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THE SUBSET-STRONG PRODUCT OF GRAPHS

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Abstract. In this paper, we introduce the subset-strong product of graphs and give a method for calculating the adjacency spectrum of this product. In addition, exact expressions for the first and second Zagreb indices of the subset-strong products of two graphs are reported. Examples are provided to illustrate the applications of this product in some growing graphs and complex networks.

1. Introduction

A graph product $G*H$ is a binary operation that is applied on two graphs G and H, such that $V(G*H) = V(G)\times V(H)$, and $E(G*H)$ is determined by a function on the edges of the factors. Graph products enable us to decompose a graph with large number of vertices into the small factors that are easier to study [14]. Graph products also apply in graphics and theoretical computer science to generate models of complex networks [2, 3], and in engineering to describe discretized structures of objects in structural mechanics [17, 18]. The study of spectra properties and topological indices (graph invariants) of graph products, by using their factor, is an attractive subject among researchers and many papers have been written on this topic [9, 16, 19, 20, 26]. In graph theory, the Cartesian, direct, strong product, and lexicographic product are

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four really important products, each with its own set of applications and theoretical interpretations. For more details, we refer the reader to [14].

Barrière, Dalfó, Fiol, and Mitjana [5] introduced a generalization of the Cartesian product of two graphs with respect to a fixed subset of vertices of one of them. In the Cartesian product $G\Box H$, two vertices (g, h) and (g', h') are adjacent if $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$, but in the generalized hierarchical product with respect to $U \subseteq V(G)$, denoted $G(U) \sqcap H$, the condition $g = g'$ is replaced by $g = g' \in U$. It follows immediately that $G(U) \sqcap H$ is a subgraph of $G\Box H$. Many papers have been devoted to the generalized hierarchical product of graphs and its applications [4, 8, 24]. Similar to the Cartesian product, the strong product, introduced by Sabidussi in 1960 [27], is one of the oldest products that has been widely investigated; see, for instance, [10, 11, 13, 14, 21, 22, 28]. The first aim of this paper is to introduce a generalization of the strong product with respect to the subsets of factors, we call it subset-strong product, similar to the Cartesian product. Then, we give a method for investigating the eigenvalues and characteristic polynomial of the subset-strong product of two graphs. We design our new methods to estimate the spectrum of the adjacency matrix of some class of growing graphs including strongly n-prism networks.

The first and second Zagreb indices were introduced more than thirty years ago by Gutman and Trinajstić [12]. These indices were found to be useful for modeling physicochemical, pharmacologic, toxicologic, biological, and other properties of chemical compounds [6, 25, 29, 30, 31]. In this paper, we compute the Zagreb indices of the subset-strong product of two graphs.

This article is organized as follows. Section 2 introduces the subset-strong product of graphs and some preliminaries. Section 3 explains our method for computing the spectra of the adjacency matrix of the subset-strong product of two graphs. Section 4 indicates how this method can be used. In section 5, we give an exact expression for the first and second Zagreb indices of the subset-strong product of two graphs. Section 6 gives us a generalization of the subset-strong product. Finally, in section 7, we summarize our conclusions.

2. Preliminaries

In this paper, we work entirely with simple graphs, with no loops or multiple edges. Let $G = (V(G), E(G))$ be a graph on a vertex set $V(G) = \{v_1, \ldots, v_n\}$. The adjacency matrix of G is defined to be the matrix $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Denote by $\deg_G(v_i)$ the degree of the vertex v_i of G and $\phi_M(x)$ the characteristic polynomial of the square matrix M. In particular, if G is a graph and $M = A(G)$, then we

write $\phi_{A(G)}(x)$ by $\phi_G(x)$. Similarly, eig(M) (eigenvalue spectrum of M) and eig(G) (adjacency spectrum of G) indicate the set of eigenvalues of M and the set of eigenvalues of $A(G)$, respectively.
The tensor product $A \otimes B$ of two matrices $A = (a_{ij})$ and B of orders $m \times p$ and $n \times q$, respectively, is the partitioned matrix $(a_{ij}B)$ of order $mn \times pq$:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1p}B \\ & \cdots & \\ a_{m1}B & \cdots & a_{mp}B \end{pmatrix}.$$

We denote by M_n, the set of $n-by-n$ real matrices. A matrix $A \in M_n$ is said to be symmetric if $A = A'$ (transpose). Let A and B be symmetric matrices. Simultaneous reduction to diagonal form shows that an orthogonal matrix P (with $P'P = I$) exists such that $P'AP$ and $P'BP$ are diagonal if and only if A and B commute [15]. P_n is the path on n vertices $\{u_1, u_2, \ldots, u_n\}$ such that $u_iu_{i+1} \in E(P_n)$, for $i = 1, \ldots, n-1$. In addition, by adding an edge between u_1 and u_n in P_n, we obtain a cycle with n vertices, denoted by C_n. Note that eig$(P_n) = \{2\cos(\frac{\pi i}{n+1}) | i = 1, \ldots, n\}$ [1].

The first and second Zagreb indices of a graph G, denoted by $M_1(G)$ and $M_2(G)$, respectively, are defined as

$$M_1(G) = \sum_{v \in V(G)} \deg_G(v)^2 = \sum_{uv \in E(G)} [\deg_G(u) + \deg_G(v)],$$
$$M_2(G) = \sum_{uv \in E(G)} \deg_G(u)\deg_G(v).$$

Given a vertex v in G, the neighborhood of v is defined as $\Gamma_G(v) = \{u \in V(G) | uv \in E(G)\}$. It is easy to see that

$$M_1(G) = \sum_{v \in V(G)} \sum_{u \in \Gamma_G(v)} \deg_G(u)$$

and

$$M_2(G) = \sum_{v \in V(G)} \sum_{u \in \Gamma_G(v)} \deg_G(u)\deg_G(v).$$

If $uv \in E(G)$, then we may sometimes write $u \sim v$ in G. For $n \geq 3$, an easy computation shows that $M_1(P_n) = 4n - 6$ and $M_2(P_n) = 4n - 8$.

The strong product $G \boxtimes H$ of two simple graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(g, h)(g', h') \in E(G \boxtimes H)$ whenever $gg' \in E(G)$ and $h = h'$ or $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $hh' \in E(H)$. Now we give the following generalization of this product.

Definition 1. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two simple graphs and $U \subseteq V(G)$. Then, the U-strong product of G and H is the graph denoted as $G(U) \boxtimes H$, and defined by

$$V(G(U) \boxtimes H) = \{(g,h) | g \in V(G) \text{ and } h \in V(H)\},$$

$$(g,h)(g',h') \in E(G(U) \boxtimes H) \Leftrightarrow \begin{cases} g = g' \in U, \ hh' \in E(H), \\ gg' \in E(G), \ h = h' \in V(H), \\ gg' \in E(G), \ hh' \in E(H). \end{cases}$$

Definition 2. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_1), E(G_2))$ be two graphs. The edge sum of these graphs is defined as follows:

$$G_1 \oplus G_2 := (V(G_1), E(G_1) \cup E(G_2)).$$

The above definition yields

$$G(U) \boxtimes H = (G(U) \sqcap H) \oplus (G \times H).$$

Therefore, for $(g,h) \in V(G(U) \boxtimes H)$,

$$\deg_{G(U)\boxtimes H}(g,h) = \deg_G(g) + \chi_U(g) \deg_H(h) + \deg_G(g) \deg_H(h), \tag{1}$$

where χ_U denotes the characteristic function of the set U.

Equation (1) leads us to the following result.

Lemma 3. *Suppose that G and H are two graphs with $|V(G)| = n_1$, $|V(H)| = n_2$, $|E(G)| = m_1$ and $|E(H)| = m_2$. If $U \subseteq V(G)$, then*

$$|E(G(U) \boxtimes H)| = m_1 n_2 + m_2 |U| + 2m_1 m_2.$$

Proof. We have

$$\begin{aligned} |E(G(U) \boxtimes H)| &= \frac{1}{2} \sum_{(g,h) \in V(G) \times V(H)} \deg_{G(U)\boxtimes H}(g,h) \\ &= \frac{1}{2} \sum_{g \in V(G)} \sum_{h \in V(H)} (\deg_G(g) + \chi_U(g) \deg_H(h) + \deg_G(g) \deg_H(h)) \\ &= \frac{1}{2} \sum_{g \in V(G)} \sum_{h \in V(H)} \deg_G(g) + \frac{1}{2} \sum_{g \in V(G)} \sum_{h \in V(H)} \chi_U(g) \deg_H(h) \\ &\quad + \frac{1}{2} \sum_{g \in V(G)} \sum_{h \in V(H)} \deg_G(g) \deg_H(h) \end{aligned}$$

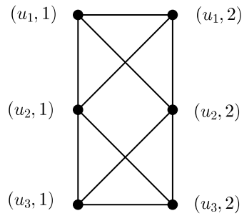

Figure 1. The graph $P_3(\{u_1, u_3\}) \boxtimes P_2$

$$
\begin{aligned}
&= \frac{1}{2}2m_1n_2 + \frac{1}{2}2m_2\sum_{u\in U}\chi_U(g) + \frac{1}{2}2m_1 2m_2 \\
&= m_1n_2 + m_2\sum_{u\in U}1 + 2m_1m_2 = m_1n_2 + m_2|U| + 2m_1m_2. \qquad \square
\end{aligned}
$$

Suppose that $V(G) = \{u_1, \ldots, u_m\}$ and $|V(H)| = n$. Then, the adjacency matrix of $G(U) \boxtimes H$ is

$$
A_{G(U)\boxtimes H} = D_U \otimes A_H + A_G \otimes I_n + A_G \otimes A_H, \tag{2}
$$

where $D_U = diag(\chi_U(u_1), \chi_U(u_2), \ldots, \chi_U(u_m))$.

For instance, let $G = P_3, H = P_2$, $V(G) = \{u_1, u_2, u_3\}$ and $U = \{u_1, u_3\}$. Then

$$
A_G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D_U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad A_H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Thus, by Equation (2), the adjacency matrix of $K = G(U) \boxtimes H$ (see Figure 1) turns out to be

$$
\begin{aligned}
A_K &= D_U \otimes A_H + A_G \otimes I_2 + A_G \otimes A_H \\
&= \begin{pmatrix} A_H & I_2 + A_H & 0_2 \\ I_2 + A_H & 0_2 & I_2 + A_H \\ 0_2 & I_2 + A_H & A_H \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

3. The spectra of the subset-strong product of graphs

Suppose that G and H are two graphs and $U \subseteq V(G)$. In this section, we give a method to compute $\mathrm{eig}(G(U) \boxtimes H)$ and $\phi_{G(U)\boxtimes H}$.

THEOREM 4. *Suppose that* $\mathrm{eig}(H) = \{\lambda_1, \ldots, \lambda_n\}$ *and* $K = G(U) \boxtimes H$. *Then*

$$\mathrm{eig}(K) = \bigcup_{i=1}^{n} \mathrm{eig}\big(A_G + \lambda_i(A_G + D_U)\big),$$
$$\phi_K(x) = \prod_{i=1}^{n} \phi_{A_G+\lambda_i(A_G+D_U)}(x).$$

PROOF. Assume that K' is a graph with $V(K') = V(H) \times V(G)$ and $(h,g)(h',g') \in E(K')$ if and only if $(g,h)(g',h') \in E(K)$. Then the function $f : V(K') \to V(K)$ defined by $f(h,g) = (g,h)$ is a graph isomorphism and, by Equation (2), the adjacency matrix of K' is $A_{K'} = A_H \otimes D_U + I_n \otimes A_G + A_H \otimes A_G$. Therefore $\mathrm{eig}(K) = \mathrm{eig}(K')$ and $\phi_K(x) = \phi_{K'}(x)$. Since A_H and I_n are commuting symmetric matrices, A_H and I_n are simultaneously diagonalizable, that is, there exists a orthogonal matrix P such that $P'A_HP$ and $P'I_nP$ are simultaneously diagonalizable. Without loss of generality, we can assume that $P'A_HP = diag(\lambda_1, \ldots, \lambda_n)$. Let $E = (P \otimes I_m)'A_{K'}(P \otimes I_m)$. Then,

$$\begin{aligned}
E &= (P' \otimes I_m)\big(A_H \otimes D_U + I_n \otimes A_G + A_H \otimes A_G\big)(P \otimes I_m)\\
&= \big(P'A_H \otimes I_mD_U + P'I_n \otimes I_mA_G + P'A_H \otimes I_mA_G\big)(P \otimes I_m)\\
&= (P'A_HP)\otimes(I_mD_UI_m) + (P'I_nP)\otimes(I_mA_GI_m) + (P'A_HP)\otimes(I_mA_GI_m)\\
&= \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \otimes D_U + \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \otimes A_G + \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \otimes A_G\\
&= \begin{pmatrix} \lambda_1 D_U & & \\ & \ddots & \\ & & \lambda_n D_U \end{pmatrix} + \begin{pmatrix} A_G & & \\ & \ddots & \\ & & A_G \end{pmatrix} + \begin{pmatrix} \lambda_1 A_G & & \\ & \ddots & \\ & & \lambda_n A_G \end{pmatrix}\\
&= \begin{pmatrix} A_G + \lambda_1(A_G + D_U) & & \\ & \ddots & \\ & & A_G + \lambda_n(A_G + D_U) \end{pmatrix}.
\end{aligned}$$

Now, $P'P = I_n$ gives that $(P \otimes I_m)'(P \otimes I_m) = I_n \otimes I_m = I_{nm}$. Consequently, if

$$M := \begin{pmatrix} A_G + \lambda_1(A_G + D_U) & & \\ & \ddots & \\ & & A_G + \lambda_n(A_G + D_U) \end{pmatrix},$$

then $\phi_K(x) = \phi_{K'}(x) = \phi_M(x)$.

M is a diagonal block matrix and, hence, $\phi_M(x) = \prod_{i=1}^{n} \phi_{A_G+\lambda_i(A_G+D_U)}(x)$, which completes the proof. $\square$

4. Some examples of the subset-strong product

Theorem 4 provides a method for calculating the eigenvalues and characteristic polynomial of the adjacency matrix of the subset-strong product of some classes of graphs and networks. In this section we explain this method.

Let $Q_0 := P_3(\emptyset) \boxtimes P_n$, $Q_1 := P_3(\{u_1\}) \boxtimes P_n$, $Q_2 := P_3(\{u_2\}) \boxtimes P_n$, $Q_{1,2} := P_3(\{u_1, u_2\}) \boxtimes P_n$ and $Q_{1,3} := P_3(\{u_1, u_3\}) \boxtimes P_n$, see Figure 2. Then, Theorem 4 yields the following statements:

$$\begin{aligned}
\operatorname{eig}(Q_0) &= \bigcup_{i=1}^{n} \operatorname{eig}\left(A_{P_3} + 2\cos\left(\frac{\pi i}{n+1}\right)(A_{P_3} + D_{\emptyset})\right) \\
&= \bigcup_{i=1}^{n} \operatorname{eig}\left(\begin{pmatrix} 0 & 1+2\cos(\frac{\pi i}{n+1}) & 0 \\ 1+2\cos(\frac{\pi i}{n+1}) & 0 & 1+2\cos(\frac{\pi i}{n+1}) \\ 0 & 1+2\cos(\frac{\pi i}{n+1}) & 0 \end{pmatrix}\right) \\
&= \bigcup_{i=1}^{n} \left\{0, \pm\sqrt{2}\left(1 + 2\cos\left(\frac{\pi i}{n+1}\right)\right)\right\}, \\
\operatorname{eig}(Q_2) &= \bigcup_{i=1}^{n} \operatorname{eig}\left(A_{P_3} + 2\cos\left(\frac{\pi i}{n+1}\right)(A_{P_3} + D_{\{u_2\}})\right) \\
&= \bigcup_{i=1}^{n} \operatorname{eig}\left(\begin{pmatrix} 0 & 1+2\cos(\frac{\pi i}{n+1}) & 0 \\ 1+2\cos(\frac{\pi i}{n+1}) & 2\cos(\frac{\pi i}{n+1}) & 1+2\cos(\frac{\pi i}{n+1}) \\ 0 & 1+2\cos(\frac{\pi i}{n+1}) & 0 \end{pmatrix}\right) \\
&= \bigcup_{i=1}^{n} \left\{0, \cos\left(\frac{\pi i}{n+1}\right) \pm \sqrt{9\cos^2\left(\frac{\pi i}{n+1}\right) + 2 + 8\cos\left(\frac{\pi i}{n+1}\right)}\right\},
\end{aligned}$$

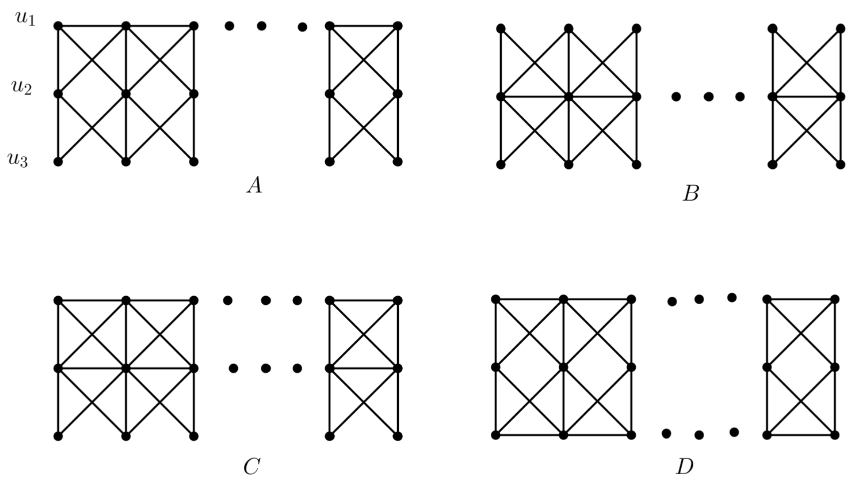

Figure 2. Graphs A: $P_3(\{u_1\}) \boxtimes P_n$, B: $P_3(\{u_2\}) \boxtimes P_n$, C: $P_3(\{u_1, u_2\}) \boxtimes P_n$ and D: $P_3(\{u_1, u_3\}) \boxtimes P_n$

$$\mathrm{eig}(Q_{1,3}) = \bigcup_{i=1}^{n} \mathrm{eig}\left(A_{P_3} + 2\cos\left(\frac{\pi i}{n+1}\right)(A_{P_3} + D_{\{u_1,u_3\}})\right)$$

$$= \bigcup_{i=1}^{n} \mathrm{eig}\left(\begin{pmatrix} 2\cos(\frac{\pi i}{n+1}) & 1+2\cos(\frac{\pi i}{n+1}) & 0 \\ 1+2\cos(\frac{\pi i}{n+1}) & 0 & 1+2\cos(\frac{\pi i}{n+1}) \\ 0 & 1+2\cos(\frac{\pi i}{n+1}) & 2\cos(\frac{\pi i}{n+1}) \end{pmatrix}\right)$$

$$= \bigcup_{i=1}^{n} \left\{2\cos\left(\frac{\pi i}{n+1}\right), \cos\left(\frac{\pi i}{n+1}\right) \pm \sqrt{9\cos^2\left(\frac{\pi i}{n+1}\right) + 2 + 8\cos\left(\frac{\pi i}{n+1}\right)}\right\}.$$

Moreover,

$$\phi_{Q_1} = \prod_{i=1}^{n} \left[x^3 - 2\cos\left(\frac{\pi i}{n+1}\right)x^2 - 2\left(1+2\cos\left(\frac{\pi i}{n+1}\right)\right)^2 x + 2\left(1+2\cos\left(\frac{\pi i}{n+1}\right)\right)^2 \cos\left(\frac{\pi i}{n+1}\right)\right],$$

$$\phi_{Q_{1,2}} = \prod_{i=1}^{n} \left[x^3 - 4\cos\left(\frac{\pi i}{n+1}\right)x^2 - \left(2 + 8\cos\left(\frac{\pi i}{n+1}\right) + 4\cos^2\left(\frac{\pi i}{n+1}\right)\right)x + 2\left(1+2\cos\left(\frac{\pi i}{n+1}\right)\right)^2 \cos\left(\frac{\pi i}{n+1}\right)\right].$$

The eigenvalues of Q_1 can be obtained by solving n cubic equations.

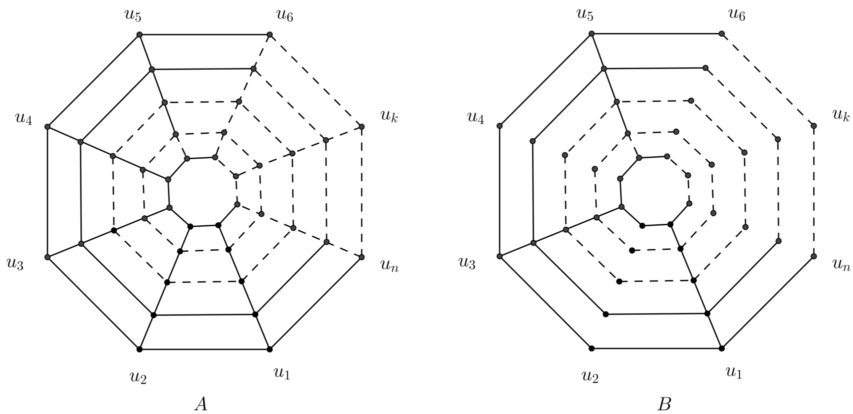

Figure 3. Networks A: An n-prism $p(g)$ and B: $p(g, n, \{u_1, u_3, u_5\})$

Strongly n-prism networks

An n-prism network is built in an iterative way [23]. Let $p(g)$ (with $g \geq 2$) be the family of this graph after $g-1$ iterations. Initially, at $g = 1$, $p(1)$ is an n-polygon. For $g \geq 2$, $P(g)$ is built from $p(g-1)$, where every existing node in $p(g-1)$ gives birth to a new node and the n new nodes form a new n-polygon, so that each new node is also connected to its corresponding mother node. Figure 3.A shows the characteristic structure of the n-prism network $p(g)$. With a suitable labeling for nodes of the n-prism network, we obtain $p(g) = P_g \Box C_n$. This observation leads us to the concept of a strongly n-prism network.

Definition 5. Let g and n be two positive integers and $V(C_n) = \{u_1, \ldots, u_n\}$. For $U \subseteq V(C_n)$, the strongly n-prism network $p(g, n, U)$ is defined as $p(g, n, U) = C_n(U) \boxtimes P_g$. In fact, we delete the edges on the interior that are bisectors of angles that not belong to U from $p(g)$ (see Figure 3.B).

The Laplacian spectra of the 3-prism network and its applications were reported in [7]. Also, Liu, Cao, Alofi, AL-Mazrooei, and Elaiw calculated the Laplacian spectra of the n-prism network [23]. Now, we consider the strongly n-prism network.

By Lemma 3, the number of vertices and edges in $p(g, n, U)$ are gn and $3ng + (g-1)|U| - 2n$, respectively.

Assume that $G = C_3$, and $V(G) = \{u_1, u_2, u_3\}$. We distingue the following cases:

Case I. Let $U = \{u_1, u_2\}$. Then, for $p(g, 3, U)$, illustrated in Figure 4.B, Theorem 4 yields

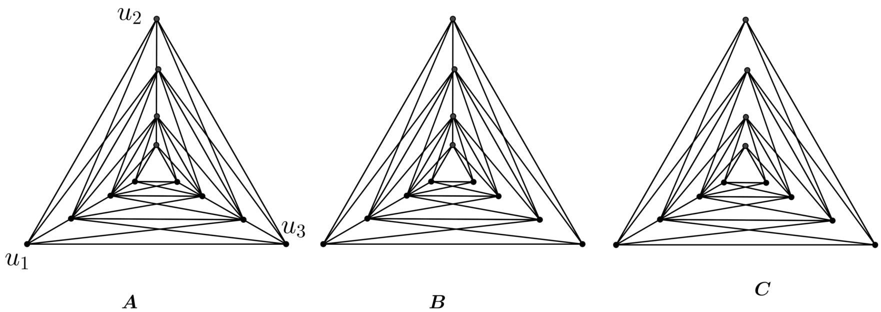

Figure 4. Networks A: 3-prism, B: $p(g,3,\{u_1,u_2\})$ and C: $p(g,3,\{u_1\})$

$$\begin{aligned}
&\mathrm{eig}(p(g{,}3,\{u_1,u_2\}))\\
&\quad=\bigcup_{i=1}^{g}\mathrm{eig}\left(\begin{pmatrix} 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1})\\ 1+2\cos(\frac{\pi i}{g+1}) & 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1})\\ 1+2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 0\end{pmatrix}\right)\\
&\quad=\bigcup_{i=1}^{g}\Bigg\{-1,\pm\frac{1}{2}\sqrt{48\cos^2\left(\frac{\pi i}{g+1}\right)+40\cos\left(\frac{\pi i}{g+1}\right)+9}\\
&\qquad+\frac{1}{2}+2+\cos\left(\frac{\pi i}{g+1}\right)\Bigg\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Phi_{p(g,3,\{u_1,u_2\})}=\prod_{i=1}^{g}\Bigg[&x^3-4\cos\left(\frac{\pi i}{g+1}\right)x^2-\Bigg(3+12\cos\left(\frac{\pi i}{g+1}\right)\\
&+8\cos\left(\frac{\pi i}{g+1}\right)^2\Bigg)x-8\cos\left(\frac{\pi i}{g+1}\right)-8\cos\left(\frac{\pi i}{g+1}\right)^2-2\Bigg].
\end{aligned}$$

Case II. Set $U=\{u_1\}$. Then, for $p(g,3,U)$, see Figure 4.C, Theorem 4 gives

$$\begin{aligned}
&\mathrm{eig}(p(g{,}3,\{u_1\}))\\
&\quad=\bigcup_{i=1}^{g}\mathrm{eig}\left(\begin{pmatrix} 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1})\\ 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1})\\ 1+2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 0\end{pmatrix}\right)\\
&\quad=\bigcup_{i=1}^{g}\Bigg\{-1-2\cos\left(\frac{\pi i}{g+1}\right),2\cos\left(\frac{\pi i}{g+1}\right)+\frac{1}{2}\\
&\qquad\pm\frac{1}{2}\sqrt{32\cos^2\left(\frac{\pi i}{g+1}\right)+32\cos\left(\frac{\pi i}{g+1}\right)+9}\Bigg\}.
\end{aligned}$$

Moreover,

$$\Phi_{p(g,3,\{u_1\})} = \prod_{i=1}^{g} \Bigg[x^3 - 2\cos\left(\frac{\pi i}{g+1}\right)x^2 - 3\left(1 + 2\cos\left(\frac{\pi i}{g+1}\right)\right)^2 x - 10\cos\left(\frac{\pi i}{g+1}\right) - 16\cos^2\left(\frac{\pi i}{g+1}\right) - 2 - 8\cos^3\left(\frac{\pi i}{g+1}\right)\Bigg].$$

Case III. Let $U = \{u_1, u_2, u_3\}$. Then, $p(g,3,U)$ is the 3-prism network, see Figure 4.A. Theorem 4 yields

$$\begin{aligned}
\operatorname{eig}(p(g,3,\{u_1,u_2,u_3\})) &= \bigcup_{i=1}^{g} \operatorname{eig}\left(L(H) + 2\cos\left(\frac{\pi i}{g+1}\right)D(\{u_1,u_2,u_3\})\right) \\
&= \bigcup_{i=1}^{g} \operatorname{eig}\left(\begin{pmatrix} 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 1+\cos(\frac{\pi i}{g+1}) \\ 1+2\cos(\frac{\pi i}{g+1}) & 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) \\ 1+2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 2\cos(\frac{\pi i}{g+1}) \end{pmatrix}\right) \\
&= \bigcup_{i=1}^{g} \left\{ -1, -1, 2 + 6\cos\left(\frac{\pi i}{g+1}\right)\right\}.
\end{aligned}$$

Moreover,

$$\Phi_{p(g,3,\{u_1,u_2,u_3\})} = \prod_{i=1}^{g} \Bigg[x^3 - 6\cos\left(\frac{\pi i}{g+1}\right)x^2 - \left(3 + 12\cos\left(\frac{\pi i}{g+1}\right)\right)x - 6\cos\left(\frac{\pi i}{g+1}\right) - 2\Bigg].$$

Assume that $G = C_4$, and $V(G) = \{u_1, u_2, u_3, u_4\}$. We consider the following cases:

Case I. Let $U = \{u_1, u_2\}$. Then, for $p(g,4,U)$, see Figure 5.C, Theorem 4 gives

$$\begin{aligned}
\operatorname{eig}(p(g,4,\{u_1,u_2\})) &= \bigcup_{i=1}^{g} \operatorname{eig}\left(L(H) + 2\cos\left(\frac{\pi i}{g+1}\right)D(\{u_1,u_2\})\right) \\
&= \bigcup_{i=1}^{g} \operatorname{eig}\left(\begin{pmatrix} 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) \\ 1+2\cos(\frac{\pi i}{g+1}) & 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 0 \\ 0 & 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) \\ 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) & 0 \end{pmatrix}\right)
\end{aligned}$$

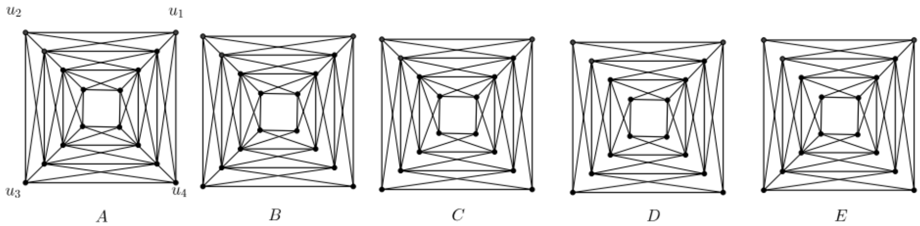

Figure 5. Networks A: The 4-prism, B: $p(g,4,\{u_1,u_2,u_3\})$, C: $p(g,3,\{u_1,u_2\})$, D: $p(g,4,\{u_1\})$ and E: $p(g,4,\{u_1,u_3\})$

$$= \bigcup_{i=1}^{g} \Bigg\{ -\cos\left(\frac{\pi i}{g+1}\right) - 1 \pm \sqrt{5\cos^2\left(\frac{\pi i}{g+1}\right) + 4\cos\left(\frac{\pi i}{g+1}\right) + 1},$$
$$1 + 3\cos\left(\frac{\pi i}{g+1}\right) \pm \sqrt{5\cos^2\left(\frac{\pi i}{g+1}\right) + 4\cos\left(\frac{\pi i}{g+1}\right) + 1}\Bigg\}.$$

Case II. Set $U = \{u_1, u_3\}$. Then, for $p(g,4,U)$, see Figure 5.E, Theorem 4 yields

$$\mathrm{eig}\Big(p(g,4,\{u_1,u_3\})\Big) = \bigcup_{i=1}^{g} \mathrm{eig}\left(L(H) + 2\cos\left(\frac{\pi i}{g+1}\right) D(\{u_1,u_3\})\right)$$
$$= \bigcup_{i=1}^{g} \mathrm{eig}\left(\begin{pmatrix} 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) \\ 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) & 0 \\ 0 & 1+2\cos(\frac{\pi i}{g+1}) & 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) \\ 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) & 0 \end{pmatrix}\right)$$
$$= \bigcup_{i=1}^{g} \left\{0, 2\cos\left(\frac{\pi i}{g+1}\right), \cos\left(\frac{\pi i}{g+1}\right) \pm \sqrt{17\cos^2\left(\frac{\pi i}{g+1}\right) + 4 + 16\cos\left(\frac{\pi i}{g+1}\right)}\right\}.$$

Case III. Set $U = \{u_1, u_2, u_3, u_4\}$. Then, $p(g,4,U)$ is the 4-prism network, see Figure 5.A, Theorem 4 implies that

$$\mathrm{eig}(p(g,4,\{u_1,u_2,u_3,u_4\})) = \bigcup_{i=1}^{g} \mathrm{eig}\left(L(H) + 2\cos\left(\frac{\pi i}{g+1}\right) D(\{u_1,u_2,u_3,u_4\})\right)$$
$$= \bigcup_{i=1}^{g} \mathrm{eig}\left(\begin{pmatrix} 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) \\ 1+2\cos(\frac{\pi i}{g+1}) & 2\cos(\frac{\pi i}{g+1}) & 1+2\cos(\frac{\pi i}{g+1}) & 0 \\ 0 & 1+2\cos(\frac{\pi i}{g+1}) & 1+\cos(\frac{\pi i}{g+1}) & 0 \\ 1+2\cos(\frac{\pi i}{g+1}) & 0 & 1+2\cos(\frac{\pi i}{g+1}) & 2\cos(\frac{\pi i}{g+1}) \end{pmatrix}\right)$$

$$= \bigcup_{i=1}^{g} \left\{ -2\cos\left(\frac{\pi i}{g+1}\right) - 2, 6\cos\left(\frac{\pi i}{g+1}\right) + 2, 2\cos\left(\frac{\pi i}{g+1}\right), \right.$$
$$\left. 2\cos\left(\frac{\pi i}{g+1}\right) \right\}.$$

Case IV. For $U = \{u_1\}$ or $U = \{u_1, u_2, u_3\}$, see Figure 5.D and B, we can compute the characteristic polynomial of $p(g, 4, U)$.

$$\Phi_{p(g,4,\{u_1\})} = \prod_{i=1}^{g} \left[x^4 - 2\cos\left(\frac{\pi i}{g+1}\right)x^3 - 4\left(1 + 2\cos\left(\frac{\pi i}{g+1}\right)^2\right)x^2 \right.$$
$$\left. + 4\cos\left(\frac{\pi i}{g+1}\right)\left(1 + 2\cos\left(\frac{\pi i}{g+1}\right)\right)x \right],$$

$$\Phi_{p(g,4,\{u_1,u_2,u_3\})} = \prod_{i=1}^{g} \left[x^4 - 6\cos\left(\frac{\pi i}{g+1}\right)x^3 \right.$$
$$- \left(4 + 16\cos\left(\frac{\pi i}{g+1}\right) + \cos^2\left(\frac{\pi i}{g+1}\right)\right)x^2$$
$$+ \left(12\cos\left(\frac{\pi i}{g+1}\right) + 48\cos^2\left(\frac{\pi i}{g+1}\right) + 40\cos^3\left(\frac{\pi i}{g+1}\right)\right)x$$
$$\left. - 8\cos^2\left(\frac{\pi i}{g+1}\right) - 32\cos^3\left(\frac{\pi i}{g+1}\right) - 32\cos^4\left(\frac{\pi i}{g+1}\right) \right].$$

5. The first and second Zagreb indices of the subset-strong product

In this section, we compute the first and second Zagreb indices of the subset-strong product of graphs.

THEOREM 6. *Suppose G and H are graphs with $|V(G)| = n_1$, $|V(H)| = n_2$, $|E(G)| = m_1$, and $|E(H)| = m_2$. If $U \subseteq V(G)$, then*

$$M_1(G(U) \boxtimes H) = [n_2 + 4m_2]M_1(G) + |U|M_1(H)$$
$$+ M_1(G)M_1(H) + [4m_2 + 2M_1(H)] \sum_{u \in U} \deg_G(u).$$

Proof. By the definition of the first Zagreb index, we have

$$M_1(G(U) \boxtimes H) = \sum_{(g,h)\in V(G(U)\boxtimes H)} \deg_{G(U)\boxtimes H}(g,h)^2$$

$$= \sum_{g\in V(G)} \sum_{h\in V(H)} [\deg_G(g) + \chi_U(g)\deg_H(h) + \deg_G(g)\deg_H(h)]^2 \qquad \text{(by (1))}$$

$$= \sum_{g\in V(G)} \sum_{h\in V(H)} \deg_G(g)^2 + \sum_{g\in V(G)} \sum_{h\in V(H)} \chi_U(g)^2 \deg_H(h)^2$$

$$+ \sum_{g\in V(G)} \sum_{h\in V(H)} \deg_G(g)^2 \deg_H(h)^2 + 2\sum_{g\in V(G)} \sum_{h\in V(H)} \chi_U(g)\deg_G(g)\deg_H(h)$$

$$+ 2\sum_{g\in V(G)} \sum_{h\in V(H)} \deg_G(g)^2\deg_H(h) + 2\sum_{g\in V(G)} \sum_{h\in V(H)} \chi_U(g)\deg_G(g)\deg_H(h)^2$$

$$= n_2M_1(G) + \sum_{g\in V(G)} \chi_U(g)^2 M_1(H) + M_1(G)M_1(H)$$

$$+ 4m_2 \sum_{g\in V(G)} \chi_U(g)\deg_G(g) + 4m_2M_1(G) + 2\sum_{h\in V(H)} \chi_U(g)\deg_G(g)M_1(H)$$

$$= n_2M_1(G) + |U|M_1(H) + M_1(G)M_1(H) + 4m_2\sum_{u\in U}\deg_G(u)$$

$$+ 4m_2M_1(G) + 2\sum_{u\in U}\deg_G(u)M_1(H)$$

$$= [n_2 + 4m_2]M_1(G) + |U|M_1(H) + M_1(G)M_1(H)$$

$$+ [4m_2 + 2M_1(H)]\sum_{u\in U}\deg_G(u).$$

□

The following corollary, already reported in [26], can be derived by direct consideration of Theorem 6:

Corollary 7. *Suppose G and H are graphs with $|V(G)| = n_1$, $|V(H)| = n_2$, $|E(G)| = m_1$, and $|E(H)| = m_2$. Then,*

$$M_1(G\boxtimes H) = [n_2+4m_2]M_1(G)+[n_1+4m_1]M_1(H)+M_1(G)M_1(H)+8m_1m_2.$$

Proof. Let $U = V(G)$. Then, $\sum_{u\in U}\deg_G(u) = 2m_1$ and the desired result is obtained from Theorem 6. □

EXAMPLE 8. Let $K := P_m(\{u_{i_1}, u_{i_2}, \dots, u_{i_t}\}) \boxtimes P_n$. Then,

$$M_1(K) = 36n_2n_1 - 54n_2 - 40n_1 + 60 + (4n_2 - 6)t + (12n_2 - 16)\sum_{j=1}^{t} \deg_{P_m}(u_{i_j}).$$

In particular, if $u_{i_j} \neq u_1, u_m$, then

$$M_1(K) = 36n_2n_1 - 54n_2 - 40n_1 + 60 + 28n_2t - 38t.$$

THEOREM 9. *Suppose G and H are graphs with $|V(G)| = n_1$, $|V(H)| = n_2$, $|E(G)| = m_1$, and $|E(H)| = m_2$. If $U \subseteq V(G)$ and $K := G(U) \boxtimes H$, then*

$$\begin{aligned} M_2(K) = {} & [m_2 + M_1(H) + M_2(H)] \sum_{u\in U} \deg_G(u)^2 \\ & + [M_1(H) + 2M_2(H)] \sum_{u\in U} \deg_G(u) \\ & + [2m_2 + 2M_1(H) + 2M_2(H)] \sum_{u\in U} \sum_{g\in\Gamma(u)} \deg_G(g) \\ & + |U|M_2(H) + 2M_2(G)M_2(H) \\ & + |\{gg' \in E(G) | g, g' \in U\}|[M_1(H) + 2M_2(H)] \\ & + M_2(G)[n_2 + 6m_2 + 3M_1(H)]. \end{aligned}$$

PROOF. By the definition of the second Zagreb index:

$$\begin{aligned} M_2(K) &= \sum_{(g,h)(g',h')\in E(K)} \deg_K(g,h) \deg_K(g',h') \\ &= \sum_{(g,h)(g',h')\in E(K)} [(\deg_G(g) + \chi_U(g)\deg_H(h) + \deg_G(g)\deg_H(h)) \\ &\qquad (\deg_G(g') + \chi_U(g')\deg_H(h') + \deg_G(g')\deg_H(h'))] \\ &= \frac{1}{2} \sum_{*,**,***} [A_1 + A_2 + \cdots + A_9], \end{aligned} \tag{3}$$

where $*: \; g = g' \in U \;\wedge\; h' \in \Gamma(h)$, $**: \; g \in \Gamma(g') \;\wedge\; h = h'$ and $***: \; g \in \Gamma(g') \;\wedge\; h' \in \Gamma(h)$, and

$A_1 = \deg_G(g)\deg_G(g')$, $\quad A_2 = \chi_U(g')\deg_G(g)\deg_H(h')$,

$A_3 = \deg_G(g)\deg_G(g')\deg_H(h')$, $\quad A_4 = \chi_U(g)\deg_G(g')\deg_H(h)$,

$A_5 = \chi_U(g)\chi_U(g') \deg_H(h) \deg_H(h')$, $\quad A_6 = \chi_U(g)\deg_H(h) \deg_G(g') \deg_H(h')$,

$A_7 = \deg_G(g) \deg_H(h) \deg_G(g')$, $\quad A_8 = \chi_U(g')\deg_G(g) \deg_H(h) \deg_H(h')$,

$A_9 = \deg_G(g) \deg_H(h) \deg_G(g') \deg_H(h')$.

We compute the above sums separately.

$$\sum_{*} A_1 = \sum_{g\in V(G)} \sum_{h\in V(H)} \sum_{h'\in\Gamma(h), g'=g\in U} \deg_G(g)^2 = 2m_2 \sum_{u\in U} \deg_G(u)^2.$$

Similarly,

$$\sum_{*} A_2 = M_1(H) \sum_{u\in U} \deg_G(u), \qquad \sum_{*} A_3 = M_1(H) \sum_{u\in U} \deg_G(u)^2,$$
$$\sum_{*} A_4 = M_1(H) \sum_{u\in U} \deg_G(u), \qquad \sum_{*} A_5 = 2|U|M_2(H),$$
$$\sum_{*} A_6 = 2M_2(H) \sum_{u\in U} \deg_G(u), \qquad \sum_{*} A_7 = M_1(H) \sum_{u\in U} \deg_G(u)^2,$$
$$\sum_{*} A_8 = 2M_2(H) \sum_{u\in U} \deg_G(u), \qquad \sum_{*} A_9 = 2M_2(H) \sum_{u\in U} \deg_G(u)^2.$$

Moreover,

$$\sum_{**} A_1 = \sum_{g\in V(G)} \sum_{h\in V(H)} \sum_{g'\in\Gamma(g), h'=h} \deg_G(g) \deg_G(g') = 2n_2 M_2(G).$$

Similarly,

$$\sum_{**} A_2 = 2m_2 \sum_{u\in U} \sum_{g\in\Gamma(u)} \deg_G(g), \qquad \sum_{**} A_3 = 4m_2 M_2(G),$$
$$\sum_{**} A_4 = 2m_2 \sum_{u\in U} \sum_{g\in\Gamma(u)} \deg_G(g), \qquad \sum_{**} A_5 = 2M_1(H)|\{gg'\in E(G) | g, g'\in U\}|,$$
$$\sum_{**} A_6 = M_1(H) \sum_{u\in U} \sum_{g\in\Gamma(u)} \deg_G(g), \qquad \sum_{**} A_7 = 4m_2 M_2(G),$$
$$\sum_{**} A_8 = M_1(H) \sum_{u\in U} \sum_{g\in\Gamma(u)} \deg_G(g), \qquad \sum_{**} A_9 = 2M_2(G) M_1(H).$$

Finally,

$$\begin{aligned}\sum_{***} A_1 &= \sum_{g\in V(G)} \sum_{h\in V(H)} \sum_{g'\in\Gamma(g)} \sum_{h'\in\Gamma(h)} \deg_G(g)\deg_G(g') \\ &= 2M_2(G)\ 2m_2 = 4m_2M_2(G).\end{aligned}$$

Likewise,

$$\sum_{***} A_2 = M_1(H)\sum_{u\in U}\sum_{g\in\Gamma(u)} \deg_G(g), \quad \sum_{***} A_3 = 2M_2(G)M_1(H),$$

$$\sum_{***} A_4 = M_1(H)\sum_{u\in U}\sum_{g\in\Gamma(u)} \deg_G(g), \quad \sum_{***} A_5 = 4M_2(H)|\{gg'\in E(G)|g,g'\in U\}|,$$

$$\sum_{***} A_6 = 2M_2(H)\sum_{u\in U}\sum_{g\in\Gamma(u)} \deg_G(g), \quad \sum_{***} A_7 = 2M_2(G)M_1(H,),$$

$$\sum_{***} A_8 = 2M_2(H)\sum_{u\in U}\sum_{g\in\Gamma(u)} \deg_G(g), \quad \sum_{***} A_9 = 4M_2(G)M_2(H).$$

Replacing the above quantities in (3) completes the proof. □

Corollary 10. *Suppose that G and H are graphs with $|V(G)| = n_1$, $|V(H)| = n_2$, $|E(G)| = m_1$, and $|E(H)| = m_2$. Then,*

$$\begin{aligned}M_2(G\boxtimes H) &= n_1M_2(G) + n_1M_2(H) + 2M_2(G)M_2(H) \\ &\quad + 3M_1(G)(m_2 + M_1(H) + M_2(H)) \\ &\quad + 3M_1(H)(m_1 + M_2(G)) + 6(m_2M_2(G) + m_1M_2(H)).\end{aligned}$$

Proof. Let $U = V(G)$. Then, $\sum_{u\in U}\deg_G(u)^2 = M_1(G)$, $\sum_{u\in U}\deg_G(u) = 2m_1$, $\sum_{u\in U}\sum_{g\in\Gamma(u)}\deg_G(g) = M_1(G)$, and $|\{gg' \in E(G)|\ g, g' \in U\}| = m_1$. By replacing these quantities in Theorem 9, we obtain the desired result. □

Corollary 10 has already been proved in [26].

Example 11. Let $U := \{u_1, u_3, \ldots, u_{2n+1}\}$, and $K := P_{2n+1}(U)\boxtimes P_{2m+1}$. Then, we have $M_1(P_{2n+1}) = 8n-2$, $M_1(P_{2m+1}) = 8m-2$, $M_2(P_{2n+1}) = 8n-4$, and $M_2(P_{2m+1}) = 8m-4$. Moreover, $\sum_{u\in U}\deg_G(u) = 2n$, $\sum_{u\in U}\deg_G(u)^2 = 4n-2$, $\sum_{u\in U}\sum_{g\in\Gamma(U)}\deg_G(u) = 4n$, and $|\{gg' \in E(G) \mid g, g' \in U\}| = 0$. Hence,

$$M_2(P_{2n+1}(\{u_1, u_3, \ldots, u_{2n+1}\}) \boxtimes P_{2m+1}) = 704mn - 200n - 244m + 60.$$

EXAMPLE 12. For strongly 3 and 4-prism networks, by Theorem 9, it may be concluded that

$$M_2(P(g,3,\{u_1\}))=460g-712, \qquad M_2(P(g,3,\{u_1,u_2\}))=608g-966,$$
$$M_2(P(g,4,\{u_1\}))=568g-872, \qquad M_2(P(g,4,\{u_1,u_2\}))=716g-1126,$$
$$M_2(P(g,4,\{u_1,u_3\}))=704g-1104, \quad M_2(P(g,4,\{u_1,u_2,u_3\}))=864g-1380,$$

where $g \geq 3$.

6. The generalized subset-strong product

DEFINITION 13. Given 3 graphs $G_i = (V_i, E_i)$ and vertex subsets $U_i \subseteq V_i$, for $i = 1, 2$. The generalized set-strong product product $G_1(U_1) \boxtimes G_2(U_2) \boxtimes G_3$ is the graph with vertex set $V_1 \times V_2 \times V_3$ and the following adjacencies:

$$(x_1,x_2,x_3) \sim \begin{cases} (y_1,x_2,x_3) & \text{if } \ y_1x_1 \in E(G_1), \\ (x_1,y_2,x_3) & \text{if } \ y_2x_2 \in E(G_2) \text{ and } x_1 \in U_1, \\ (x_1,x_2,y_3) & \text{if } \ y_3x_3 \in E(G_3),\ x_1 \in U_1, \text{ and } x_2 \in U_2, \\ (y_1,y_2,x_3) & \text{if } \ y_1x_1 \in E(G_1) \text{ and } y_2x_2 \in E(G_2), \\ (x_1,y_2,y_3) & \text{if } \ y_2x_2 \in E(G_2),\ y_3x_3 \in E(G_3), \text{ and } x_1 \in U_1, \\ (y_1,x_2,y_3) & \text{if } \ y_1x_1 \in E(G_1) \text{ and } y_3x_3 \in E(G_3), \\ (y_1,y_2,y_3) & \text{if } y_1x_1 \in E(G_1),\ y_2x_2 \in E(G_2),\ y_3x_3 \in E(G_3). \end{cases}$$

From Definition 13, it follows that:

$$G_1(U_1) \boxtimes G_2(U_2) \boxtimes G_3 = \big(G_1(U_1) \sqcap G_2(U_2) \sqcap G_3\big) \oplus \big(G_1 \times G_2 \times G_3\big).$$

THEOREM 14. *For $i = 1, 2$, let G_i be a graph and $U_i \subseteq V_i$. The generalized subset-strong product satisfies*

$$\begin{aligned} G_1(U_1) \boxtimes G_2(U_2) \boxtimes G_3 &= \big(G_1(U_1) \boxtimes G_2\big)(U_1 \times U_2) \boxtimes G_3 \\ &= G_1(U_1) \boxtimes \big(G_2(U_2) \boxtimes G_3\big). \end{aligned}$$

PROOF. To prove the first equality, we show that in the subset-strong product $\big(G_1(U_1) \boxtimes G_2\big)(U_1 \times U_2) \boxtimes G_3$ vertex $(\big(x_1, x_2\big), x_3)$ has the same adjacencies as vertex (x_1, x_2, x_3) in $G_1(U_1) \boxtimes G_2(U_2) \boxtimes G_3$. Indeed,

$$((x_1,x_2),x_3) \sim \begin{cases} ((y_1,y_2),x_3) & \text{if } (y_1,y_2)(x_1,x_2) \in E(G_1(U_1) \boxtimes G_2), \\ ((x_1,x_2),y_3) & \text{if } y_3x_3 \in E(G_3) \text{ and } (x_1,x_2) \in U_1 \times U_2, \\ ((y_1,y_2),y_3) & \text{if } (y_1,y_2)(x_1,x_2) \in E(G_1(U_1) \boxtimes G_2) \\ & \quad \text{and } y_3x_3 \in E(G_3). \end{cases}$$

This is equivalent to

$$((x_1,x_2),x_3) \sim \begin{cases} ((y_1,x_2),x_3) & \text{if } y_1x_1 \in E(G_1), \\ ((x_1,y_2),x_3) & \text{if } y_2x_2 \in E(G_2) \text{ and } x_1 \in U_1, \\ ((x_1,x_2),y_3) & \text{if } y_3x_3 \in E(G_3), x_1 \in U_3, \text{ and } x_2 \in U_2, \\ ((y_1,y_2),x_3) & \text{if } y_1x_1 \in E(G_1) \text{ and } y_2x_2 \in E(G_2), \\ ((x_1,y_2),y_3) & \text{if } y_2x_2 \in E(G_2),\ y_3x_3 \in E(G_3),\ x_1 \in U_1, \\ ((y_1,x_2),y_3) & \text{if } y_1x_1 \in E(G_1) \text{ and } y_3x_3 \in E(G_3), \\ ((y_1,y_2),y_3) & \text{if } y_1x_1 \in E(G_1),\ y_2x_2 \in E(G_2), \\ & \quad \text{and } y_3x_3 \in E(G_3). \end{cases}$$

Thus, the required isomorphism is simply $((x_1,x_2),x_3) \to (x_1,x_2,x_3)$. Analogously, we can prove the second equality. $\square$

EXAMPLE 15. Let $G = P_3(\{u_1,u_2\}) \boxtimes P_3(\{u_1,u_2\}) \boxtimes P_3$ (see Figure 6). By Theorem 4, we have

$$\begin{aligned} \operatorname{eig}(P_3(\{u_1,u_2\}) \boxtimes P_3) &= \bigcup_{i=1}^{3} \operatorname{eig}\big(A_{P_3} + \mu_i(P_3)(A_{P_3} + D_{\{u_1,u_2\}})\big) \\ &= \bigcup_{i=1}^{3} \operatorname{eig}\left(\begin{pmatrix} 2\cos(\frac{i\pi}{4}) & 1+2\cos(\frac{i\pi}{4}) & 0 \\ 1+2\cos(\frac{i\pi}{4}) & 2\cos(\frac{i\pi}{4}) & 1+2\cos(\frac{i\pi}{4}) \\ 0 & 1+2\cos(\frac{i\pi}{4}) & 0 \end{pmatrix}\right) \\ &\approx \{4.552, -2.459, 0.736, 0, 1.414, -1.414, 0.12, -1.876, -1.072\}. \end{aligned}$$

Hence, again by Theorem 4, we obtain

$$\begin{aligned} \operatorname{eig}(Q(3,(u_1,u_2))^3) &= \bigcup_{i=1}^{9} \operatorname{eig}\Big(A_{P_3} + \mu_i\big(P_3(\{u_1,u_2\}) \boxtimes P_3\big)\big(A_{P_3} + D_{\{u_1,u_2\}}\big)\Big) \\ &\approx \{11.59, 2.451, -4.938, -4.195, -1.536, 0.812, 3.0378, 0.3761, -1.941, \\ &\quad 0, 1.414, -1.414, 4.552, 0.736, -2.459, -1.876, -1.072, 0.12, 1.676, \\ &\quad 0.0603, -1.495, -2.895, -1.254, 0.397, -1.147, -1.002, 0.004\}. \end{aligned}$$

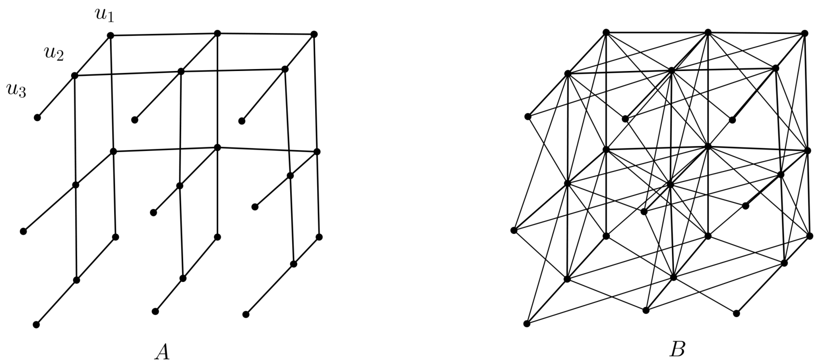

Figure 6. Graph B: $= P_3(\{u_1, u_2\}) \boxtimes P_3(\{u_1, u_2\}) \boxtimes P_3$

7. Conclusion

In this work, we introduced subset-strong products of graphs and gave a method for computing the adjacency spectra or the characteristic polynomial of this product. Our method enabled us to compute the spectra of some growing graphs and networks. Also, we deduced an exact expression for the first and second Zagreb indices of the subset-strong product of two graphs.

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