{2n-4}, \qquad n \geq 2,$$

so to complete the proof of (3.1), it suffices to define a bijection between $\mathcal{A}_n$ and $\mathcal{T}_n$. Let $\lambda \in \mathcal{A}_n$. We decompose λ as $\lambda = \lambda_1 \cdots \lambda_j$ for some $j \geq 1$, where λ_i for each $i \in [j]$ ends in a marked s and contains no other marked s. Let $\lambda_i = \lambda_i' s$, where $\lambda_i' \in \mathcal{L}_r$ with $r = |\lambda_i'|$ and the terminal s is marked. If λ_i' does not start with d_ℓ, then let $g(\lambda_i)$ be obtained from λ_i by replacing each s in λ_i' with d and each d with sds and then appending s^2 to the tiling that results from these replacements. If λ_i' starts with d_ℓ, then let $g(\lambda_i)$ be obtained by replacing s and d in λ_i' with d and sds as before and then appending $sd^\ell s$ to the resulting tiling. Finally, define $g(\lambda)$ as the concatenation of the various $g(\lambda_i)$ tilings, i.e., $g(\lambda) = g(\lambda_1) \cdots g(\lambda_j)$. Note that $g(\lambda) \in \mathcal{T}_n$ for all λ since it is of length $2n$ and ends in either s^2 or $sd^\ell s$ for some $\ell \geq 2$.

To reverse g, note first that $\rho \in \mathcal{T}_n$ may be regarded as a sequence consisting of the following larger "pieces": $sd^m s$ for any $m \geq 1$, d or s^2, where the final piece cannot be sds or d. To see this, note that the first s within an $sd^m s$ or s^2 piece corresponds to an s which covers an odd number within the original square-and-domino tiling ρ, whereas the second s corresponds to the subsequent s within ρ which would then cover an even number. We decompose ρ and $\rho = \rho_1 \cdots \rho_j$, where ρ_i for each $i \in [j]$ ends in either s^2 or $sd^\ell s$ for some $\ell \geq 2$. Suppose ρ_i ends in $sd^\ell s$, and we transform ρ_i as follows. First replace each sds with d and each d with s, going from left to right within ρ_i. To the tiling that results from these replacements, we prepend a d_ℓ piece and append an s, and subsequently mark the appended s. Let $h(\rho_i)$ denote the (marked) tiling that results. On the other hand, if ρ_i ends in s^2, then to obtain $h(\rho_i)$, we perform the same replacements as before but only append s, which is again marked, to the tiling that results. Let $h(\rho) = h(\rho_1) \cdots h(\rho_j)$ and note that $h(\rho) \in \mathcal{A}_n$. One may verify that the mappings g and h are inverses to one another, which completes the proof of (3.1). $\square$

5.2. Proof of (3.2)

Define the sign of a member of $\mathcal{M}_n$ by $(-1)^{n-j}$, where j denotes the number of marked tiles. Then it is seen that $D_+(\ell_1, \ldots, \ell_n) = \sigma(\mathcal{M}_n)$ for all $n \geq 1$. We define a sign-changing involution on $\mathcal{M}_n$ as follows, where we may assume $n \geq 3$. Consider the rightmost non-terminal tile that is *not* directly followed by d_ℓ for some $\ell \geq 2$ (if it exists) and either mark this tile or remove the marking from it. Then the set $\mathcal{M}'_n$ of survivors of the foregoing involution consists of those members of $\mathcal{M}_n$ that contain at most one tile that is not a d_ℓ, with this tile at the very beginning, if it occurs.

We now define an involution on $\mathcal{M}'_n$ as follows. Consider, if it exists, the leftmost d_ℓ piece such that $\ell \neq 3$, where we require further that this piece be non-terminal if $\ell = 2$. If $\ell \geq 4$, then we replace this d_ℓ with d_2, $d_{\ell-2}$, and perform the reverse operation if $\ell = 2$, leaving all other tiles undisturbed in either case. Note that since every tile within a member of $\mathcal{M}'_n$ must be marked (as each non-terminal tile directly precedes a d_ℓ), the preceding involution ϕ is seen to always reverse the sign. Let S denote the set of survivors of ϕ. If $n = 3m$ for some $m \geq 1$, then $S = \{d_3^m, sd_3^{m-1}d_2\}$, with the two members of the doubleton S seen to be of opposite sign. Therefore, each member of $\mathcal{M}'_{3m}$, and thus also of $\mathcal{M}_{3m}$, is paired with another of opposite parity, which implies $\sigma(\mathcal{M}_{3m}) = 0$ and hence the first case of formula (3.2). If $n = 3m+1$, then $S = \{sd_3^m, dd_3^{m-1}d_2\}$, with both members of S having sign $(-1)^{n-(m+1)} = (-1)^{2m} = 1$. Hence, $\sigma(\mathcal{M}_{3m+1}) = 2$, which yields the second case of (3.2). Finally, if $n = 3m+2$, then $S = \{dd_3^m, d_3^m d_2\}$, with both members of S now having sign -1, whence $\sigma(\mathcal{M}_{3m+2}) = -2$, which implies the last case of (3.2) and completes the proof. $\square$

5.3. Proof of (3.3)

Given $n \geq 2$ and $1 \leq j \leq n$, let $\mathcal{P}_{n,j}$ denote the set of j-tuples $(\lambda_1, \ldots, \lambda_j)$ of Leonardo tilings such that $|\lambda_i| \geq 2$ for $1 \leq i \leq j$ and $\sum_{i=1}^{j} (|\lambda_i| - 1) = n$ (i.e., $\sum_{i=1}^{j} |\lambda_i| = n + j$). Define the sign of a member of $\mathcal{P}_{n,j}$ by $(-1)^{n-j}$ and let $\mathcal{P}_n = \cup_{j=1}^{n} \mathcal{P}_{n,j}$. Then we have $D_+(\ell_2, \ldots, \ell_{n+1}) = \sigma(\mathcal{P}_n)$ and we seek to define a sign-changing involution on $\mathcal{P}_n$. Consider pairing members of $\mathcal{P}_n$ of opposite sign based on alterations made to the final few components, leaving all others unchanged. In the pairings below, only the relevant components of $\lambda = (\lambda_1, \ldots, \lambda_j) \in \mathcal{P}_{n,j}$ for arbitrary j are indicated and how they change:

(i) $\lambda_j = \rho s \leftrightarrow \lambda_j = \rho$, $\lambda_{j+1} = d_2$, $\quad |\rho| \geq 2$,
(ii) $\lambda_j = \rho d \leftrightarrow \lambda_j = \rho s$, $\lambda_{j+1} = d$, $\quad |\rho| \geq 1$,
(iii) $\lambda_j = d_\ell \leftrightarrow \lambda_j = d_{\ell-1}$, $\lambda_{j+1} = d$, $\quad \ell \geq 3$,
(iv) $\lambda_{j-1} = \rho d$, $\lambda_j = d \leftrightarrow \lambda_{j-1} = \rho$, $\lambda_j = sd$, $\lambda_{j+1} = d$, $\quad |\rho| \geq 2$,

where ρ in each case denotes a Leonardo tiling. Note that (iv) only applies if $n \geq 4$.

Let α denote the composite mapping on $\mathcal{P}_n$ defined by (i)–(iv). If $n \geq 4$, then let $\mathcal{P}_n^*$ denote the set of survivors of the involution α, with

$$\mathcal{P}_2^* = \{(s^2, s^2),\ (d, s^2),\ (d_2, s^2),\ (d, d)\}$$

and

$$\mathcal{P}_3^* = \{(x, y, s^2) : (x, y) \in \mathcal{P}_2\} \cup \{(s^2, d, d),\ (d, d, d)\}.$$

Note that in the cases $n = 2$ and $n = 3$, we need only apply the pairings (i)–(iii), along with the pairing $(sd,\, d) \leftrightarrow (d_2,\, d,\, d)$ in the $n = 3$ case, to obtain the respective sets of survivors. We wish to extend α to members of the set $\mathcal{P}_n^*$ for $n \geq 3$. To do so, given $\lambda = (\lambda_1, \ldots, \lambda_j) \in \mathcal{P}_n^* \cap \mathcal{P}_{n,j}$ for a fixed j, let $\lambda^{(i)} = (\lambda_1, \ldots, \lambda_i)$ for each $i \in [j]$. Consider an index $i \in [j-1]$, if it exists, such that $\lambda^{(i)} \in \mathcal{P}_r - \mathcal{P}_r^*$ for some $r \in [2, n-1]$, with the remaining components $\lambda_{i+1}, \ldots, \lambda_j$ of λ forming a sequence made up solely of the 2-tilings s^2 and d such that all runs of d are of even length. Note that the index i is uniquely determined when it exists, which we denote by i_0. Suppose $\lambda^{(i_0)} \in \mathcal{P}_{r_0} - \mathcal{P}_{r_0}^*$, where $2 \leq r_0 < n$. We apply the involution α (in the case $n = r_0$) to $\lambda^{(i_0)}$, and then append the remaining components $\lambda_{i_0+1}, \ldots, \lambda_j$ of λ to $\alpha(\lambda^{(i_0)})$. We pair with λ the resulting member of $\mathcal{P}_n^*$, which is seen to have parity opposite that of λ.

One can show that the set S of survivors in $\mathcal{P}_n^*$ of the extended involution comprises those $\lambda = (\lambda_1, \lambda_2, \ldots)$ wherein each component λ_i is s^2 or d, except for possibly the first, which may also be d_2, such that (I) all runs of d components are of even length, except for a possible run of d starting with the first component, which can have odd length, and (II) if $\lambda_1 = d_2$, then $\lambda_2 = s^2$. For example, when $n = 2$ and $n = 3$, we have $S = \mathcal{P}_2^*$ and

$$S = \{(s^2, s^2, s^2),\ (d, s^2, s^2),\ (d_2, s^2, s^2),\ (d, d, s^2),\ (s^2, d, d),\ (d, d, d)\},$$

respectively. Note that all members of S necessarily belong to $\mathcal{P}_{n,n}$, and hence have positive sign, as all components in each survivor are of length two. Thus, we have $\sigma(\mathcal{P}_n) = |S|$, so to complete the proof, we must enumerate S. To aid in doing so, let T denote the set of marked compositions of n with parts in $\{1, 2\}$ where the first part may be marked. To define a correspondence between S and T, consider making the following replacements within $\lambda = (\lambda_1, \ldots, \lambda_n) \in S$: (a) replace an initial run, if it occurs, of d components in λ of length $2\ell + 1$ for some $\ell \geq 0$ with $1'2^\ell$, where the prime indicates that a part is marked, (b) replace $\lambda_1 = d_2$, $\lambda_2 = s^2$, if it occurs, with $2'$, and (c) replace all other s^2 components with 1 and all other runs of d, which must be of even length, say $2t$ for some $t \geq 1$, with the string 2^t. Let $\beta(\lambda)$ denote the member of T that results from concatenating the various sequences of parts arising from the replacements in (a)–(c) applied to λ, going from left to right. For example, if $n = 10$ and

$$\lambda = (d,\, d,\, d,\, s^2,\, s^2,\, d,\, d,\, d,\, d,\, s^2) \in S,$$

then $\beta(\lambda) = 1'21^22^21 \in T$. One may verify that β is a bijection between S and T, and hence $|S| = |T| = 2F_{n+1}$, which implies (3.3). □

5.4. Proof of (3.4)

Let $\mathcal{B}_n$ denote the subset of $\mathcal{M}_{2n}$ consisting of those tilings in which the only tiles that may be marked are squares covering even numbers. Note that by the definitions members of $\mathcal{B}_n$ must end in a marked square. Consider overlaying each part of size r within a composition of n with a member of $\mathcal{L}_{2r-1}$ followed by a marked square. By the discussion at the beginning of the section, this implies $D_-(\ell_1, \ldots, \ell_{2n-1}) = |\mathcal{B}_n|$ for all $n \geq 1$. We now recall a combinatorial interpretation of the sequence $A010903[n]$. By an $(r+b)$-*color composition* of n, where b is a fixed integer, we mean one in which each part of size r for all $r \geq 1$ is colored in one of $r+b$ ways. Then it is known that $A010903[n]$ enumerates the set of $(r+2)$-color compositions of $n+1$ for $n \geq 0$ (see [6]). To establish (3.4), we then need to demonstrate that $|\mathcal{B}_n|$ equals twice the number of $(r+2)$-color compositions of $n-1$ for $n \geq 2$. Let $\mathcal{V}_n$ denote the set of marked $(r+2)$-color compositions of $n-1$ wherein the first part may be marked. Let $b_n = |\mathcal{B}_n|$ and $v_n = |\mathcal{V}_n|$. Then $v_n = 2 \cdot A010903[n-2]$ and we need to show $b_n = v_n$ for $n \geq 2$.

First note $b_2 = 6$, the enumerated set being

$$\mathcal{B}_2 = \{dss',\, d_2ss',\, d_3s',\, sds',\, s^3s',\, ss'ss'\},$$

where a marked s is indicated by s'. We wish to write a recurrence for b_n, where $n \geq 3$. To do so, we consider several cases based on the final sequence of tiles within $\lambda \in \mathcal{B}_n$. First, observe that if λ ends in -$s'ss'$, -s^2s' or -ds', then there are b_{n-1} possibilities in each case (for the last, consider inserting a d directly prior to the terminal s within a member of $\mathcal{B}_{n-1}$). Now suppose it is the case that λ ends in -ss' or -$d_\ell s'$ for some ℓ and contains at least one other s such that the third (second, if λ ends in -$d_\ell s'$) rightmost s, either marked or unmarked, occurs in position $2n-2k$ for some $2 \leq k \leq n-2$. Note that if λ ends in -$d_\ell s'$, then the second rightmost s must be marked, with $\ell \geq 3$ odd. Thus, λ must end in either -$s'd_{2k-1}s'$, -$s'd_\ell d^i ss'$, -$s'd^{k-1}ss'$ or -$sd^{k-1}ss'$, where $\ell + 2i = 2k-2$ in the second case with $\ell \geq 2$ even and $i \geq 0$. Hence, there are $k+2$ possibilities altogether for the endings of λ, which implies there are $(k+2)b_{n-k}$ such λ with one of the given endings for each k. Summing over k yields $\sum_{k=2}^{n-2}(k+2)b_{n-k}$ members of $\mathcal{B}_n$ of the form stated above for λ.

The remaining unaccounted for members of $\mathcal{B}_n$ must then be of one of the following forms: (i) $d_\ell d^i ss'$, where $\ell+2i=2n-2$, $\ell\geq 2$ and $i\geq 0$, (ii) $ss'd_\ell d^j ss'$, where $\ell+2j=2n-4$, $\ell\geq 2$ and $j\geq 0$, or (iii) one of $\{d^{n-1}ss',\, s^2d^{n-2}ss',\, ss'd^{n-2}ss',\, d_{2n-1}s',\, ss'd_{2n-3}s'\}$. This gives $2n+2$ additional possibilities and combining with the prior cases yields the recurrence

$$b_n = 2n+2+\sum_{k=1}^{n-2}(k+2)b_{n-k}, \quad n\geq 3, \tag{5.2}$$

with initial value $b_2=6$.

To complete the proof of (3.4), we argue that v_n also satisfies recurrence (5.2) for $n\geq 3$, with the same initial condition. First note $v_2=6$, as $\mathcal{V}_2=\{1_1,\, 1_2,\, 1_3,\, 1_1^*,\, 1_2^*,\, 1_3^*\}$, where a marked initial part is starred and a part of size r receiving the ℓ-th color for some $\ell\in[r+2]$ is denoted by r_ℓ. Now let $n\geq 3$ and suppose $\rho\in\mathcal{V}_n$ contains at least two parts, the last of which is k_ℓ for some $1\leq k\leq n-2$ and $\ell\in[k+2]$. Then there are b_{n-k} options regarding the remaining parts of ρ and allowing k to vary gives $\sum_{k=1}^{n-2}(k+2)b_{n-k}$ possibilities for ρ. Otherwise, ρ consists of a single part $(n-1)_\ell$ for some $\ell\in[n+1]$ which may be marked, yielding $2(n+1)$ additional possibilities. Combining with the prior case implies v_n satisfies recurrence (5.2), as desired. $\square$

Remark. If preferred, it is possible to re-express the preceding argument that $b_n=v_n$, which used (5.2), in terms of a recursive bijection between the sets $\mathcal{B}_n$ and $\mathcal{V}_n$ for $n\geq 2$.

5.5. Proof of (3.5)

Let $\mathcal{D}_n$ denote the subset of $\mathcal{M}_{2n}$ consisting of those members where only tiles ending in an even position may be marked. Define the sign of a member of $\mathcal{D}_n$ by $(-1)^{n-j}$, where j denotes the number of marked tiles. Then it is seen $D_+(\ell_2,\ldots,\ell_{2n})=\sigma(\mathcal{D}_n)$ for all $n\geq 1$. We define a series of sign-changing involutions on $\mathcal{D}_n$ for $n\geq 2$. Consider, if it exists, the leftmost non-terminal tile ending in an even position and not directly followed by d_ℓ for some ℓ, and either mark this tile or remove the marking from it. Let $\mathcal{D}_n^{(1)}$ denote the set of survivors of this operation. We decompose $\lambda\in\mathcal{D}_n^{(1)}$ as $\lambda=\lambda^{(1)}\cdots\lambda^{(j)}$ for some $j\geq 1$, where $\lambda^{(i)}$ for each $i\in[j]$ ends in a marked tile and contains no other marked tiles. We will refer to a subtiling $\lambda^{(i)}$ as a *unit* of λ. Note that $\lambda^{(1)}$ must have one of the following forms:

(a) $d_\ell d^i s$, where $\ell \geq 3$ is odd and $i \geq 0$,
(b) a single d_ℓ, where $\ell \geq 2$ is even,
(c) $sd^i s$, where $i \geq 0$, or a single d.

Further, a $\lambda^{(i)}$ where $i > 1$ can only be of the form (a) or (b) above, for otherwise membership in $\mathcal{D}_n^{(1)}$ would be violated.

We now define an involution on $\mathcal{D}_n^{(1)}$ as follows. If $\lambda^{(1)}$ is of the form (a) with $i \geq 1$, then replace $\lambda^{(1)} = d_\ell d^i s$ with d_2, directly followed by $d_\ell d^{i-1} s$. Note that d_2 itself now comprises its own unit since d_2 must be marked, as it is followed by a d_ℓ, and hence this operation reverses the sign. If $\lambda^{(1)}$ is of the form (b) with $\ell \geq 4$, then replace d_ℓ with d_2, $d_{\ell-2}$, which is again seen to reverse the sign as the number of units changes by one. We perform the inverse of one of the preceding two operations if $\lambda^{(1)} = d_2$, upon considering the form of $\lambda^{(2)}$; note that $n \geq 2$ implies $\lambda^{(2)}$ exists if $\lambda^{(1)} = d_2$. So assume $\lambda^{(1)}$ is of the form $d_\ell s$, $sd^i s$ or d, where $\ell \geq 3$ is odd and $i \geq 0$. In this case, consider the smallest index $r > 1$, if it exists, such that $\lambda^{(r)} \neq d_\ell s$ for some ℓ, which we denote by r_0. If $r_0 < j$, consider applying to $\lambda^{(r_0)}$ one of the two operations (or its inverse) defined above where $\lambda^{(1)}$ was of the form (a) or (b) by either breaking apart $\lambda^{(r_0)}$ into two units or combining $\lambda^{(r_0)}$ and $\lambda^{(r_0+1)}$ into a single unit. If $r_0 = j$, then proceed in the same manner unless $\lambda^{(j)} = d_2$.

We extend the involution on $\mathcal{D}_n^{(1)}$ in a few cases when r_0 does not exist or $r_0 = j$ with $\lambda^{(j)} = d_2$. We will describe as *main* a unit of the form $d_\ell s$, where $\ell \geq 3$ is odd. Suppose $\lambda^{(1)} = sd^i s$ where $i \geq 0$, $\lambda^{(2)}$ through $\lambda^{(j-1)}$ are each main units where $j \geq 2$ and $\lambda^{(j)} = d_2$. Then combine $\lambda^{(1)}$ and $\lambda^{(j)}$ into the single unit $sd^{i+1}s$, deleting $\lambda^{(j)}$ from λ and leaving all other units unchanged. Conversely, if $\lambda^{(1)} = sd^i s$ with $i \geq 1$, and $\lambda^{(2)}$ through $\lambda^{(j)}$ are all main with $j = 1$ allowed in this case, then replace $\lambda^{(1)}$ with $sd^{i-1}s$ and append the unit $\lambda^{(j+1)} = d_2$.

Let $\mathcal{D}_n^{(2)}$ denote the set of survivors of the (composite) sign-changing involution defined in the preceding two paragraphs on $\mathcal{D}_n^{(1)}$. Then members $\lambda = \lambda^{(1)} \cdots \lambda^{(j)} \in \mathcal{D}_n^{(2)}$ are of one of the following forms:

(i) $\lambda^{(i)}$ is a main unit for $1 \leq i \leq j$,
(ii) $\lambda^{(j)} = d_2$, with $\lambda^{(i)}$ main for $1 \leq i < j$,
(iii) $\lambda^{(1)} = s^2$ or d, with $\lambda^{(i)}$ main for $1 < i \leq j$,
(iv) $\lambda^{(1)} = d$ and $\lambda^{(j)} = d_2$, with $\lambda^{(i)}$ main for $1 < i < j$.

By the preceding involutions, we have $\sigma(\mathcal{D}_n) = \sigma(\mathcal{D}_n^{(2)})$. To determine $\sigma(\mathcal{D}_n^{(2)})$, note first that members of $\mathcal{D}_n^{(2)}$ of the form (i) may be viewed, upon halving, as compositions of n with no parts of size 1. Similarly, members of $\mathcal{D}_n^{(2)}$ in (ii) or (iii) may be viewed as compositions of $n-1$, while those in (iv) are synonymous with compositions of $n-2$, where there are again no parts of size 1 in each case.

We now consider cases on n mod 3 and first let $n = 3m$, where $m \geq 1$. By Lemma 5.1 below, the sum of the signs of the members of $\mathcal{D}_{3m}^{(2)}$ in (i) is given by $(-1)^{3m-m} = 1$. Further, members of $\mathcal{D}_{3m}^{(2)}$ of the form (ii) or (iii) are seen to contribute $3 \cdot (-1)^{3m-(m+1)} = -3$ towards $\sigma(\mathcal{D}_{3m}^{(2)})$ in total, whereas those in (iv) contribute zero. Combining the contributions from (i)–(iv) then gives $\sigma(\mathcal{D}_{3m}^{(2)}) = -2$, which implies the first case of formula (3.5). If $n = 3m+1$, then Lemma 5.1 implies that we get contributions towards $\sigma(\mathcal{D}_{3m+1}^{(2)})$ of 0, 3 and -1 from (i), (ii)/(iii) together and (iv), respectively. Thus, we have $\sigma(\mathcal{D}_{3m+1}^{(2)}) = 2$, which implies the second case of (3.5). Finally, if $n = 3m+2$, then we get respective contributions of -1, 0 and 1, whence $\sigma(\mathcal{D}_{3m+2}^{(2)}) = 0$, which implies the third case of (3.5) and completes the proof. $\square$

Let $\mathcal{K}_n$ denote the set of compositions of n with no parts of size 1. Define the sign of $\rho \in \mathcal{K}_n$ by $(-1)^r$, where r denotes the number of parts of ρ. We have the following formula for the sum of signs of the members of $\mathcal{K}_n$.

Lemma 5.1. *If $n \geq 1$, then*

$$\sigma(\mathcal{K}_n) = \begin{cases} (-1)^m, & n = 3m; \\ 0, & n = 3m+1; \\ (-1)^{m+1}, & n = 3m+2. \end{cases} \tag{5.3}$$

Proof. Let $\rho = \rho_1 \cdots \rho_r$ denote a member of $\mathcal{K}_n$ with r parts for some $r \geq 1$. Consider the smallest $j \in [r]$, which we will denote by j_0, such that $\rho_j \neq 3$, where we require $j < r$ if $\rho_j = 2$. If $\rho_{j_0} \geq 4$, then replace the part ρ_{j_0} with the two parts 2, $\rho_{j_0} - 2$, leaving all other parts of ρ undisturbed. If $\rho_{j_0} = 2$, whence $j_0 < r$, we perform the reverse operation of combining ρ_{j_0} and ρ_{j_0+1} into a single part of size ≥ 4. Note that these two operations taken together yield a sign-changing involution that is defined on all members of $\mathcal{K}_n$ except for $\rho = 3^m$ if $n = 3m$ or $\rho = 3^m 2$ if $n = 3m+2$. Considering cases on n mod 3 then gives formula (5.3). $\square$

Remark. It is well-known (see, e.g., [26, p. 46]) that $|\mathcal{K}_n| = F_{n-1}$ for all $n \geq 1$. Hence, the argument given above for Lemma 5.1 provides a quick combinatorial proof of the fact that F_n is even if and only if n is divisible by 3.

5.6. Proof of (3.6)

We proceed in a manner analogous to the proof of (3.3) above and consider a set of vectors whose components are tilings of a specific form. Given $n \geq 2$ and $1 \leq j \leq n$, let $\mathcal{Q}_{n,j}$ denote the set of j-tuples $(\lambda_1, \ldots, \lambda_j)$ of Leonardo tilings such that $|\lambda_i| \geq 3$ is odd for each $i \in [j]$ and $\sum_{i=1}^{j} \binom{\frac{|\lambda_i|-1}{2}}{} = n$. Define the sign of a member of $\mathcal{Q}_{n,j}$ by $(-1)^{n-j}$ and let $\mathcal{Q}_n = \cup_{j=1}^{n} \mathcal{Q}_{n,j}$. Then we have $D_+(\ell_3, \ldots, \ell_{2n+1}) = \sigma(\mathcal{Q}_n)$ and we seek to define a sign-changing involution on $\mathcal{Q}_n$. We first pair members λ of $\mathcal{Q}_n$ based on their final components as indicated:

(i) $\lambda_j = \rho d \leftrightarrow \lambda_j = \rho,\ \lambda_{j+1} = sd$,
(ii) $\lambda_j = \rho s^2 \leftrightarrow \lambda_j = \rho,\ \lambda_{j+1} = s^3$,
(iii) $\lambda_j = \gamma ds \leftrightarrow \lambda_j = \gamma s,\ \lambda_{j+1} = ds$,
(iv) $\lambda_j = d_{\ell+1}s \leftrightarrow \lambda_j = d_\ell,\ \lambda_{j+1} = ds, \quad \ell \geq 3$,
(v) $\lambda_{j-1} = \rho d,\ \lambda_j = ds \leftrightarrow \lambda_{j-1} = \rho,\ \lambda_j = sd,\ \lambda_{j+1} = ds$,
(vi) $\lambda_j = d_\ell \leftrightarrow \lambda_j = d_{\ell-2},\ \lambda_{j+1} = d_2 s, \quad \ell \geq 5$,

where ρ and γ denote Leonardo tilings with $|\rho| \geq 3$ odd and $|\gamma| \geq 2$ even and ℓ is odd. Note that (v) only applies if $n \geq 3$.

If $n \geq 3$, then the set of survivors of the composite involution obtained by combining (i)–(vi) consists of those members of $\mathcal{Q}_n$ whose final component is d_3 or $d_2 s$, where in the latter case, the penultimate component is not a single d_ℓ for some $\ell \geq 3$. We extend the involution slightly as follows. Let $\lambda = (\lambda_1, \ldots, \lambda_j) \in \mathcal{Q}_{n,j}$, where $n \geq 3$ and $j \geq 2$ with $\lambda_{j-1} \neq d_\ell$ and $\lambda_j = d_2 s$. Apply (i)–(v) above to the member of $\mathcal{Q}_{n-1}$ comprising the first $j-1$ components of λ. Then append the component $d_2 s$ to the member of $\mathcal{Q}_{n-1}$ that results and it is seen that the member of $\mathcal{Q}_n$ that arises in this way has sign opposite that of λ.

Let $\mathcal{Q}_n^{(1)}$ denote the set of remaining members of $\mathcal{Q}_n$ that have not been paired by any of the previously defined operations. If $n \geq 4$, then $\mathcal{Q}_n^{(1)}$ consists of those members of $\mathcal{Q}_n$ having last component d_3 or last two components both equal to $d_2 s$. If $n = 3$, then $\mathcal{Q}_3^{(1)}$ contains these same members of $\mathcal{Q}_3$, together with $(sd,\, ds,\, d_2 s)$. If $n = 2$, then

$$\mathcal{Q}_2^{(1)} = \{(x, d_3) : x \in \mathcal{L}_3\} \cup \{(x, d_2 s) : x \in \mathcal{L}_3,\ x \neq d_3\} \cup \{(sd,\, ds)\}.$$

We define an involution on $\mathcal{Q}_n^{(1)}$ for $n \geq 3$ as follows. Suppose $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{Q}_n^{(1)} \cap \mathcal{Q}_{n,m}$, where $1 \leq m \leq n$. Let $\lambda^{(i)} = (\lambda_1, \ldots, \lambda_i)$ for each $i \in [m]$. Assume there exists an index $i \in [m-1]$ such that $\lambda^{(i)} \in \mathcal{Q}_r - \mathcal{Q}_r^{(1)}$ for some $r \in [2, n-1]$, with the remaining components $\lambda_{i+1}, \ldots, \lambda_m$ of λ forming a sequence made up exclusively of the 3-tilings d_3 and $d_2 s$ such that all runs of $d_2 s$ are of even length. We will denote this index by i_0; one can show that i_0 is uniquely determined when it exists. Suppose $\lambda^{(i_0)} \in \mathcal{Q}_{r_0} - \mathcal{Q}_{r_0}^{(1)}$, where $2 \leq r_0 < n$. Let α denote the composite involution on $\mathcal{Q}_n$ whose set of survivors was $\mathcal{Q}_n^{(1)}$. Then apply α (in the case $n = r_0$) to $\lambda^{(i_0)}$ and append the remaining components $\lambda_{i_0+1}, \ldots, \lambda_m$ of λ to $\alpha(\lambda^{(i_0)})$. We pair with λ the resulting member of $\mathcal{Q}_n^{(1)}$, which has sign opposite that of λ.

Let $\mathcal{Q}_n^{(2)}$ denote the set of survivors of this involution on $\mathcal{Q}_n^{(1)}$. Then $\mathcal{Q}_2^{(2)} = \mathcal{Q}_2^{(1)}$ and $\mathcal{Q}_3^{(2)}$ consists of the following:

(a) $\lambda = (\lambda_1, \lambda_2, d_3)$, where $(\lambda_1, \lambda_2) \in \mathcal{Q}_2^{(2)}$,
(b) $\lambda = (\rho, d_2s, d_2s)$, where $\rho \in \mathcal{L}_3$,
(c) $\lambda = (sd, ds, d_2s)$.

Further, one can show that $\mathcal{Q}_n^{(2)}$ for $n \geq 4$ consists of all $\lambda = (\lambda_1, \lambda_2, \ldots)$ that can be obtained from members of $\mathcal{Q}_n^{(2)}$ or $\mathcal{Q}_n^{(3)}$ by appending a sequence of components each equal to d_3 or d_2s such that all runs of d_2s are of even length. Each member of $\mathcal{Q}_n^{(2)}$ belongs to $\mathcal{Q}_{n,n}$, and hence has positive sign, since all components are of length three. Thus, we have $\sigma(\mathcal{Q}_n) = \sigma(\mathcal{Q}_n^{(2)}) = q_n$ for all $n \geq 2$, where $q_n = |\mathcal{Q}_n^{(2)}|$. Then $q_n = q_{n-1}+q_{n-2}$ for $n \geq 4$, with $q_2 = 10 = 2F_5$ and $q_3 = 16 = 2F_6$, which implies $q_n = 2F_{n+3}$ for all $n \geq 2$ and completes the proof. $\square$

References

[1] Y. Alp and E.G. Koçer, *Hybrid Leonardo numbers*, Chaos Solitons Fractals **150** (2021), Paper No. 111128, 5 pp.

[2] Y. Alp and E.G. Koçer, *Some properties of Leonardo numbers*, Konuralp J. Math. **9** (2021), no. 1, 183–189.

[3] A.T. Benjamin and J.J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, Washington, DC, 2003.

[4] M. Bicknell-Johnson, *Divisibility properties of the Fibonacci numbers minus one, generalized to $C_n = C_{n-1} + C_{n-2} + k$*, Fibonacci Quart. **28** (1990), no. 2, 107–112.

[5] M. Bicknell-Johnson and G.E. Bergum, *The generalized Fibonacci numbers $\{C_n\}$, $C_n = C_{n-1} + C_{n-2} + k$*, in: A.N. Philippou et al. (eds.), *Applications of Fibonacci Numbers*, Kluwer Academic Publishers, Dordrecht, 1988, pp. 193–205.

[6] D. Birmajer, J.B. Gil, and M.D. Weiner, *$(an + b)$-color compositions*, Congr. Numer. **228** (2017), 245–251.

[7] P. Catarino and A. Borges, *A note on incomplete Leonardo numbers*, Integers **20** (2020), Paper No. A43, 7 pp.

[8] E.W. Dijkstra, *Fibonacci numbers and Leonardo numbers*, EWD797, University of Texas at Austin, 1981. Available at www.cs.utexas.edu.

[9] E.W. Dijkstra, *Smoothsort, an alternative for sorting in situ*, Sci. Comput. Programming **1** (1981), no. 3, 223–233.

[10] T. Goy and M. Shattuck, *Determinant formulas of some Toeplitz–Hessenberg matrices with Catalan entries*, Proc. Indian Acad. Sci. Math. Sci. **129** (2019), no. 4, Paper No. 46, 17 pp.

[11] T. Goy and M. Shattuck, *Determinants of Toeplitz–Hessenberg matrices with generalized Fibonacci entries*, Notes Number Theory Discrete Math. **25** (2019), no. 4, 83–95.

[12] T. Goy and M. Shattuck, *Determinants of some Hessenberg–Toeplitz matrices with Motzkin number entries*, J. Integer Seq. **26** (2023), no. 3, Art. 23.3.4, 21 pp.

[13] T. Goy and M. Shattuck, *Hessenberg–Toeplitz matrix determinants with Schröder and Fine number entries*, Carpathian Math. Publ. **15** (2023), no. 2, 420–436.

[14] Z. İşbilir, M. Akyiğit, and M. Tosun, *Pauli–Leonardo quaternions*, Notes Number Theory Discrete Math. **29** (2023), no. 1, 1–16.

[15] N. Kara and F. Yilmaz, *On hybrid numbers with Gaussian Leonardo coefficients*, Mathematics **11** (2023), no. 6, Paper No. 1551, 12 pp.

[16] A. Karataş, *On complex Leonardo numbers*, Notes Number Theory Discrete Math. **28** (2022), no. 3, 458–465.

[17] K. Kuhapatanakul and J. Chobsorn, *On the generalized Leonardo numbers*, Integers **22** (2022), Paper No. A48, 7 pp.

[18] F. Kürüz, A. Dağdeviren, and P. Catarino, *On Leonardo Pisano hybrinomials*, Mathematics **9** (2021), no. 22, Paper No. 2923, 9 pp.

[19] M. Merca, *A note on the determinant of a Toeplitz–Hessenberg matrix*, Spec. Matrices **1** (2013), 10–16.

[20] T. Muir, *The Theory of Determinants in the Historical Order of Development. Vol. 3*, Dover Publications, Mineola, NY, 1960.

[21] A.G. Shannon, *A note on generalized Leonardo numbers*, Notes Number Theory Discrete Math. **25** (2019), no. 3, 97–101.

[22] A.G. Shannon and Ö. Deveci, *A note on generalized and extended Leonardo sequences*, Notes Number Theory Discrete Math. **28** (2022), no. 1, 109–114.

[23] M. Shattuck, *Combinatorial proofs of identities for the generalized Leonardo numbers*, Notes Number Theory Discrete Math. **28** (2022), no. 4, 778–790.

[24] N.J.A. Sloane (ed.), *The On-Line Encyclopedia of Integer Sequences*. Published electronically at https://oeis.org, 2023.

[25] Y. Soykan, *Generalized Leonardo numbers*, J. Progressive Res. Math. **18** (2021), no. 4, 58–84.

[26] R.P. Stanley, *Enumerative Combinatorics. Vol. 1*, Cambridge University Press, Cambridge, 1997.

[27] E. Tan and H.-H. Leung, *On Leonardo p-numbers*, Integers **23** (2023), Paper No. A7, 11 pp.

[28] R.P.M. Vieira, M.C.S. Mangueira, F.R.V. Alves, and P.M.M.C. Catarino, *The generalization of Gaussians and Leonardo's octonions*, Ann. Math. Sil. **37** (2023), no. 1, 117–137.

Taras Goy
Faculty of Mathematics and Computer Science
Vasyl Stefanyk Precarpathian National University
76018 Ivano-Frankivsk
Ukraine
e-mail: tarasgoy@pnu.edu.ua

Mark Shattuck
Department of Mathematics
University of Tennessee
37996 Knoxville, TN
USA
e-mail: mark.shattuck2@gmail.com