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NEW UPPER BOUNDS FOR THE WEIGHTED CHEBYSHEV FUNCTIONAL

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Dedicated to Professor Kazimierz Nikodem on the occasion of his 70th birthday

Abstract. New upper bounds for the weighted Chebyshev functional under various conditions, including those of Steffensen type, are given. The obtained results are used to establish some new bounds for the Jensen functional.

1. Introduction

Let f be a convex function defined on a real interval $I \subset \mathbb{R}$. It is well known that if $x_1, \ldots, x_n \in I$, $n \in \mathbb{N}$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

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for all $p_1, \ldots, p_n \geq 0$ such that $P_n = p_1 + \cdots + p_n = 1$. For f strictly convex and $p_1, \ldots, p_n > 0$ (1.1) is strict unless all x_i are equal [4, p. 43]. Inequality (1.1), known as the *Jensen inequality* for convex functions, is in fact an inductive extension of the definition of convexity and undoubtedly one of the most important inequalities in convex analysis with various applications in mathematics, statistics and engineering.

It is also known that the assumption $p_1, \ldots, p_n \geq 0$ can be relaxed at the expense of restricting $x_1, \ldots, x_n$ more severely [5]. Namely, if $\boldsymbol{p} = (p_1, \ldots, p_n)$ is a real n-tuple such that

$$0 \leq P_k = p_1 + \cdots + p_k \leq P_n = 1, \quad k \in \{1, \ldots, n-1\}, \tag{1.2}$$

then for any monotonic n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ we get

$$\overline{x} = \sum_{i=1}^{n} p_i x_i \in I,$$

and for any function f convex on I, inequality (1.1) still holds. Under such assumptions, inequality (1.1) is referred to as *the Jensen–Steffensen inequality* for convex functions, and (1.2) with the monotonicity condition as *Steffensen's conditions* due to J.F. Steffensen. Again, for a strictly convex f, inequality (1.1) remains strict under certain additional assumptions on $\boldsymbol{x}$ and $\boldsymbol{p}$ [1].

Another important inequality in mathematical analysis is the *Chebyshev inequality* (*Čebyšev inequality*), [4, p. 197] or [2, p. 240], which states that

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i$$

holds whenever $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (a_1, \ldots, b_n)$ are real n-tuples monotonic in the same direction, and $\boldsymbol{p} = (p_1, \ldots, p_n)$ is a positive n-tuple [4, p. 43]. Many authors also considered so-called *Chebyshev functional* (or Chebyshev difference) D defined by

$$D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}) = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i.$$

By the Chebyshev inequality we know that

$$D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}) \geq 0$$

when $\boldsymbol{p}$ is positive and $\boldsymbol{a}$, $\boldsymbol{b}$ are monotonic in the same direction. For the special case $\boldsymbol{a} = \boldsymbol{b}$, we immediately obtain

$$D(\boldsymbol{a}, \boldsymbol{a}; \boldsymbol{p}) \geq 0.$$

It is also interesting to note that

$$D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}) = \sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j)(b_i - b_j), \tag{1.3}$$

where

$$I_n = \{1, 2, \ldots, n\}, \quad \Delta = \{(i,j) \in I_n \times I_n \mid i < j\},$$

holds as a weighted version of the Korkine identity [2, p. 242].

The Ostrowski inequality [4, p. 209] provides an upper bound for the absolute value of the integral Chebyshev functional in terms of the *sup* norm or $\|\cdot\|_\infty$. The goal of this paper is to establish some new Ostrowski-like bounds for the discrete weighted Chebyshev functional with positive weights $\boldsymbol{p}$ as well as weights $\boldsymbol{p}$ satisfying (1.2).

2. Bounds for the Chebyshev functional

In the rest of the paper we denote

$$P_k = p_1 + \cdots + p_k, \quad \bar{P}_k = p_k + \cdots + p_n, \quad k \in \{1, \ldots, n\}, \quad \boldsymbol{e} = (1, \ldots, n).$$

To prove our main results, we need the following lemma.

LEMMA 2.1. *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a real n-tuple. Then for all nonnegative n-tuples $\boldsymbol{p} = (p_1, \ldots, p_n)$ the following inequality holds*

$$D(\boldsymbol{a}, \boldsymbol{a}; \boldsymbol{p}) = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2 \leq \sum_{i=1}^{n-1} \tilde{P}_i (a_{i+1} - a_i)^2, \tag{2.1}$$

where

$$\tilde{P}_i = \sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j.$$

Inequality (2.1) *is sharp.*

Proof. Suppose that $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$ are two real n-tuples, and that $\boldsymbol{p} = (p_1, \ldots, p_n)$ is a nonnegative n-tuple. The following identity holds [3, Theorem 3]

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \tag{2.2}$$

$$= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j (a_{i+1} - a_i)(b_{j+1} - b_j) + \sum_{j=i+1}^{n} P_i \bar{P}_j (a_{i+1} - a_i)(b_j - b_{j-1}) \right).$$

For the special case $\boldsymbol{b} = \boldsymbol{e}$, we get

$$D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{p}) = \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right) (a_{i+1} - a_i) \tag{2.3}$$

$$= \sum_{i=1}^{n-1} \tilde{P}_i (a_{i+1} - a_i).$$

By the Korkine identity (1.3), we know that

$$D(\boldsymbol{a}, \boldsymbol{a}; \boldsymbol{p}) = \sum_{(i,j) \in \Delta} p_i p_j (a_i - a_j)^2,$$

and

$$D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{p}) = \sum_{(i,j) \in \Delta} p_i p_j (i - j)(a_i - a_j).$$

We can write

$$D(\boldsymbol{a}, \boldsymbol{a}; \boldsymbol{p}) = \sum_{(i,j) \in \Delta} p_i p_j (a_i - a_j)^2 = \sum_{(i,j) \in \Delta} p_i p_j (j - i)^2 \left(\frac{a_j - a_i}{j - i} \right)^2$$

$$= \sum_{(i,j) \in \Delta} p_i p_j (j - i)^2 \left(\frac{1}{j - i} \sum_{k=i}^{j-1} (a_{k+1} - a_k) \right)^2$$

$$= \sum_{(i,j) \in \Delta} p_i p_j (j - i)^2 \left(\sum_{k=i}^{j-1} \frac{1}{j - i} (a_{k+1} - a_k) \right)^2.$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\begin{aligned} D(\boldsymbol{a},\boldsymbol{a};\boldsymbol{p}) &\le \sum_{(i,j)\in\Delta} p_i p_j (j-i)^2 \sum_{k=i}^{j-1} \frac{1}{(j-i)^2} \sum_{k=i}^{j-1} (a_{k+1}-a_k)^2 \\ &= \sum_{(i,j)\in\Delta} p_i p_j \sum_{k=i}^{j-1} 1 \sum_{k=i}^{j-1} (a_{k+1}-a_k)^2 \\ &= \sum_{(i,j)\in\Delta} p_i p_j (j-i) \left(\sum_{k=1}^{j-1} (a_{k+1}-a_k)^2 - \sum_{k=1}^{i-1} (a_{k+1}-a_k)^2 \right) \\ &= \sum_{(i,j)\in\Delta} p_i p_j (j-i)(\xi_j-\xi_i) = D(\boldsymbol{\xi},\boldsymbol{e};\boldsymbol{p}), \end{aligned}$$

where

$$\xi_m = \sum_{k=1}^{m-1} (a_{k+1}-a_k)^2 .$$

Using (2.3), we now obtain

$$\begin{aligned} D(\boldsymbol{a},\boldsymbol{a};\boldsymbol{p}) \le D(\boldsymbol{\xi},\boldsymbol{e};\boldsymbol{p}) &= \sum_{i=1}^{n-1} \tilde{P}_i (\xi_{i+1}-\xi_i) \\ &= \sum_{i=1}^{n-1} \tilde{P}_i \left(\sum_{k=1}^{i} (a_{k+1}-a_k)^2 - \sum_{k=1}^{i-1} (a_{k+1}-a_k)^2 \right) \\ &= \sum_{i=1}^{n-1} \tilde{P}_i (a_{i+1}-a_i)^2 . \end{aligned}$$

To prove that (2.1) is sharp, assume that

$$D(\boldsymbol{a},\boldsymbol{a};\boldsymbol{p}) \le C \sum_{i=1}^{n-1} \tilde{P}_i (a_{i+1}-a_i)^2 ,$$

for some $C > 0$. Consider $\boldsymbol{a} = \boldsymbol{e}$. Then

$$\begin{aligned} &D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}) \\ &= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j (e_{i+1}-e_i)(e_{j+1}-e_j) + \sum_{j=i+1}^{n} P_i \bar{P}_j (e_{i+1}-e_i)(e_j-e_{j-1}) \right) \end{aligned}$$

$$= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right) = \sum_{i=1}^{n-1} \tilde{P}_i$$

$$\leq C \sum_{i=1}^{n-1} \tilde{P}_i \left(e_{i+1} - e_i \right)^2 = C \sum_{i=1}^{n-1} \tilde{P}_i,$$

hence $C \geq 1$. □

COROLLARY 2.2. *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a real n-tuple. Then*

$$D(\boldsymbol{a}, \boldsymbol{a}; \mathbf{1}) = n \sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i \right)^2 \leq \frac{1}{2} n \sum_{i=1}^{n-1} i (n-i) (a_{i+1} - a_i)^2,$$

where $\mathbf{1} = (1, \ldots, 1)$. The constant $1/2$ is the best possible.

PROOF. Using (2.1) with $\boldsymbol{p} = \mathbf{1}$ we obtain

$$D(\boldsymbol{a}, \boldsymbol{a}; \mathbf{1}) \leq \sum_{i=1}^{n-1} \tilde{\mathbf{1}}_i (a_{i+1} - a_i)^2,$$

where

$$\tilde{\mathbf{1}}_i = \sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j$$

$$= (n-i) \sum_{j=1}^{i-1} j + i \sum_{j=i+1}^{n} (n+1-j) = \frac{1}{2} n i (n-i).$$ □

In the next theorem, we use Lemma 2.1 to obtain an upper bound for the Chebyshev functional.

THEOREM 2.3. *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be real n-tuples. Then for all nonnegative n-tuples $\boldsymbol{p} = (p_1, \ldots, p_n)$ the following inequalities hold*

$$|D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p})| \leq D(\boldsymbol{a}, \boldsymbol{a}; \boldsymbol{p})^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1} - b_i)^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^{n-1} \tilde{P}_i (a_{i+1} - a_i)^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1} - b_i)^2 \right)^{\frac{1}{2}}.$$

These inequalities are sharp.

PROOF. This is a simple consequence of the Cauchy-Bunyakovsky-Schwarz inequality and Lemma 2.1. We have

$$\begin{aligned}
|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| &= \left| \sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j)(b_i - b_j) \right| \\
&\le \left(\sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j)^2 \right)^{\frac{1}{2}} \left(\sum_{(i,j)\in\Delta} p_i p_j (b_i - b_j)^2 \right)^{\frac{1}{2}} \\
&= D(\boldsymbol{a},\boldsymbol{a};\boldsymbol{p})^{\frac{1}{2}} D(\boldsymbol{b},\boldsymbol{b};\boldsymbol{p})^{\frac{1}{2}} \le D(\boldsymbol{a},\boldsymbol{a};\boldsymbol{p})^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1} - b_i)^2 \right)^{\frac{1}{2}} \\
&\le \left(\sum_{i=1}^{n-1} \tilde{P}_i (a_{i+1} - a_i)^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1} - b_i)^2 \right)^{\frac{1}{2}}. \qquad \square
\end{aligned}$$

COROLLARY 2.4. *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *and* $\boldsymbol{b} = (b_1, \ldots, b_n)$ *be real* n*-tuples. Then for all nonnegative* n*-tuples* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *the following inequalities hold*

$$\begin{aligned}
|D(\boldsymbol{a},\boldsymbol{b};\mathbf{1})| &\le \frac{\sqrt{n}}{\sqrt{2}} D(\boldsymbol{a},\boldsymbol{a};\mathbf{1})^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} i(n-i)(b_{i+1} - b_i)^2 \right)^{\frac{1}{2}} \\
&\le \frac{n}{2} \left(\sum_{i=1}^{n-1} i(n-i)(a_{i+1} - a_i)^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} i(n-i)(b_{i+1} - b_i)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The constants $1/\sqrt{2}$ *and* $1/2$ *are the best possible.*

Our next goal is to establish some Ostrowski-like upper bounds for the Chebyshev functional under various conditions on the n-tuples $\boldsymbol{a}$ and $\boldsymbol{b}$. In the discrete case as here

$$\max \left\{ \left| \frac{a_i - a_j}{i - j} \right| \mid (i,j) \in \Delta \right\}$$

takes role of $\|f'\|_\infty$ which appears in Ostrowski-like upper bounds for the integral Chebyshev functional.

THEOREM 2.5. *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *and* $\boldsymbol{b} = (b_1, \ldots, b_n)$ *be real* n*-tuples, and assume that* $\boldsymbol{b}$ *is nondecreasing. Then for all nonnegative* n*-tuples* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *the following inequalities hold*

$$|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| \leq \delta D(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p}) = \delta \sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1} - b_i) \leq \delta (b_n - b_1) D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}), \tag{2.4}$$

and

$$\delta = \max\left\{\left|\frac{a_i - a_j}{i-j}\right| \mid (i,j) \in \Delta\right\}. \tag{2.5}$$

The first inequality is sharp.

PROOF. By the Korkine identity we have

$$|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| = \left|\sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j)(b_i - b_j)\right| \leq \sum_{(i,j)\in\Delta} p_i p_j \left|\frac{a_i - a_j}{i-j}\right| |(i-j)(b_i - b_j)|.$$

Observe that since $\boldsymbol{b}$ is nondecreasing, we know that

$$(i-j)(b_i - b_j) \geq 0, \quad (i,j) \in \Delta.$$

Now we have

$$|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| \leq \delta \sum_{(i,j)\in\Delta} p_i p_j (i-j)(b_i - b_j) = \delta D(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p}),$$

where the middle term in (2.4) follows from (2.3) and the fact that $D(\boldsymbol{b},\boldsymbol{e};\boldsymbol{p}) = D(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p})$.

To prove sharpness, assume that there exist some $C > 0$ such that

$$|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| \leq \delta C \sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1} - b_i).$$

If we choose $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{e}$, we have

$$\delta = \max\left\{\left|\frac{a_i - a_j}{i-j}\right| \mid (i,j) \in \Delta\right\} = 1, \quad D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}) = \sum_{i=1}^{n-1} \tilde{P}_i \leq C \sum_{i=1}^{n-1} \tilde{P}_i,$$

hence $C \geq 1$. □

Corollary 2.6. *Let* $\boldsymbol{a}=(a_1,\ldots,a_n)$ *and* $\boldsymbol{b}=(b_1,\ldots,b_n)$ *be real* n*-tuples, and assume that* $\boldsymbol{b}$ *is nondecreasing. Then*

$$\begin{aligned}\left|D\left(\boldsymbol{a},\boldsymbol{b};\boldsymbol{1}\right)\right| &= \left|n\sum_{i=1}^{n}a_ib_i-\sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i\right| \\ &\leq \delta D\left(\boldsymbol{e},\boldsymbol{b};\boldsymbol{1}\right)=\frac{\delta n}{2}\sum_{i=1}^{n-1}i\left(n-i\right)\left(b_{i+1}-b_i\right) \\ &\leq \delta\left(b_n-b_1\right)\frac{n^2\left(n^2-1\right)}{12},\end{aligned}$$

and δ *is defined as in* (2.5)*. The constant* $1/2$ *is the best possible.*

It is easy to see that we can eliminate the term $b_{i+1}-b_i$ from the upper bound in (2.4) in the same way as we did with the term $a_{i+1}-a_i$.

Theorem 2.7. *Let* $\boldsymbol{a}=(a_1,\ldots,a_n)$ *and* $\boldsymbol{b}=(b_1,\ldots,b_n)$ *be real* n*-tuples. Then for all nonnegative* n*-tuples* $\boldsymbol{p}=(p_1,\ldots,p_n)$ *the following inequality holds*

$$\left|D\left(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p}\right)\right|\leq\delta_1\delta_2D\left(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}\right),$$

and

$$\delta_1=\max\left\{\left|\frac{a_i-a_j}{i-j}\right|\mid(i,j)\in\Delta\right\},\quad\delta_2=\max\left\{\left|\frac{b_i-b_j}{i-j}\right|\mid(i,j)\in\Delta\right\}.$$

If we additionally assume that $\boldsymbol{b}$ *is nondecreasing then*

$$\left|D\left(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p}\right)\right|\leq\delta_1D\left(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p}\right)\leq\delta_1\delta_2D\left(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}\right).$$

All inequalities are sharp.

Proof. As in the proof of Theorem 2.5, we know that

$$\begin{aligned}\left|D\left(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p}\right)\right| &\leq \delta_1\sum_{(i,j)\in\Delta}p_ip_j\left|\left(i-j\right)\left(b_i-b_j\right)\right|\leq \\ &\leq \delta_1\delta_2\sum_{(i,j)\in\Delta}p_ip_j\left(i-j\right)^2=\delta_1\delta_1D\left(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}\right).\end{aligned}$$

If $\boldsymbol{b}$ is nondecreasing then for $i<j$ we have that $|(i-j)(b_i-b_j)|=(i-j)(b_i-b_j)$, and

$$\begin{aligned}|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| &\le \delta_1 \sum_{(i,j)\in\Delta} p_i p_j (i-j)(b_i-b_j) = \delta_1 D(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p}) \\ &= \delta_1 \sum_{(i,j)\in\Delta} p_i p_j (i-j)^2 \frac{b_i-b_j}{i-j} \le \delta_1\delta_2 \sum_{(i,j)\in\Delta} p_i p_j (i-j)^2 = \delta_1\delta_1 D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}). \ \square\end{aligned}$$

Of course, we can formulate the special case $\boldsymbol{p}=\mathbf{1}$ as it was done in the previous corollaries.

Our next goal is to establish new Ostrowski-like upper bounds for the Chebyshev functional under Steffensen's conditions (1.2).

THEOREM 2.8. *Let $\boldsymbol{a}=(a_1,\ldots,a_n)$ and $\boldsymbol{b}=(b_1,\ldots,b_n)$ be real n-tuples, and assume that $\boldsymbol{b}$ is nondecreasing. Then for all n-tuples $\boldsymbol{p}=(p_1,\ldots,p_n)$ satisfying* (1.2) *the following inequalities hold*

$$|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| \le \delta D(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p}) = \delta \sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1}-b_i) \le \delta (b_n-b_1) D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}),$$

and

$$\delta = \max\{|a_{i+1}-a_i| \mid i\in\{1,\ldots,n-1\}\}.$$

The first inequality is sharp.

PROOF. Recall (2.2)

$$\begin{aligned}D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p}) &= \sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \\ &= \sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j (b_{j+1}-b_j) + \sum_{j=i+1}^{n} P_i \bar{P}_j (b_j-b_{j-1})\right)(a_{i+1}-a_i).\end{aligned}$$

Similarly as in the proof of Theorem 2.5, we have

$$\begin{aligned}|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| \le \sum_{i=1}^{n-1} &\left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j (e_{i+1}-e_i)(b_{j+1}-b_j)\right. \\ &\left.+ \sum_{j=i+1}^{n} P_i \bar{P}_j (e_{i+1}-e_i)(b_j-b_{j-1})\right)|a_{i+1}-a_i| \\ &\le \delta D(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p}) = \delta \sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1}-b_i),\end{aligned}$$

since by (1.2) all P_i and $\bar{P}_j$ are nonnegative. Sharpness can be proved in a similar way as before. □

THEOREM 2.9. *Let $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ be real n-tuples. Then for all n-tuples $\boldsymbol{p} = (p_1, \dots, p_n)$ satisfying (1.2) the following inequality holds*

$$\left|D\left(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}\right)\right| \leq \delta_1 \delta_2 D\left(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{p}\right),$$

and

$$\delta_1 = \max\left\{\left|a_{i+1} - a_i\right| \mid i \in \{1, \dots, n-1\}\right\},$$

$$\delta_2 = \max\left\{\left|b_{i+1} - b_i\right| \mid i \in \{1, \dots, n-1\}\right\}.$$

If we additionally assume that $\boldsymbol{b}$ is nondecreasing then

$$\left|D\left(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}\right)\right| \leq \delta_1 D\left(\boldsymbol{e}, \boldsymbol{b}; \boldsymbol{p}\right) \leq \delta_1 \delta_2 D\left(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{p}\right).$$

All inequalities are sharp.

PROOF. Similarly as in the previous proof. □

THEOREM 2.10. *Let $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ be real n-tuples, and assume that $\boldsymbol{a}$ satisfies*

$$a_{i+1} \neq a_i, \quad i \in \{1, \dots, n-1\}.$$

Then for all n-tuples $\boldsymbol{p} = (p_1, \dots, p_n)$ satisfying (1.2) the following inequality holds

$$\left|D\left(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}\right)\right| \leq \delta \sum_{i=1}^{n-1} \tilde{P}_i \left(a_{i+1} - a_i\right)^2,$$

where

$$\delta = \max\left\{\left|\frac{b_{j+1} - b_j}{a_{i+1} - a_i}\right| \mid i, j \in \{1, \dots, n-1\}\right\}.$$

This inequality is sharp.

Proof. We have

$$
\begin{aligned}
&|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| \\
&\le \sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}\bar{P}_{i+1}P_j\left|\frac{b_{j+1}-b_j}{a_{i+1}-a_i}\right|+\sum_{j=i+1}^{n}P_i\bar{P}_j\left|\frac{b_j-b_{j-1}}{a_{i+1}-a_i}\right|\right)(a_{i+1}-a_i)^2 \\
&\le \delta\sum_{i=1}^{n-1}\tilde{P}_i(a_{i+1}-a_i)^2 .
\end{aligned}
$$

Sharpness can be proved in a similar way as before. □

In [6] (or see [4, p. 199]), Steffensen noticed that the Chebyshev inequality also holds when $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{p}$ are such that $\boldsymbol{a}$ is nondecreasing and

$$
P_n\sum_{i=1}^{k}p_ib_i\le P_k\sum_{i=1}^{n}p_ib_i,\quad k\in\{1,\dots,n-1\}.
$$

In the next theorem, we give an upper bound for the Chebyshev functional under similar conditions.

Theorem 2.11. *Let $\boldsymbol{a}=(a_1,\dots,a_n)$ and $\boldsymbol{b}=(b_1,\dots,b_n)$ be real n-tuples. Then for all n-tuples $\boldsymbol{p}=(p_1,\dots,p_n)$ satisfying*

$$
P_n\sum_{i=1}^{k}p_ib_i\le P_k\sum_{i=1}^{n}p_ib_i,\quad k\in\{1,\dots,n-1\} \tag{2.6}
$$

the following inequality holds

$$
|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})|\le\delta\sum_{i=1}^{n-1}\left(\sum_{k=1}^{i}p_k\sum_{j=1}^{n}p_j(b_j-b_k)\right)=\delta D(\boldsymbol{e},\boldsymbol{b};\boldsymbol{p}),
$$

where

$$
\delta=\max\{|a_{i+1}-a_i|\mid i\in\{1,\dots,n-1\}\}.
$$

This inequality is sharp.

Proof. It can be easily proved (using summation by parts, sometimes called the Abel transformation) that for real n-tuples $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{p} = (x_1, \ldots, x_n)$ and any $k \in \{2, \ldots, n-1\}$

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{k-1} P_i \left(x_i - x_{i+1}\right) + P_k x_k + \bar{P}_{k+1} x_{k+1} + \sum_{i=k+2}^{n} \bar{P}_i \left(x_i - x_{i-1}\right), \tag{2.7}$$

and in border cases $k = 1$ or $k = n$

$$\sum_{i=1}^{n} p_i x_i = \bar{P}_1 x_1 + \sum_{i=2}^{n} \bar{P}_i \left(x_i - x_{i-1}\right),$$

$$\sum_{i=1}^{n} p_i x_i = P_n x_n - \sum_{i=1}^{n-1} P_i \left(x_{i+1} - x_i\right). \tag{2.8}$$

The following identities hold (it could be checked directly)

$$\begin{aligned} D\left(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}\right) &= \sum_{i=1}^{n} p_i b_i \sum_{j=1}^{n} p_j a_i - \sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j b_j \\ &= \sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j \left(b_i - b_j\right). \end{aligned}$$

Using (2.8) with $x_i = a_i$ and weights $p_i \sum_{j=1}^{n} p_j \left(b_i - b_j\right)$ we get

$$\begin{aligned} &\sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j \left(b_i - b_j\right) \\ &= a_n \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \left(b_i - b_j\right) - \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_k - b_j\right) \right) \left(a_{i+1} - a_i\right). \end{aligned}$$

Since

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \left(b_i - b_j\right) = \sum_{i=1}^{n} p_i b_i \sum_{j=1}^{n} p_j - \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j b_j = 0,$$

we obtain

$$D\left(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}\right) = \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k\right) \right) \left(a_{i+1} - a_i\right).$$

From (2.6) we have

$$\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k\right) = \sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j b_j - P_n \sum_{k=1}^{i} p_k b_k$$

$$= P_i \sum_{j=1}^{n} p_j b_j - P_n \sum_{k=1}^{i} p_k b_k \geq 0,$$

hence

$$|D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p})| \leq \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k\right) \right) |a_{i+1} - a_i|$$

$$\leq \delta \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k\right) \right) = \delta D(\boldsymbol{e}, \boldsymbol{b}; \boldsymbol{p}). \qquad \square$$

Remark 2.12. We can prove that standard Steffensen's conditions imply (2.6).

Suppose that $\boldsymbol{b}$ is nondecreasing and $\boldsymbol{p}$ satisfies (1.2). By (2.8), with

$$x_i = \sum_{j=1}^{n} p_j \left(b_j - b_i\right),$$

we get

$$P_i \sum_{j=1}^{n} p_j b_j - P_n \sum_{k=1}^{i} p_k b_k = \sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k\right) = P_i \sum_{j=1}^{n} p_j \left(b_j - b_i\right)$$

$$- \sum_{k=1}^{i-1} P_k \left(\sum_{j=1}^{n} p_j \left(b_j - b_{k+1}\right) - \sum_{j=1}^{n} p_j \left(b_j - b_k\right) \right)$$

$$= P_i \left(\sum_{j=1}^{n} p_j b_j - P_n b_i \right) - P_n \sum_{k=1}^{i-1} P_k \left(b_k - b_{k+1}\right).$$

From that, using (2.7) with $x_i = b_i$, we obtain

$$P_i \sum_{j=1}^{n} p_j b_j - P_n \sum_{k=1}^{i} p_k b_k = P_i \left(\sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1}\right) + P_i b_i + \bar{P}_{i+1} b_{i+1} \right.$$

$$\left. + \sum_{j=i+2}^{n} \bar{P}_j \left(b_j - b_{j-1}\right) - P_n b_i \right) - P_n \sum_{k=1}^{i-1} P_k \left(b_k - b_{k+1}\right)$$

$$= P_i\left(\sum_{j=1}^{i-1} P_j\left(b_j - b_{j+1}\right) - \bar{P}_{i+1}b_i + \bar{P}_{i+1}b_{i+1} + \sum_{j=i+2}^{n} \bar{P}_j\left(b_j - b_{j-1}\right)\right)$$

$$- P_n \sum_{j=1}^{i-1} P_j\left(b_j - b_{j+1}\right)$$

$$= P_i \sum_{j=1}^{i-1} P_j\left(b_j - b_{j+1}\right) + P_i \sum_{j=i+1}^{n} \bar{P}_j\left(b_j - b_{j-1}\right) - P_n \sum_{j=1}^{i-1} P_j\left(b_j - b_{j+1}\right)$$

$$= P_i \sum_{j=i+1}^{n} \bar{P}_j\left(b_j - b_{j-1}\right) + \bar{P}_{i+1} \sum_{j=1}^{i-1} P_j\left(b_{j+1} - b_j\right).$$

Recall that if $\boldsymbol{p}$ satisfies (1.2), all P_i and $\bar{P}_i$ are nonnegative, and since $\boldsymbol{b}$ is nondecreasing, we get

$$P_i \sum_{j=1}^{n} p_j b_j - P_n \sum_{k=1}^{i} p_k b_k \geq 0.$$

It is easy to see that the other implication is not true, which means that condition (2.6) is weaker.

3. Bounds for the Jensen functional

In this section, we show how some of the results from the previous section can be used to obtain new Ostrowski-like upper bounds for the Jensen functional (i.e., the Jensen difference). In the rest of the paper we denote $I = (a, b) \subseteq \mathbb{R},\ a < b$, and

$$\overline{x} = \sum_{i=1}^{n} p_i x_i.$$

THEOREM 3.1. *Let $f : I \to \mathbb{R}$ be a differentiable function and $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$. Suppose that there exist some $m, M \in \mathbb{R}$ such that*

$$m \leq f'(x) \leq M, \quad \textit{for all } x \in I.$$

Then for all nonnegative n-tuples $\boldsymbol{p} = (p_1, \ldots, p_n)$ such that $P_n = 1$ the following inequalities hold

$$\left| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \right| \leq (M - m) D(\boldsymbol{x}, \boldsymbol{x}; \boldsymbol{p})^{\frac{1}{2}} D(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{p})^{\frac{1}{2}}$$

$$\leq (M - m) D(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{p})^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i (x_{i+1} - x_i)^2 \right)^{\frac{1}{2}}.$$

Proof. From the mean-value theorem we know that for any $x, y \in I$ there exist some z between them such that

$$f(y) - f(x) = f'(z)(y - x).$$

Choosing $x = \bar{x}$ and $y = x_i$, we get

$$f(x_i) - f(\bar{x}) = f'(z_i)(x_i - \bar{x})$$

for some z_i between $\bar{x}$ and x_i (observe that $\bar{x}$ and z_i are both in I). If we multiply the above equality by p_i, and sum over i, we obtain

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) = \sum_{i=1}^{n} p_i x_i f'(z_i) - \bar{x} \sum_{i=1}^{n} p_i f'(z_i)$$

$$= \sum_{i=1}^{n} p_i x_i f'(z_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(z_i) = D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}),$$

where $\boldsymbol{a} = (f'(z_1), \ldots, f'(z_n))$ and $\boldsymbol{b} = \boldsymbol{x} = (x_1, \ldots, x_n)$. Note that

$$\max \left\{ (f'(z_{i+1}) - f'(z_i))^2 \mid i \in \{1, \ldots, n-1\} \right\} \leq (M - m)^2,$$

and

$$D(\boldsymbol{e}, \boldsymbol{e}; \boldsymbol{p}) = \sum_{i=1}^{n-1} \tilde{P}_i.$$

By Theorem 2.3, we know that

$$|D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p})| = \left| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \right|$$

$$\leq D(\boldsymbol{b}, \boldsymbol{b}; \boldsymbol{p})^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i (f'(z_{i+1}) - f'(z_i))^2 \right)^{\frac{1}{2}}.$$

Using Lemma 2.1 we get

$$\begin{aligned}&\left|\sum_{i=1}^{n} p_i f(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big)\right| \\ &\leq (M-m)\, D(\boldsymbol{b},\boldsymbol{b};\boldsymbol{p})^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i\right)^{\frac{1}{2}} = (M-m)\, D(\boldsymbol{b},\boldsymbol{b};\boldsymbol{p})^{\frac{1}{2}}\, D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p})^{\frac{1}{2}} \\ &\leq (M-m)\, D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p})^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} \tilde{P}_i (x_{i+1}-x_i)^2\right)^{\frac{1}{2}}. \qquad \square\end{aligned}$$

THEOREM 3.2. *Let $f : I \to \mathbb{R}$ be a differentiable function, and let $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ be nondecreasing. Suppose that there exist some $m, M \in \mathbb{R}$ such that*

$$m \leq f'(x) \leq M, \text{ for all } x \in I.$$

Then for all nonnegative n-tuples $\boldsymbol{p} = (p_1, \ldots, p_n)$ such that $P_n = 1$ the following inequalities hold

$$\begin{aligned}\left|\sum_{i=1}^{n} p_i f(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big)\right| &\leq (M-m) \sum_{i=1}^{n-1} \tilde{P}_i (x_{i+1}-x_i) \\ &\leq (M-m)(x_n - x_1)\, D(\boldsymbol{e},\boldsymbol{e};\boldsymbol{p}).\end{aligned}$$

PROOF. As in the proof of the previous theorem, we have

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) = D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p}), \tag{3.1}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{p}$ are the same as there.

From Theorem 2.5, we know that

$$|D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{p})| \leq \delta \sum_{i=1}^{n-1} \tilde{P}_i (b_{i+1} - b_i).$$

Since all z_i are in I, and $m \leq f'(x) \leq M$ for $x \in I$, we have

$$\delta = \max\left\{ \frac{|f'(z_j) - f'(z_i)|}{j-i} \mid (i,j) \in \Delta \right\} \leq M - m.$$

Hence, by (3.1),

$$\left|\sum_{i=1}^{n} p_i f(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big)\right| \le (M-m)\sum_{i=1}^{n-1} \tilde{P}_i (x_{i+1}-x_i)$$
$$\le (M-m)(x_n - x_1)\sum_{i=1}^{n-1} \tilde{P}_i. \qquad \square$$

4. Bounds for the Jensen–Steffensen functional

In the previous section, all weights $\boldsymbol{p}$ were nonnegative. However, we know that the Jensen inequality remains valid under slightly different conditions for the weights $\boldsymbol{p}$ as proposed by Steffensen [5]. In this section, we give an Ostrowski-like upper bound for the Jensen functional under Steffensen's conditions (2.6). This difference can be called the Jensen–Steffensen difference or the Jensen–Steffensen functional.

Theorem 4.1. *Let $f\colon I \to \mathbb{R}$ be a differentiable function and $\boldsymbol{x} = (x_1, \dots, x_n) \in I^n$. Suppose that there exist some $m, M \in \mathbb{R}$ such that*

$$m \le f'(x) \le M, \quad \text{for all } x \in I.$$

Then for all $\boldsymbol{p} = (p_1, \dots, p_n)$ satisfying

$$\sum_{i=1}^{k} p_i x_i \le P_k \sum_{i=1}^{n} p_i x_i, \quad k \in \{1, \dots, n-1\} \text{ and } \bar{x} \in I,$$

with $P_n = 1$, the following inequality holds

$$\left|\sum_{i=1}^{n} p_i f(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big)\right| \le (M-m)\sum_{i=1}^{n-1}\Big(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j (x_j - x_k)\Big)$$
$$= (M-m)\, D(\boldsymbol{e}, \boldsymbol{x}; \boldsymbol{p}).$$

Proof. As in the proof of Theorem 3.1, we start from

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) = \sum_{i=1}^{n} p_i x_i f'(z_i) - \bar{x}\sum_{i=1}^{n} p_i f'(z_i) = D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{p}),$$

where $\boldsymbol{a} = (f'(z_1), \ldots, f'(z_n))$ and $\boldsymbol{b} = \boldsymbol{x} = (x_1, \ldots, x_n)$. Note that

$$\max \{|f'(z_{i+1}) - f'(z_i)| \mid i \in \{1, \ldots, n-1\}\} \leq M - m.$$

By Theorem 2.11, with

$$\begin{aligned}\delta &= \max \{|a_{i+1} - a_i| \mid i \in \{1, \ldots, n-1\}\} \\ &= \max \{|f'(z_{i+1}) - f'(z_i)| \mid i \in \{1, \ldots, n-1\}\},\end{aligned}$$

we get

$$\begin{aligned}\left| \sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) \right| &\leq \delta \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j (x_j - x_k) \right) \\ &\leq (M - m) D(\boldsymbol{e}, \boldsymbol{x}; \boldsymbol{p}),\end{aligned}$$

which is the desired result. □

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