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CLOSURE OPERATIONS ON INTUITIONISTIC LINEAR ALGEBRAS

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Abstract. In this paper, we introduce the notions of radical filters and extended filters of Intuitionistic Linear algebras (IL-algebras for short) and give some of their properties. The notion of closure operation on an IL-algebra is also introduced as well as the study of some of their main properties. The radical of filters and extended filters are examples of closure operations among several others provided. The class of stable closure operations on an IL-algebra **L** is used to study the unifying properties of some subclasses of the lattice of filters of **L**. In particular, we obtain that for a stable closure operation c on an IL-algebra, the collection of c-closed elements of its lattice of filters forms a complete Heyting algebra. Hyperarchimedean IL-algebras are also characterized using closure operations. It is shown that the image of a semi-prime closure operation on an IL-algebra is isomorphic to a quotient IL-algebra. Some properties of the quotients induced by closure operations on an IL-algebra are explored.

1. Introduction

In commutative ring theory and lattice theory, closure operations have been thoroughly investigated in literature [4, 5, 9, 18, 22]. Girard in 1987 [7] introduced linear logic whose algebraic counterpart is the notion of Intuitionistic Linear Algebra (IL-algebra for short) initiated by Troelstra (see [20]). In Galatos et al. [6], IL-algebras are viewed as FL_e-algebras. For properties

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regarding IL-algebras the reader may refer to [2, 20]. One vital difference between IL-algebras and commutative residuated lattices is that the top of the lattice and the monoidal identity are different in the former structure, whereas these two coincide in the latter one.

Filter theory plays a crucial role in the study of algebraic structures and associated logics. The filter theory of IL-algebras has been introduced in [2, 10] and some important results have been obtained. Recently, Tenkeu and Nganteu have characterized the filter generated by a subset and show that the lattice of filters of an IL-algebra is algebraic, pseudocomplemented and is endowed with a structure of Heyting algebra [19]. Given that several concepts of commutative rings theory have their analogs in the theory of ordered structures, the ring theory notion of closure operation was transported to the MV-algebra context in [8] and relations between closure operations and BL-algebras were studied in [14]. In the same order of idea, we consider closure operations on IL-algebras for a common treatment of classes of IL-algebras.

Closure operations play an important role in (fuzzy) topological spaces (see [13]) and form interesting tools for constructing topological spaces, e.g in [21] Stone's topology for pseudocomplemented lattices is investigated using closure operations on the lattice of filters. In ring theory, there are several examples of closure operations used to characterize different classes of rings and some of their algebraic properties [5]. Similarly, in [8], examples of closure operations on MV-algebras such as radical of ideals have been provided along with their main properties as well as some characterizations of classes of MV-algebras using closure operations. In this framework, we consider the radical of a filter as the intersection of all maximal filters containing it. The radical of a filter satisfies the three axioms of a closure operation: it is extensive, non-decreasing, and idempotent. This motivates the introduction of closure operations on IL-algebras. Extended filters are also introduced in Section 2 along with some preliminary results to be used in later sections.

In Section 3, the notion of closure operation on an IL-algebra L is introduced and defined as a map on the lattice of filters of L that is extensive, non-decreasing, and idempotent. We provide several examples of closure operations on IL-algebras and their properties including the investigation of properties of the class of stable closure operations. It is shown that if the set of filters of an IL-algebra $\mathbf{L}$ is linearly ordered by inclusion, then every closure operation on this lattice is stable. We prove that the set of closed filters with stable closure operations forms a complete Heyting algebra. Characterization of Hyperarchimedean IL-algebras in terms of closure operations is also proven. Note that these precedent results generalize those existing in IL-algebra subclasses such as MV-algebras, BL-algebras and residuated lattices.

In Section 4, We turn our attention to semi-prime closure operations on IL-algebras and their relationships with homomorphisms. We extend some results from [14] to IL-algebras. We study the first isomorphism theorem induced by closure operations on IL-algebras and the properties of quotients induced by closure operations on IL-algebras. We also explore some relationship between closure operations on $\mathbf{L}$ and its powers $\mathbf{L}^X$ where X is an arbitrary non-empty set.

2. Preliminaries

For convenience, we gather in this section some definitions and results on IL-algebras from [12, 19, 20] and provide some preliminary results to be used in later sections.

2.1. Definition and some algebraic properties of IL-algebras

Definition 2.1 ([11, 20]). An Intuitionistic Linear algebra is an algebraic system $\mathbf{L} = (L, \wedge, \vee, *, \to, e, \perp, \top)$ with four binary operations $\wedge, \vee, *$ and $\to$ and three constants $\perp, e$ and $\top$ such that:

(1) $(L, \wedge, \vee, \perp, \top)$ is a bounded lattice.
(2) $(L, *, e)$ is a commutative monoid with unit e.
(3) For any $x, y, z \in L, x * y \leq z$ if and only if $x \leq y \to z$ (residuation property).

In what follows, $\mathbf{L}$ denotes any IL-algebra $(L, \wedge, \vee, *, \to, e, \perp, \top)$.

Example 2.2 ([11, Example 2]). Let $L_7 = \{\perp, a, b, c, d, e, \top\}$, where the lattice diagram is given in Figure 1, $*$ and $\to$ tables are given below:

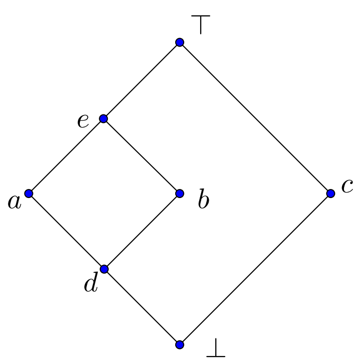

Figure 1. Lattice diagram

$*$	$\perp$	a	b	c	d	e	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	a	d	c	d	a	$\top$
b	$\perp$	d	b	c	d	b	$\top$
c	$\perp$	c	c	d	c	c	$\top$
d	$\perp$	d	d	c	d	d	$\top$
e	$\perp$	a	b	c	d	e	$\top$
$\top$	$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\perp$	a	b	c	d	e	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\perp$	1	b	c	b	e	$\top$
b	$\perp$	a	e	c	a	e	$\top$
c	$\perp$	c	c	e	c	e	$\top$
d	$\perp$	e	e	c	e	e	$\top$
e	$\perp$	a	b	c	d	e	$\top$
$\top$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\top$

The IL-algebra $\mathbf{L}_7$ is not a residuated lattice, since we have $b \wedge c < b * c$.

EXAMPLE 2.3 ([12, Example 10]). Consider $L = [-1, 2]$, where

$$x \vee y = \max\{x, y\}, \quad x \wedge y = \min\{x, y\}, \quad x \to y = 1 - (x * (1 - y))$$

and

$$x * y = \begin{cases} \max\{0, x + y - 1\} & \text{when } x, y \in [0, 1], \\ \min\{x, y\} & \text{when } x + y \leq 1 \text{ and at least one of } x, y \notin [0, 1], \\ \max\{x, y\} & \text{when } x + y > 1 \text{ and at least one of } x, y \notin [0, 1]. \end{cases}$$

Then $(L, \wedge, \vee, *, \to, \perp, e, \top)$ is an IL-algebra with $\perp = -1, \top = 2$ and $e = 1$.

EXAMPLE 2.4 ([12]). If $(L, \vee, \wedge, *, \to, e, \perp, \top)$ is an IL-algebra and X is a non-empty set, then the set $L^X := \{f : X \to L \mid f \text{ is a map}\}$ becomes an IL-algebra $(L^X, \vee, \wedge, *, \to, \underline{\perp}, \underline{e}, \underline{\top})$ with the operations defined pointwise and $\underline{\perp}, \underline{\top}, \underline{e} : X \to L$ are the constant functions associated with $\perp, \top, e$.

The ordering $\leq$ and negation $\neg$ in $\mathbf{L}$ are defined as follows: for all x and y in L:

$$x \leq y \quad \text{iff} \quad x \wedge y = x \quad (\text{or equivalently, iff} \quad x \vee y = y), \quad \neg x = x \to \perp.$$

Let $n \geq 1$ be an integer. For any $x \in L$, define $x^n = x^{n-1} * x$ and $x^0 = e$.

In the case $e = \top$, $\mathbf{L}$ is a commutative residuated lattice.

The following theorem provides some known rules in IL-algebras.

THEOREM 2.5 ([10, 12, 20]). *Let $\mathbf{L}$ be an IL-algebra, I a non-empty set and x, y, x_1, y_1, y_i $(i \in I), z \in L$, then the following statements hold:*

(c_1) $x * (y \vee z) = (x * y) \vee (x * z)$, *if* $\bigvee_{i \in I} y_i$ *exists, then* $x * (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x * y_i)$,

(c_2) $\perp \to \perp = \top$,

(c_3) $\top * \top = \top$,

(c_4) *If* $x, y \leq e$, *then* $x * y \leq x \wedge y$,

(c_5) *If* $e \leq x, y$, *then* $x \vee y \leq x * y$,

(c_6) $(x \to y) * (y \to z) \leq (x \to z)$,

(c_7) $e \to x = x$,

(c_8) *If* $x \leq y, x_1 \leq y_1$, *then* $x * x_1 \leq y * y_1$ *and* $y \to x_1 \leq x \to y_1$,

(c_9) $x \to (y \to z) = (x * y) \to z$,

(c_{10}) $x * (x \to y) \leq y$,

(c$_{11}$) $e \leq x \to x$,
(c$_{12}$) $(z \to x) \wedge (z \to y) = z \to (x \wedge y)$,
(c$_{13}$) $(x \to z) \wedge (y \to z) = (x \vee y) \to z$.

Some known useful inequalities on IL-algebras are presented in the following proposition.

Proposition 2.6 ([19]). *Let **L** be an IL-algebra. For any positive integers, $n, m \geq 1$ and $g, h, k, h_1, \ldots, h_n \in L$, the following statements hold:*

(c$_{14}$) $(g \wedge e) \vee [(h \wedge e) * (k \wedge e)] \geq ((g \wedge e) \vee (h \wedge e)) * ((g \wedge e) \vee (k \wedge e))$,
(c$_{15}$) $(g\wedge e)\vee[(h_1\wedge e)*\ldots*(h_n\wedge e)] \geq [(g\wedge e)\vee(h_1\wedge e)]*\ldots*[(g\wedge e)\vee(h_n\wedge e)]$,
(c$_{16}$) $(g \wedge e) \vee ((h \wedge e)^n) \geq ((g \wedge e) \vee (h \wedge e))^n$,
(c$_{17}$) $(g \wedge e)^n \vee (h \wedge e)^m \geq ((g \wedge e) \vee (h \wedge e))^{mn}$.

Definition 2.7 ([12]). Let **L** be an IL-algebra. A subset F of L is called a *filter* if the following conditions hold:

(1) $e \in F$, for any $x, y \in F$, $x * y$ and $x \wedge y \in F$,
(2) if $x \leq y$ and $x \in F$, then $y \in F$.

Proposition 2.8. *Let **L** be an IL-algebra. A subset F of **L** is a filter if and only if*

(1) *$e \in F$,*
(2) *if $x, y \in F$, then $(x \wedge e) * (y \wedge e) \in F$,*
(3) *if $x \leq y$ and $x \in F$, then $y \in F$.*

Proof. Using $(x \wedge e) * (y \wedge e) \leq x * y$, $x \wedge y$ we obtain the desired equivalence. □

Definition 2.9 ([12]). Let **L** be an IL-algebra. A subset D of L, is called a *deductive system* if the following conditions are satisfied for all $x, y \in L$:

(1) $e \in D$,
(2) if $x, x \to y \in D$, then $y \in D$,
(3) if $e \leq x$, then $x \in D$.

It is easy to see that every filter of **L** is a deductive system. A filter F of **L** is called *proper* if $F \neq L$, in that case $\perp \notin F$; it is called *maximal* if F is proper and F is not contained in another proper filter of **L**. An IL-algebra is called a *local IL-algebra* when it contains a unique maximal filter. For each non-empty subset S of L, the filter generated by S denoted $\langle S \rangle$ is the intersection of all filters of **L** containing S. A filter F of **L** is finitely generated if $F = \langle S \rangle$, where

S is a finite subset of L. If $\{F_i\}_{i\in I}$ is a family of filters of $\mathbf{L}$, we define

$$\wedge_{i\in I}F_i = \cap_{i\in I}F_i \quad \text{and} \quad \vee_{i\in I} F_i = \langle(\cup_{i\in I}F_i)\rangle.$$

We denote by $F(\mathbf{L})$ the set of all filters of $\mathbf{L}$ and $\mathcal{F}(\mathbf{L})$ the algebra $(F(\mathbf{L});\wedge,\vee,\langle e\rangle, L)$ where $F_1\wedge F_2 = F_1\cap F_2$, $F_1\vee F_2 = \langle F_1\cup F_2\rangle$.

Example 2.10.
(1) For the IL-algebra $\mathbf{L}_7$, the set $F = \{d, a, b, e, \top\}$ is the only maximal filter.
(2) Let $\mathbf{L}$ be the IL-algebra of Example 2.3, for any positive integer $n \geq 2$, set $F_n = [\frac{1}{n} + \frac{1}{2}, 2]$. Then F_n is a filter of $\mathbf{L}$.

Lemma 2.11 ([19]). *Let F be a filter of $\boldsymbol{L}$ and $a, b\in L$. If $(a\wedge e)\vee(b\wedge e)\in F$, then $\langle F\cup\{a\}\rangle\cap\langle F\cup\{b\}\rangle = F$.*

Proposition 2.12 ([19]). *If $F\in F(\boldsymbol{L})$, then the following conditions are equivalent:*
(i) *If $F = F_1\cap F_2$ with $F_1, F_2\in F(\boldsymbol{L})$, then $F = F_1$ or $F = F_2$.*
(ii) *If $a, b\in L$ with $(a\wedge e)\vee(b\wedge e)\in F$, then $a\in F$ or $b\in F$.*
(iii) *If $F_1\cap F_2\subseteq F$ with $F_1, F_2\in F(\boldsymbol{L})$, then $F_1\subseteq F$ or $F_2\subseteq F$.*

Definition 2.13. Let $\mathbf{L}$ be an IL-algebra. A filter F of $\mathbf{L}$ is called a *prime filter* if $F\neq L$ and F satisfies one of the conditions of Proposition 2.12.

Example 2.14. Consider the IL-algebra $\mathbf{L}_7$. Then $F_1 = \{d, a, b, e, \top\}$, $F_2 = \{a, e, \top\}$ and $F_3 = \{b, e, \top\}$ are prime filters of $\mathbf{L}_7$.

It is easy to check that an arbitrary intersection of filters of $\mathbf{L}$ is a filter. According to [19], we have the following two propositions.

Proposition 2.15 ([19]). *Let $\boldsymbol{L}$ be an IL-algebra, $F, F_1, F_2\in F(\boldsymbol{L})$, $a, b\in L$ and $S\subseteq L, S\neq\emptyset$. Then the following statements hold:*
(1) $\langle a\wedge e\rangle = \langle a\rangle = \{x\in L : (a\wedge e)^n \leq x\wedge e$ *for some* $n\geq 1\}$,
(2) $\langle S\rangle = \{x\in L : (s_1\wedge e)*\ldots*(s_n\wedge e)\leq x\wedge e$, *for some* $s_1,\ldots,s_n\in S, n\geq 1\}$,
(3) $F_1\vee F_2 = \langle F_1\cup F_2\rangle = \{x\in L : \exists f_1\in F_1, f_2\in F_2, (f_1\wedge e)*(f_2\wedge e)\leq x\wedge e\}$,
(4) $\langle F\cup\{a\}\rangle = \{x\in L : \exists f\in F, n\geq 1, (f\wedge e)*(a\wedge e)^n\leq x\wedge e\}$,
(5) $\langle a\rangle\cap\langle b\rangle = \langle(a\wedge e)\vee(b\wedge e)\rangle$.

Recall that a *Heyting algebra* $(H,\wedge,\vee,0,1,\to)$ is a lattice with least element 0 and greatest element 1 such that, for all $a, b, x\in H$, $x\wedge a\leq b$ if and only if $x\leq a\to b$. Heyting algebra is necessarily distributive ([16]).

Proposition 2.16 ([16, Lemma 1.1]). *The following standard properties hold in Heyting algebras:*

(c_{18}) $b \leq c$ *implies* $a \to b \leq a \to c$,
(c_{19}) $a \to (b \to c) = (a \wedge b) \to c$,
(c_{20}) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Proposition 2.17 ([19]). *Let L be an IL-algebra. The algebra $(F(\boldsymbol{L}), \wedge, \vee, \to, \langle e \rangle)$ is a complete Heyting algebra, where for all $F_1, F_2 \in F(L)$,*

$$F_1 \wedge F_2 = F_1 \cap F_2, \quad F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle,$$

$$F_1 \to F_2 = \{x \in L : F_1 \cap \langle x \rangle \subseteq F_2\}.$$

To end this subsection, we state some properties of homomorphisms of IL-algebras and some relationships between homomorphisms of IL-algebras and filters.

Definition 2.18. Let $\mathbf{L}_1, \mathbf{L}_2$ be two IL-algebras. A function $h\colon L_1 \to L_2$ is said to be an *IL-homomorphism* (or simply, *homomorphism*) if $h(x\gamma y) = h(x)\gamma h(y)$, for all $x, y \in L_1$ and $\gamma \in \{*, \to, \wedge, \vee\}$ and $h(c_{L_1}) = c_{L_2}$, for any $c \in \{\perp, \top, e\}$.

If $\mathbf{L}_1 = \mathbf{L}_2$, then h is called an *endomorphism of IL-algebra.*

If h is a homorphism of IL-algebras, then the set

$$coker(h) = \{x \in L_1 \mid h(e_1) \leq h(x)\}$$

is called the *cokernel* of h (called in [12] kernel of h).

If $h\colon L_1 \to L_2$ is an IL- homomorphism, then one can easily show that

$$h(\neg x) = \neg h(x) \quad \text{and if} \quad x \leq y, \quad \text{then} \quad h(x) \leq h(y) \quad \text{for all } x, y \in L_1.$$

It was claimed in [11], and the proof is stated in [19], that for any filter F of $\mathbf{L}$, the binary relation $\Theta(F)$ defined by

$$\Theta(F) := \{(x, y) \in L^2 \mid x \to y, y \to x \in F\}$$

is a congruence relation on $\mathbf{L}$. For any $a \in L$, let $[a]_F$ be the congruence class of a. If we denote by $\frac{L}{F}$ the quotient set $\{[a]_F \mid a \in L\}$, then $\frac{\mathbf{L}}{F}$ becomes an IL-algebra with the natural operations induced by those of $\mathbf{L}$ and the correspondence $\pi_F\colon L \to \frac{L}{\Theta(F)}, x \mapsto [x]_{\Theta(F)}$ is a surjective IL-homomorphism ([12]). One can check that $coker(\pi_F) \subseteq F$ and $F = \langle [e]_{\Theta(F)} \rangle$ (see [19]).

PROPOSITION 2.19. *Let $h\colon L_1 \to L_2$ be a homomorphism of IL-algebras, $F \in F(\boldsymbol{L}_2)$, $G \in F(\boldsymbol{L}_1)$. Set $h^{-1}(F) = \{x \in L_1 \mid h(x) \in F\}$ and $h(G) = \{h(x) \mid x \in G\}$. Then:*
(1) *$h^{-1}(F) \in F(\boldsymbol{L}_1)$ and $coker(h) \subseteq h^{-1}(F)$,*
(2) *$h(G)$ is a filter of $\boldsymbol{L}_2$ if h is surjective.*

PROOF. The proof is trivial. □

EXAMPLE 2.20. We consider the IL-algebra $\mathbf{L}_7$ and define the map $h\colon L_7 \to L_7$, $h(a) = b$, $h(b) = a$ and for all $x \in L_7 \setminus \{a, b\}$, $h(x) = x$. It is easy to check that h is an endomorphism of $\mathbf{L}_7$. It is clear that h is injective and surjective, hence bijective. $F(\mathbf{L}_7) = \{F_1 = \{a, e, \top\}, F_2 = \{b, e, \top\}, F_3 = \{a, b, d, \top\}, F_4 = \{e, \top\}, F_5 = L\}$. We have $h(F_1) = F_2$, $h(F_3) = F_3$ and $coker(h) = \{e, \top\}$.

2.2. Radical of filters in IL-algebras

Radicals have arisen from structural investigations in rings, but later they infiltrated into various branches of algebras as well as in topology and in relational structures, in particular on MV-algebras and residuated lattices, but in the context of ideals. Radicals provide structure theorems for ordered algebraic structures which are semi-simple and provide also a context for studying and comparing classes of algebras via closure operations. We introduce and state some useful properties of radical in IL-algebras in the context of filters, since there is a bijective correspondence between filters and congruences on IL-algebras [11, 19]. We denote by $max(\mathbf{L})$ the set of all maximal filters of $\mathbf{L}$.

DEFINITION 2.21. Let $F \in F(\mathbf{L})$, the *radical of F* denoted $Rad(F)$ is defined as follows: If $F = L$, then $Rad(F) = L$, if $F \neq L$, then $Rad(F)$ is the intersection of all maximal filters of $\mathbf{L}$ contained it, that is

$$Rad(F) := \cap\{G : G \in max(\mathbf{L}) \text{ and } F \subseteq G\}.$$

Clearly $Rad(F)$ is a filter of $\mathbf{L}$ for any $F \in F(\mathbf{L})$.

PROPOSITION 2.22. *Let $\boldsymbol{L}$ be an IL-algebra and $F, G \in F(\boldsymbol{L})$. Then the following statements hold:*
(1) *$F \subseteq Rad(F)$,*
(2) *if $F \subseteq G$, then $Rad(F) \subseteq Rad(G)$,*
(3) *$Rad(Rad(F)) = Rad(F)$,*
(4) *$Rad(F) \cap Rad(G) = Rad(F \cap G)$,*
(5) *if $F \in max(\boldsymbol{L})$, then $Rad(F) = F$.*

PROOF. (1) and (5) are obvious.

(2) Let $F, G \in F(\mathbf{L})$ such that $F \subseteq G$. Let us show that $Rad(F) \subseteq Rad(G)$. If $F = L$ or $G = L$ we are done; since $Rad(L) = L$. Assume that F and G are proper filters. Let M be maximal filter of $\mathbf{L}$ such that $G \subseteq M$, then $F \subseteq G \subseteq M$ and $Rad(F) \subseteq M$. Therefore $Rad(F) \subseteq \cap\{M \in \max(\mathbf{L}) \mid G \subseteq M\} = Rad(G)$.

For (3), if $F = L$ we are done. Assume that $F \neq L$, let us show that $Rad(Rad(F)) = Rad(F)$. Set

$$J = \{G \in max(\mathbf{L}) \mid Rad(F) \subseteq G\} \quad \text{and} \quad K = \{G \in max(\mathbf{L}) \mid F \subseteq G\}.$$

Let $G \in K$, then by definition $Rad(F) \subseteq G$, so $G \in J$, hence $K \subseteq J$. Let $G \in J$, then $G \in max(\mathbf{L})$ and $Rad(F) \subseteq G$, since $F \subseteq Rad(F)$ we have $F \subseteq G$ and $G \in K$, so $J \subseteq K$. Therefore $J = K$. By definition of $Rad(Rad(F))$ and $J = K$ we have $Rad(Rad(F)) = Rad(F)$.

(4) Let $F, G \in F(\mathbf{L})$. Since $F \cap G \subseteq F, G$, we have $Rad(F \cap G) \subseteq Rad(F) \cap Rad(G)$ by using (2). Recall that each maximal filter in $\mathbf{L}$ is a prime filter ([19, Corollary 3.21]). Let M be a maximal filter containing $F \cap G$, then by the primness of M (Proposition 2.12), we have $F \subseteq M$ or $G \subseteq M$. If $F \subseteq M$, then $F \subseteq Rad(F) \subseteq M$ and $Rad(F) \cap Rad(G) \subseteq M$. If $G \subseteq M$, then a similar argument shows that $Rad(F) \cap Rad(G) \subseteq M$; since M is arbitrary chosen, we get $Rad(F) \cap Rad(G) \subseteq M$ for all maximal filter containing $F \cap G$, that is $Rad(F) \cap Rad(G) \subseteq Rad(F \cap G)$. Hence the desired equality holds. $\square$

2.3. Extended filters in IL-algebras

Extended filters were investigated on residuated lattices in [15, 17]. In [17], it is proved that, given a commutative residuated lattice $\mathbf{L}$, the set of extended filters on $\mathbf{L}$ forms a Heyting algebra. In this subsection, we introduce and investigate some properties of extended filters on IL-algebras. Later, using closure operations, we will show that one can derive some properties of the collection of extended filters. Given a fixed non-empty subset B of L, for any filter F of $\mathbf{L}$, set

$$E_F(B) = \{x \in L \mid (x \wedge e) \vee (y \wedge e) \in F, \text{ for all } y \in B\}.$$

The set $E_F(B)$ is called *extended filter of* F relative to B.

LEMMA 2.23. *For two non-empty subsets B and C of an IL-algebra $\mathbf{L}$ and any filter F of $\mathbf{L}$, if $B \subseteq C$, then $E_F(C) \subseteq E_F(B)$.*

PROOF. Clear. $\square$

THEOREM 2.24. *For any non-empty subset B of L and $F \in F(\boldsymbol{L})$, $E_F(B) = E_F(\langle B\rangle)$ where $\langle B\rangle$ is the filter generated by B.*

PROOF. By Lemma 2.23 we have $E_F(\langle B\rangle) \subseteq E_F(B)$. It remains to show that $E_F(B) \subseteq E_F(\langle B\rangle)$. Let $x \in E_F(B)$, we will show that $x \in E_F(\langle B\rangle)$. Let $y \in \langle B\rangle$, then by (2) of Proposition 2.15 there are $y_1, \ldots, y_n \in B$ such that $(y\wedge e) \geq (y_1\wedge e)*\ldots*(y_n\wedge e)$; since $x \in E_F(B)$, we have $(x\wedge e)\vee(y_i\wedge e) \in F$, for $1 \leq i \leq n$. Using (c_{15}) we have $(x\wedge e)\vee(y\wedge e) \geq (x\wedge e)\vee[(y_1\wedge e)*\ldots*(y_n\wedge e)] \geq [(x\wedge e)\vee(y_1\wedge e)]*\ldots*[(x\wedge e)\vee(y_n\wedge e)] \in F$, we deduce that $(x\wedge e)\vee(y\wedge e) \in F$; hence $x \in E_F(\langle B\rangle)$. □

PROPOSITION 2.25. *Let $\boldsymbol{L}$ be an IL-algebra, $F, G \in F(\boldsymbol{L})$, B a non-empty set and $\rightarrow$ defined on $F(\boldsymbol{L})$ as above, then in the Heyting algebra $(F(\boldsymbol{L}), \wedge, \vee, \rightarrow, \langle e\rangle, L)$ the following statements hold:*
(1) $E_F(B) = \langle B\rangle \rightarrow F \in F(\boldsymbol{L})$,
(2) $F \subseteq E_F(B)$,
(3) *if F is a prime filter, then $E_F(B)$ is a prime filter,*
(4) *if $F \subseteq G$, then $E_F(B) \subseteq E_G(B)$,*
(5) $E_F(B) \cap E_G(B) = E_{F\cap G}(B)$,
(6) $E_{E_F(B)}(B) = E_F(B)$.

PROOF. Recall that $E_F(B) = E_F(\langle B\rangle)$ and $\langle B\rangle \rightarrow F = \{x \in L \mid \langle x\rangle \cap \langle B\rangle \subseteq F\}$.

(1) Let $x \in E_F(B)$, we will show that $x \in \langle B\rangle \rightarrow F$, i.e $\langle x\rangle \cap \langle B\rangle \subseteq F$. Let $t \in \langle x\rangle \cap \langle B\rangle$, then there are $n, m \geq 1$ and $b_1, \ldots, b_m \in B$ such that, using (c_{15}) and (c_{16}), one obtains

$$\begin{aligned}(t\wedge e) &\geq (x\wedge e)^n \vee [(b_1\wedge e) * \ldots * (b_m\wedge e)]\\ &\geq [(x\wedge e)^n \vee (b_1\wedge e)] * \ldots * [(x\wedge e)^n \vee (b_m\wedge e)]\\ &\geq [(x\wedge e) \vee (b_1\wedge e)]^n * \ldots * [(x\wedge e) \vee (b_m\wedge e)]^n.\end{aligned}$$

Since $(x\wedge e)\vee(b_i\wedge e) \in F$, $1 \leq i \leq m$, we deduce that $t \in F$. So $E_F(B) \subseteq \langle B\rangle \rightarrow F$. Let $x \in \langle B\rangle \rightarrow F$. We will show that $x \in E_F(B)$. Let $y \in B$, we will show that $(x\wedge e)\vee(y\wedge e) \in F$. Since $\langle x\rangle \cap \langle B\rangle \subseteq F$, we get $(x\wedge e)\vee(y\wedge e) \in \langle x\rangle \cap \langle B\rangle$, so $x \in E_F(B)$. Hence $\langle B\rangle \rightarrow F \subseteq E_F(B)$. Therefore $E_F(B) = \langle B\rangle \rightarrow F$. Since $E_F(B) = \langle B\rangle \rightarrow F$, we deduce by Lemma 3.26 in [19] that $E_F(B) \in F(\mathbf{L})$.

(2) It is easy to see that $F \subseteq E_F(B)$.

(3) Assume that F is a prime filter. Assume that $(x \wedge e) \vee (y \wedge e) \in E_F(B)$. We show that x or y belongs to $E_F(B)$. We have

$$
\begin{aligned}
&\langle B\rangle \cap \langle (x \wedge e) \vee (y \wedge e)\rangle \\
&\qquad = \langle B\rangle \cap \langle x \wedge e\rangle \cap \langle y \wedge e\rangle \quad \text{(by (5) and (1) of Proposition 2.15)} \\
&\qquad = [\langle B\rangle \cap (\langle (x \wedge e)] \cap [\langle B\rangle \cap \langle (y \wedge e)\rangle] \subseteq F.
\end{aligned}
$$

Since F is a prime filter, by (iii) of Proposition 2.12 $\langle B\rangle \cap \langle x \wedge e\rangle \subseteq F$ or $\langle B\rangle \cap \langle y \wedge e\rangle \subseteq F$, and hence $x \in \langle B\rangle \to F$ or $y \in \langle B\rangle \to F$. Therefore $E_F(B)$ is a prime filter.

(4) By using (c_{18}), we are done.

(5) $(E_F(B) \cap E_G(B)) = ((\langle B\rangle \to F) \cap (\langle B\rangle \to G) \overset{(c_{20})}{=} \langle B\rangle \to (F \cap G) \overset{(\mathrm{def})}{=} E_{F\cap G}(B)$.

(6) $E_{E_F(B)}(B) = \langle B\rangle \to (\langle B\rangle \to F) \overset{(c_{19})}{=} (\langle B\rangle \cap \langle B\rangle) \to F = E_F(B)$. □

Corollary 2.26. *$E_F(B) = \bigcap_{x\in B} (F : x)$, where*

$$(F : x) = \{y \in L \mid (x \wedge e) \vee (y \wedge e) \in F\}.$$

3. Closure operations on the lattice of filters of IL-algebras

Closure operations have intensively been used to characterize rings, for instance, many classifications of rings have been obtained using v-closure, star closure and semi-star closure operations on the lattice of ideals of rings ([4, 5]). Some of the concerned rings among others are for example, completely integrally closed rings, Noetherian rings, Dedekind rings, and Krull rings. Recently, a similar work on MV-algebras has been carried out by O.A. Heubo-Kwegna and J.B. Nganou [8] who characterized hyperarchimedean MV-algebras. Since MV-algebras, BL-algebras, commutative residuated lattices form subclasses of IL-algebras and these classes of algebras have commutative ring counterpart notions, it motivates us to consider closure operations on IL-algebras as a map on the set of the lattice of filters of IL-algebras.

3.1. Definition, examples and construction of closure operations

Definition 3.1. Let $\mathbf{L}$ be an IL-algebra and $F(\mathbf{L})$ be the set of all filters of $\mathbf{L}$. A *closure operation* on $F(\mathbf{L})$ is a map $c\colon F(\mathbf{L}) \to F(\mathbf{L})$, $F \mapsto F^c$ satisfying the following conditions where F^c is used in place of $c(F)$:
(CO-1) $F \subseteq F^c$ (extension),
(CO-2) $F^c = (F^c)^c$ (idempotence),
(CO-3) If $F \subseteq G$, then $F^c \subseteq G^c$(order-preservation).

A filter is called *c-closed* if $F^c = F$. It is immediate that F^c is c-closed for any filter of $\mathbf{L}$. The set of c-closed filters is denoted $F(\mathbf{L})^c$.

A typical reason that a closure operation is studied is often that the closedness of certain classes of filters (ideals) is related (and often equivalent) to the ordered algebraic structures having desired properties.

Definition 3.2.
(1) A closure operation c is called *proper closure operation* on $F(\mathbf{L})$ if $c(F) \neq L$ for all proper filter F.
(2) Two filters F and G of $\mathbf{L}$ are called *co-maximal* if $F \vee G = L$.

We proceed to give examples of closure operations.

Example 3.3.
(1) The identity map as well as the constant map that sends every filter to L are closure operations.
(2) From (1), (2) and (3) of Proposition 2.22, for every IL-algebra $\mathbf{L}$, Rad is a nontrivial example of closure operation on $\mathbf{L}$.
(3) By using (1), (3), (5) and (7) of Proposition 2.25, for any non-empty subset B of L, the map $c_B\colon F \mapsto E_F(B)$ is a closure operation.
(4) Let $\mathcal{F}$ be a subset of $F(\mathbf{L})$ that is closed under (arbitrary) intersection. For each filter F of $\mathbf{L}$, let

$$F^c = \bigcap\{G \mid G \in \mathcal{F}, F \subset G\}.$$

A similar argument used in (6) of [8, Example 3.2] shows that c is a closure operation on $\mathbf{L}$. It is clear that $\mathcal{F}$ is the set of all c-closed filters of $\mathbf{L}$.
(5) For any filter G of $\mathbf{L}$, the map $c_G\colon F(\mathbf{L}) \to F(\mathbf{L}), F \mapsto F \vee G$, where $F \vee G = \langle F \cup G\rangle$ is a closure operation (easy to check).
(6) For any filter F, set $F^* = \{x \in L \mid \text{for any } y \in F, (x \wedge e) \vee (y \wedge e) = e\}$. The correspondence $F \mapsto F^{**}$ is a closure operation on $\mathbf{L}$.

By [19, Proposition 3.31], $F(\mathbf{L})$ is a pseudocomplemented distributive lattice, and using Theorem I of [21] we get that the concerned correspondence is a closure operation on $F(\mathbf{L})$.

We state some properties that follow from the axiomatic definition of closure operations.

PROPOSITION 3.4. *Let $\boldsymbol{L}$ be an IL-algebra, c be a closure operation, $\{F_\alpha \mid \alpha \in \Lambda\}$ be a nonempty set of filters of $\boldsymbol{L}$ and F be a filter of $\boldsymbol{L}$.*

(i) *F^c is the intersection of all c-closed filters that contain F, that is*

$$F^c = \bigcap\{G \in F(\boldsymbol{L})^c \mid F \subseteq G\}. \tag{3.1}$$

(ii) *If every F_α is c-closed, so is $\cap_\alpha F_\alpha$.*
(iii) *$\cap_\alpha F_\alpha^c$ is c-closed.*
(iv) *$(\bigvee_{\alpha\in\Lambda} F_\alpha^c)^c = (\bigvee_{\alpha\in\Lambda} F_\alpha)^c$, where $\bigvee_{\alpha\in\Lambda} F_\alpha$ is the filter generated by $\bigcup_{\alpha\in\Lambda} F_\alpha$.*

PROOF. Since $F(\mathbf{L})$ is a lattice, it follows from Proposition 7.2 of [3]. $\square$

COROLLARY 3.5 ([3]). *For every IL-algebra $\boldsymbol{L}$, there is a one-to-one correspondence between the closure operations on $F(\boldsymbol{L})$ and the subsets of $F(\boldsymbol{L})$ that are closed under arbitrary intersections.*

COROLLARY 3.6.
(1) *If F and G are two co-maximal filters of $\boldsymbol{L}$ and c is a proper closure operation on $\boldsymbol{L}$, then F^c and G^c are comaximal.*
(2) *If c is a proper closure operation and F is a maximal filter of $\boldsymbol{L}$, then $c(F) = F$.*

PROOF. (1) Clear.
(2) Let c be a proper closure operation and F a maximal filter, then by extensivity $F \subseteq F^c$ and since F is maximal, we have $F^c = F$ or $F^c = L$. Since c is proper we get $c(F) = F$. $\square$

COROLLARY 3.7. *Let $\boldsymbol{L}$ be an IL-algebra, $\emptyset \neq B \subseteq L$ and $\{F_\alpha \mid \alpha \in \Lambda\}$ be a family of filters on $\boldsymbol{L}$. Let $Rad\colon F \mapsto Rad(F)$ and $c_B\colon F \mapsto E_F(B)$ as above. Then:*

(1) (i) $Rad(\bigcap_{\alpha\in\Lambda} Rad(F_\alpha)) = \bigcap_{\alpha\in\Lambda} Rad(F_\alpha)$,
(ii) $Rad(\bigvee_\alpha Rad(F_\alpha)) = Rad(\bigvee F_\alpha)$.

(2) (i) $E_{\bigcap_{\alpha\in\Lambda} E_{F_\alpha}(B)}(B) = E_{\bigcap_{\alpha\in\Lambda} F_\alpha}(B)$,
(ii) $E_{\bigvee_{\alpha\in\Lambda}(E_{F_\alpha(B)})}(B) = E_{\bigvee_{\alpha\in\Lambda} F_\alpha}(B)$.

DEFINITION 3.8 ([5, p.8]). A closure operation c on $F(\mathbf{L})$ is said to be of *finite type* if $F^c := \bigcup\{J^c \mid J$ is a finitely generated filter such that $J \subseteq F\}$.

We end this subsection by giving further characterizations of closure operations.

LEMMA 3.9. *Given a collection $\{c_\lambda \mid \lambda \in \Lambda\}$ of closure operations on $F(\boldsymbol{L})$, if we denote by c its infirmum, we have*

$$F^c = \bigcap_{\lambda\in\Lambda} F^{c_\lambda}.$$

PROOF. Since each F^{c_λ} is a filter, F^c is also a filter as an arbitrary intersection of filters. The extensive and order preserving properties follow from that of $c_\lambda, \lambda \in \Lambda$. The idempotent property follows because rewriting the intersection using (3.1), we have

$$\bigcap_{\lambda\in\Lambda} F^{c_\lambda} = \bigcap_{\lambda\in\Lambda} \bigcap_{\substack{G\in F(L)^{c_\lambda}\\ F\subseteq G}} G = \bigcap_{F\subseteq G} \bigcap_{\substack{G\in F(L)^{c_\lambda}\\ \lambda\in\Lambda}} G = \bigcap_{\substack{F\subseteq G\\ G\in C}} G$$

where $C = \bigcup_{\lambda\in\Lambda} F(\mathbf{L})^{c_\lambda}$, that is the closure operation generated by the set $C = \bigcup_{\lambda\in\Lambda} F(\mathbf{L})^{c_\lambda}$, namely c. □

Let $B \subseteq L$, $B \neq \emptyset$, and $x \in B$, set

$$C_x(F) = \{y \in L \mid (y \wedge e) \vee (x \wedge e) \in F\},$$

then one can see that $E_F(B) = \underset{x\in B}{\cap}(C_x(F))$, hence $C_B = \inf\{C_x : x \in B\}$.

One of very productive ways to construct closure operations on an IL-algebra $\mathbf{L}$ is by using homomorphisms from $\mathbf{L}$ to other IL-algebras.

PROPOSITION 3.10 (Transport of closure operations). *Let $\phi\colon \boldsymbol{L}_1 \to \boldsymbol{L}_2$ be an IL-homomorphism and d a closure operation on $F(\boldsymbol{L}_2)$. The map*

$$c\colon F \mapsto F^c := \phi^{-1}(\langle\phi(F)\rangle^d)$$

is a closure operation on $\boldsymbol{L}_1$ where $\langle\phi(F)\rangle$ is the filter of $\boldsymbol{L}_2$ generated by $\phi(F)$. Moreover, if d is of finite type, then so is c.

PROOF. Let $F \in F(\mathbf{L}_1)$, since $\langle\phi(F)\rangle^d$ is a filter of $\mathbf{L}_2$ and ϕ a homomorphism of IL-algebras, F^c is a filter of $\mathbf{L}_1$(using Proposition 2.19). Extension follows from $F \subseteq \phi^{-1}(\phi(F)) \subseteq \phi^{-1}(\langle\phi(F)\rangle^d)$ and order preservation follows from that of d. For idempotence, clearly $g \in (F^c)^c$ implies $\phi(g) \in \langle\phi(F^c)\rangle^d$, but $\phi(F^c) = \phi(\phi^{-1}(\langle\phi(F)\rangle^d)) \subseteq \langle\phi(F)\rangle^d$; hence $\phi(F^c) \subseteq \langle\phi(F)\rangle^d$. Then $\langle\phi(F^c)\rangle^d \subseteq \langle\langle\phi(F)\rangle^d\rangle^d = \langle\phi(F)\rangle^d$. So $\phi(g) \in \langle\phi(F^c)\rangle^d \subseteq \langle\phi(F)\rangle^d$. Thus

$g \in \phi^{-1}(\langle\phi(F)\rangle^d)$, i.e $g \in F^c$. Thus $F^{cc} \subseteq F^c$ and $F^{cc} = F^c$. Hence c is a closure operation on $F(\mathbf{L}_1)$.

Suppose that d is of finite type. We will show that c is of finite type, i.e.

$$F^c = \bigcup\{G^c \mid G \subseteq F, G \text{ is finitely generated}\}.$$

For every $y \in \langle\phi(F)\rangle$, there are $y_1, \ldots, y_n \in \phi(F)$ such that $(y \wedge e) \geq (y_1 \wedge e) * \ldots * (y_n \wedge e)$ and there are $x_i \in F, y_i = \phi(x_i)$, $i = 1, \ldots, n$. So there exists a finitely generated filter G of $\mathbf{L}_1$ contained in F namely $G = \langle\{x_1, \ldots, x_n\}\rangle$ such that $y \in \langle\phi(G)\rangle$. Let $z \in F^c$, then $\phi(z) \in \langle\phi(F)\rangle^d$, so there is a finitely generated filter $H \subseteq \langle\phi(F)\rangle$ such that $\phi(z) \in H^d$. Let $H = \langle\{h_1, \ldots, h_m\}\rangle$. Each $h_i = s_i$ for some $s_i \in F$ and each s_i is contained in a finitely generated filter $F_i \subseteq F$; hence $\phi(z) \in H^d \subseteq \langle\phi(F_1) \cup \ldots \cup \phi(F_m)\rangle^d$, so that $z \in (\phi(F_1) \vee \phi(F_2) \vee \ldots \vee \phi(F_m))^c$. Since $F_1 \vee \ldots \vee F_m$ is finitely generated, c is of finite type. □

Proposition 3.11.

(1) *If $L_2 = L_1/F$, where F is a filter of L_1, the closure operation induced by the quotient map $\pi\colon L \to L/F$ and by the closure operation d on $\boldsymbol{L}/F$ is the closure operation c defined on $F(\boldsymbol{L})$ whose c-filters are the filters containing F that project to the d-filters of L/F.*

(2) *If we have a whole family of IL-algebras $\boldsymbol{S}_\alpha$, closure operations d_α and homomorphisms $\phi_\alpha\colon L \to S_\alpha$, we can take the infirmum of the corresponding closure c_α, obtaining*

$$F^c := \bigcap_{\alpha\in\Lambda} \phi_\alpha^{-1}(\langle\phi_\alpha(F)\rangle^{d_\alpha})$$

which is a closure operation.

Proof. (1) It follows from Proposition 3.10.
(2) Follows from Lemma 3.9 and Proposition 3.10. □

3.2. Relations between some subclasses of $\mathcal{F}(\mathbf{L})$ and stable closure operations

We consider now the closure operations that preserve finite meets in the lattice $F(\mathbf{L})$.

Definition 3.12 ([8]). We call a closure operation c on $\mathbf{L}$ *stable* if for all filters F, G of $\mathbf{L}$, $(F \cap G)^c = F^c \cap G^c$.

One should observe that as $(F \cap G)^c \subseteq F^c \cap G^c$ for any closure operation c, showing that c is stable only requires the inclusion $F^c \cap G^c \subseteq (F \cap G)^c$ to hold.

Example 3.13.

(1) Obviously, the identity map as well as the constant map that sends every filter to L are stable closure operations.
(2) By (4) of Proposition 2.22, the closure operation *Rad* is stable.
(3) By (5) of Proposition 2.25 for any $\emptyset \neq B \subseteq L$, the closure c_B: $c_B(F) = E_F(B)$ is a stable closure operation.
(4) Since $F(\mathbf{L})$ is a distributive lattice, one has that $C^G\colon F \mapsto F \vee G$ is a stable closure operation.
(5) Since $\mathbf{L}_7$ is finite all its filters are principal and we have $F(\mathbf{L}_7) = \{\langle e\rangle, \langle a\rangle, \langle b\rangle, \langle d\rangle, L\}$. Let $c_i\colon F(\mathbf{L}_7) \to F(\mathbf{L}_7)$ be defined as follows:
   (5.1) $c_1(L) = L, c_1(\langle e\rangle) = \langle e\rangle$ and $c_1(\langle x\rangle) = L$, $x \in \{a, b, d\}$. Then c_1 is a closure operation that is not stable, since $c_1(\langle a\rangle) \cap c_1(\langle b\rangle) = L \neq c_1(\langle a\rangle \cap \langle b\rangle) = \langle e\rangle$.
   (5.2) $c_2(L) = L, c_2(\langle x\rangle) = \langle d\rangle$, $x \in \{a, b, d\}$ and $c_2(\langle e\rangle) = \langle e\rangle$. Then c_2 is a closure operation that is not stable, since $c_2(\langle a\rangle) \cap c_2(\langle b\rangle) = \langle d\rangle \neq c_2(\langle a\rangle \cap \langle b\rangle) = \langle e\rangle$.
(6) A two element Boolean algebra $\mathbf{L}_2$ is a bounded commutative residuated lattice with $x * y = x \wedge y, x \to y = \neg x \vee y$ for any $x, y \in L_2 = \{0, 1\}$, and so an IL-algebra in which $e = 1$. Let $\underline{L}_3$ be the subalgebra of $\mathbf{L}_7$, with $L_3 = \{\perp, e, \top\}$. Following Cayley tabular of $\mathbf{L}_7$, one can extract that of $\mathbf{L}_3$. Then $\mathbf{L}_6 = \mathbf{L}_2 \times \mathbf{L}_3$ is an IL-algebra as product of IL-algebras $\mathbf{L}_2$ and $\mathbf{L}_3$. Setting $\overline{\top} = (1, \top), \overline{e} = (1, e), \overline{d} = (0, \top), \overline{b} = (0, e), \overline{a} = (1, 0)$ and $\overline{\perp} = (0, \perp)$. We have $L_6 = \{\underline{\perp}, \overline{a}, \overline{b}, \overline{e}, \overline{d}, \overline{\top}\}$ and $F(\mathbf{L}_6) = \{\langle\overline{e}\rangle, \langle\overline{a}\rangle, \langle\overline{b}\rangle, \langle\underline{\perp}\rangle = L_6\}$, where $\mathbf{L}_6$ is given by Figure 2 below.

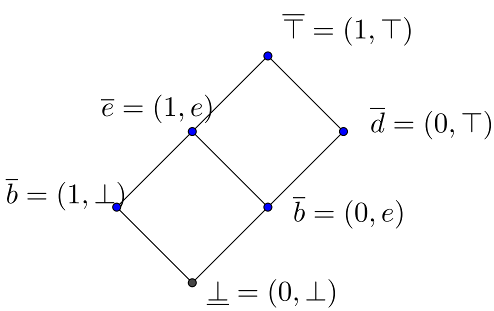

Figure 2. L_6

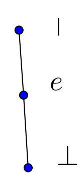

Figure 3. L_3

We define the operation c on $F(\mathbf{L}_6)$ by $c(\langle\overline{e}\rangle) = \langle\overline{e}\rangle$ and $c(\langle\overline{a}\rangle) = c(\langle\overline{b}\rangle) = L$. Then c is a closure operation on $F(\mathbf{L})$ that is not stable. Indeed: $\langle\overline{e}\rangle = c(\langle\overline{a}\rangle \cap \langle\overline{b}\rangle) \neq c(\langle\overline{a}\rangle) \cap c(\langle\overline{b}\rangle) = L$.

We can use stable closure operations c to derive properties of c-closed elements of $F(\mathbf{L})$.

THEOREM 3.14. *Let $c\colon F(\boldsymbol{L}) \to F(\boldsymbol{L})$ be a stable closure operation, and $c(\boldsymbol{L}) = \{F \in F(\boldsymbol{L}) \mid F^c = F\}$. For any $F, G \in F(\boldsymbol{L})$, set $F \vee^c G = (F \vee G)^c$ and $F^c \cap G^c = (F \cap G)^c$. Then $(c(\boldsymbol{L}), \cap, \vee^c, \langle e\rangle^c, L)$ is a complete Heyting algebra.*

PROOF. Let $F, G \in c(\mathbf{L})$, then $c(F) = F$ and $c(G) = G$ and $F \to G \subseteq c(F \to G)$. Since $F \cap (F \to G) \subseteq G$, using order preservation and stable property, we have $c(F) \cap c(F \to G) \subseteq c(G)$, so residuation property yields $c(F \to G) \subseteq c(F) \to c(G)$. Since $c(F) = F$ and $c(G) = G$ we have $c(F \to G) \subseteq F \to G \subseteq c(F \to G)$ by extensivity of c; therefore $c(F \to G) = F \to G$.

Since c is a stable closure operation, one can show that $c(\mathbf{L})$ is stable with $\cap$ and $\vee$. Since $(F(\mathbf{L}), \vee, \cap, \to \langle e\rangle, L)$ is a complete Heyting algebra, it is easy to see that $c(\mathbf{L})$ is complete by using (ii) of Proposition 3.4. □

COROLLARY 3.15. *Let $\boldsymbol{L}$ be an IL-algebra and $B \subseteq L$, $B \neq \emptyset$.*

(i) *The class of extended filters relative to B $(c_B(F(\boldsymbol{L})), \vee^{c_B}, \cap, \to, \langle B\rangle \to \langle e\rangle, L)$ is a complete Heyting algebra with least (resp. greatest) element $\langle B\rangle \to \langle e\rangle$ (resp. L).*

(ii) *The class of radical filters of $\boldsymbol{L}$, $(Rad(F(\boldsymbol{L})); \vee^{Rad}, \cap, Rad(\langle e\rangle), L)$ is a complete Heyting algebra with least (resp. greatest) element $Rad(\langle e\rangle)$ (resp. L).*

PROOF. (i) Let $B \subseteq L, B \neq \emptyset$. Let us show that $\langle B\rangle \to \langle e\rangle$ is the least element of $c_B(F(\mathbf{L}))$. We have $\langle B\rangle \to \langle e\rangle = c_B(\langle e\rangle) \in c_B(F(\mathbf{L}))$. Let $\langle B\rangle \to F \in c_B(F(\mathbf{L}))$, with $F \in F(\mathbf{L})$. Since $\langle e\rangle \subseteq F$, from (c_{18}) $\langle B\rangle \to \langle e\rangle \subseteq \langle B\rangle \to F$. Therefore, $\langle B\rangle \to \langle e\rangle$ is the least element of $c_B(F(\mathbf{L}))$. Clearly L is the greatest element of $c_B(F(\mathbf{L}))$. Since c_B is a stable closure operation on $F(\mathbf{L})$, from Theorem 3.14, we are done.

(ii) Clearly $Rad(\langle e\rangle)$ (resp. L) is the least (resp. the greatest) element of $Rad(F(\mathbf{L}))$ and the latter is a complete Heyting algebra by Theorem 3.14. □

Given an IL-algebra $\mathbf{L}$, we recall that $\mathcal{C}(F(\mathbf{L}))$ is the set of closure operations on $F(\mathbf{L})$. We define on $\mathcal{C}(F(\mathbf{L}))$ the binary relation $\preceq$ by $c \preceq c'$ if $F^c \subseteq F^{c'}$ for each filter F of $\mathbf{L}$.

PROPOSITION 3.16. *Let $\boldsymbol{L}$ be an IL-algebra such that $F(\boldsymbol{L})$ is linearly ordered by inclusion. Then every closure operation on $\boldsymbol{L}$ is stable.*

PROOF. Let c be a closure operation on $\mathbf{L}$ and $F, G \in F(\mathbf{L})$. Then $F \subseteq G$ or $G \subseteq F$. Assume $F \subseteq G$, then $F^c \subseteq G^c$ and $(F \cap G)^c = F^c = F^c \cap G^c$. □

The Proposition 3.16 is verified for local IL-algebras.

Semi-prime closure operation on the lattice of ideal in a ring R is a closure operation such that $c(I).c(J) \subseteq c(I.J)$ for any two ideals I and J. We observe that, since $F(\mathbf{L})$ is a Heyting algebra, semi-prime and stable closure operations on $F(\mathbf{L})$ coincide.

REMARK 3.17. Let $\mathcal{PC}(\mathbf{L})$ denote the set of all proper closure operations on $F(\mathbf{L})$. Note that for every $c \in \mathcal{PC}(\mathbf{L})$, every filter F and maximal filter M containing F, we have $F^c \subseteq M$. Hence $F^c \subseteq Rad(F)$ and $c \preceq Rad$. As observed above, both Id and Rad belong to $\mathcal{PC}(\mathbf{L})$ and are its minimum and maximum element respectively. Indeed, it is clear that the meet and join of proper closure operations are proper. Therefore, $(\mathcal{PC}(\mathbf{L}), \preceq)$ is a bounded sublattice of $(C(F(\mathbf{L})), \preceq)$.

In the following, we prove that every closure operation induces a quotient that preserves stability.

PROPOSITION 3.18. *Let $\boldsymbol{L}$ be an IL-algebra and F be a filter of $\boldsymbol{L}$. Then every closure operation c on $F(\boldsymbol{L})$ induces a closure operation $\overline{c}$ on $F(\boldsymbol{L}/F)$. Moreover, if c is stable, then $\overline{c}$ is stable.*

PROOF. For any filter G of $\mathbf{L}$ containing F, define $(G/F)^{\overline{c}} = \frac{G^c}{F}$. That $\overline{c}$ satisfies (CO1, CO2,CO3) is immediate from the fact that c satisfies the same conditions. In addition, if c is stable, then for any filters G, K containing F, $(G/F \cap K/F)^{\overline{c}} = ((G \cap K)/F)^{\overline{c}} = (G \cap K)^c/F = (G^c \cap K^c)/F = (G^c/F) \cap (K^c/F) = (G/F)^{\overline{c}} \cap (K/F)^{\overline{c}}$. □

COROLLARY 3.19. *For any filter G containing F, the following statement holds for $\gamma \in \{Rad, c_B \mid B \subseteq L, B \neq \emptyset\}$:*

$$\gamma\left(\frac{G_1}{F} \cap \frac{G_2}{F}\right) = \gamma\left(\frac{G_1 \cap G_2}{F}\right) = \frac{\gamma(G_1)}{F} \cap \frac{\gamma(G_2)}{F}.$$

To end this section, we provide some characterizations of maximal filters of IL-algebras.

THEOREM 3.20. *If F is a proper filter of $\boldsymbol{L}$, then the following statements are equivalent:*

(1) *$F \in max(\boldsymbol{L})$.*

(2) *For any $x \notin F$, there are $f \in F$, $n \in \mathbb{N}, n \geq 1$ such that*

$$(f \wedge e) * (x \wedge e)^n = \perp.$$

PROOF. Assume that F is a maximal filter of $\mathbf{L}$. Let $x \in L \setminus F$. Since F is a maximal filter, we have $\langle F \cup \{x\} \rangle = L$, so $\perp \in \langle F \cup \{x\} \rangle$ and by (4) of Proposition 2.15 there exist $n \geq 1, f \in F$ such that $(x \wedge e)^n * (f \wedge e) \leq \perp$. Hence $(f \wedge e) * (x \wedge e)^n = \perp$.

Conversely, assume that (2) holds. We will show that F is a maximal filter. Assume that there is a proper filter E of $\mathbf{L}$ such that $F \subseteq E$ and $F \neq E$. Then there exists an $x \in E$ such that $x \notin F$. By hypothesis, there exist $f \in F$, $n \geq 1$ such that $(f \wedge e) * (x \wedge e)^n = \perp$. Since $f, x \in E$, it follows that $\perp \in E$. Therefore $E = L$ which is a contradiction. Hence $F \in max(\mathbf{L})$. □

COROLLARY 3.21. *If F is a proper filter of $\boldsymbol{L}$, then the following are equivalent:*

(1) $F \in max(\boldsymbol{L})$.
(2) *For any $x \in L$, $x \notin F$ if and only if $\neg((x \wedge e)^n) \in F$ for some $n \in \mathbb{N}$.*

PROOF. Assume that F is a maximal filter. Let $x \in L \setminus F$. By (2) of Theorem 3.20, there exist $f \in F$, $n \geq 1$ such that $(f \wedge e) * (x \wedge e)^n = \perp$; therefore $f \wedge e \leq (x \wedge e)^n \to \perp = \neg((x \wedge e)^n))$ (by residuation property). Since $f \in F$, we deduce that $\neg((x \wedge e)^n) \in F$.

Conversely, assume that $\neg((x \wedge e)^n) \in F$ for some $n \in \mathbb{N}$. Let us show that $x \notin F$. By contradiction, assume that $x \in F$, then $(x \wedge e)^n * [\neg((x \wedge e)^n] = \perp \in F$, which is not possible. Therefore $x \notin F$ and we are done.

Conversely, assume that condition (2) holds for F. Let us show that F is maximal. Let $x \in L \setminus F$, then by assumption there exists $n \in \mathbb{N}$ such that $\neg[(x \wedge e)^n] \in F$. Set $f = (x \wedge e)^n \to \perp$, then $(f \wedge e) * (x \wedge e)^n = \perp$. Thus, by Theorem 3.20 we conclude that F is a maximal filter. □

We end this section by showing that, one can characterize some classes of IL-algebras using closure operations.

DEFINITION 3.22. Let $\mathbf{L}$ be an IL-algebra. An element $a \in L$ is called *archimedean* if it satisfies the condition: there exists $n \geq 1$ such that $(\neg[(a \wedge e)^n] \wedge e) \vee (a \wedge e) = e$. An IL-algebra is called *hyperarchimedean* if all its elements are archimedean.

EXAMPLE 3.23 ([12, Example 1]). Let $\mathbf{L}_5$ with $L_5 = \{\perp, a, b, e, \top\}$. The lattice diagram is given in Figure 4, and $*$ and $\to$ tables are given below:

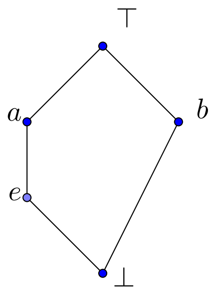

Figure 4. L_5

$*$	$\perp$	a	b	e	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\top$	$\top$	a	$\top$
b	$\perp$	$\top$	$\top$	b	$\top$
e	$\perp$	a	b	e	$\top$
$\top$	$\perp$	$\top$	$\top$	$\top$	$\top$

$\rightarrow$	$\perp$	a	b	e	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\perp$	e	$\perp$	$\perp$	$\top$
b	$\perp$	$\perp$	e	$\perp$	$\top$
e	$\perp$	a	b	e	$\top$
$\top$	$\perp$	$\perp$	$\perp$	$\perp$	$\top$

This IL-algebra is not a residuated lattice, since $a * b = \top \nleq a \wedge b = \perp$. An easy calculation shows that $\mathbf{L}$ is hyperarchimedean.

Let $\mathbf{L}_6 = \mathbf{L}_2 \times \mathbf{L}_3$ be the above IL-algebra. Then $max(\mathbf{L}) = Spec(\mathbf{L})$.

Let $Spec_c(\mathbf{L}) = F(\mathbf{L})^c \cap Spec(\mathbf{L})$, i.e. the set of all c-closed prime filters of $\mathbf{L}$. Note that the identity closure and the radical closure are both proper. It is straightforward to see that the prime spectra of the identity closure operation and the radical closure operation are $Spec(\mathbf{L})$ and $max(\mathbf{L})$, respectively. We can characterize hyperarchimedean IL-algebras using the prime spectra of proper closure operations.

Theorem 3.24 ([1, Theorem 50]). *For a residuated lattice $\boldsymbol{L}$ the following conditions are equivalent:*

(1) *$\boldsymbol{L}$ is hyperarchimedean.*

(2) *$Spec(L) = max(\boldsymbol{L})$.*

Proposition 3.25. *If the IL-algebra $\boldsymbol{L}$ is hyperarchimedean, then*

$$Spec(\boldsymbol{L}) = max(\boldsymbol{L}).$$

Proof. Assume that $\mathbf{L}$ is hyperarchimedean. Since $Max(\mathbf{L}) \subseteq Spec(\mathbf{L})$, we only have to show that $Spec(\mathbf{L}) \subseteq \max(\mathbf{L})$. Let P be a prime filter of $\mathbf{L}$, we will show that P is a maximal filter. Let $x \in L$. Assume that $x \notin F$ (since F is proper). Since $\mathbf{L}$ is hyperarchimedean, there is $n \geq 1$ such that $(x \wedge e) \vee (\neg[(x \wedge e)^n] \wedge e) = e \in F$. Then $\neg[(x \wedge e)^n] \in F$ by the primness of F. Conversely, assume that $\neg[(x \wedge e)^n] \in F$. We will show that $x \notin F$. By contradiction, suppose that $x \in F$, then $(x \wedge e)^n * [\neg(x \wedge e)^n] = (x \wedge e)^n * ((x \wedge e)^n \rightarrow \perp) \leq \perp \in F$, which is not possible, because F is proper, therefore $x \notin F$. Hence F is maximal by Corollary 3.21. Thus $Spec(\mathbf{L}) = max(\mathbf{L})$. $\square$

The converse of Proposition 3.25 is an open question.

THEOREM 3.26. *Let* $\boldsymbol{L}$ *be an IL-algebra. We consider the following assertions:*

(i) $\boldsymbol{L}$ *is hyperarchimedean.*
(ii) $Spec_c(\boldsymbol{L}) = max(\boldsymbol{L})$ *for every proper closure operation* c *on* $F(\boldsymbol{L})$.
(iii) *The only proper closure operation on* $F(\boldsymbol{L})$ *is the identity closure.*

Then

(1) (a) (i) $\Rightarrow$ (ii),
(b) (i) $\Rightarrow$ (iii).
(2) *If* $\boldsymbol{L}$ *is a commutative residuated lattice, then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

PROOF. (1) (i) $\Rightarrow$ (ii). Assume that $\mathbf{L}$ is hyperarchimedean and let c be a proper closure operation on $F(\mathbf{L})$. Then for each maximal filter M, $M \subseteq M^c \neq L$. It follows that $M = M^c$ and $Max(\mathbf{L}) \subseteq Spec_c(\mathbf{L}) \subseteq Spec(\mathbf{L})$. Since $\mathbf{L}$ is hyperarchimedean, by Proposition 3.25 $Spec(\mathbf{L}) = max(\mathbf{L})$. Thus $Spec_c(\mathbf{L}) = max(\mathbf{L})$.

(i) $\Rightarrow$ (iii). Assume that $\mathbf{L}$ is hyperarchimedean. Let c be a proper closure operation on $\mathbf{L}$ and P a prime filter, then $P \subseteq P^c$. Since $\mathbf{L}$ is hyperarchimedean and P prime, then P is maximal and $P^c = P$. That is P is c-closed. Let F be a filter of $\mathbf{L}$, then $F = \cap\{P \in Spec(\mathbf{L}) \mid F \subseteq P\}$ (see [19, Corollary 3.20 (ii)]). By (ii) of Proposition 3.4 we have $c(F) = F$. Thus c is the identity closure.

(2) We only need to show that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (i). Assume that $Spec_c(L) = \max(\mathbf{L})$ for every proper closure operation c on $\mathbf{L}$. Consider the identity closure Id, which is proper and also verified to have for spectrum $Spec(\mathbf{L})$. Therefore $Spec_{Id}(\mathbf{L}) = Spec(\mathbf{L}) = max(\mathbf{L})$ and $\mathbf{L}$ is hyperarchimedean by Theorem 3.24.

(iii) $\Rightarrow$ (i). Assume that $\mathbf{L}$ is not hyperarchimedean, then by Theorem 3.24 $\mathbf{L}$ has a prime filter that is not maximal. Consider the closure Rad on $\mathbf{L}$, then Rad is clearly proper and $Rad(P)$ is equal to the unique maximal filter containing P. In particular, $Rad(P) \neq P$ and Rad is not the identity closure. $\square$

PROPOSITION 3.27.

(1) *If an IL-algebra* $\boldsymbol{L}$ *is hyperarchimedean, then every filter of* $\boldsymbol{L}$ *is Rad-closed.*
(2) *A commutative residuated lattice* $\boldsymbol{L}$ *is hyperarchimedean if and only if all filters of* $\boldsymbol{L}$ *are Rad-closed.*

PROOF. (1) Recall that any filter F of $\mathbf{L}$ verifies $F = \cap\{P \in Spec(\mathbf{L}) \mid F \subseteq P\}$ (a) (see [19]). Assume that $\mathbf{L}$ is hyperarchimedean, then $Spec(\mathbf{L}) = max(\mathbf{L})$. By using (a) we get $F = Rad(F)$.

(2) Assume that $\mathbf{L}$ is a commutative residuated lattice. By (1) it is clear that if $\mathbf{L}$ is hyperarchimedean, then each filter of $\mathbf{L}$ is Rad-closed.

Conversely, assume that each filter of $\mathbf{L}$ is Rad closed. Let P be a prime filter on $\mathbf{L}$. Since P is a prime filter, by Proposition 2.12 there exists a unique maximal filter M such that $P \subseteq M$. By definition of Rad we have $Rad(P) = M$ and by assumption $Rad(P) = P$, hence $M = P$ and each prime filter is maximal. Therefore $Spec(\mathbf{L}) = max(\mathbf{L})$, by Theorem 3.24, we deduce that $\mathbf{L}$ is hyperarchimedean. □

4. Semi-prime closure operations and IL-algebras homomorphisms

In this section, we study relationship between IL-algebras and semi-prime closure operations in the sense of [14, 20]. We investigate the properties of closure operations and relationship between semi-prime closure operations and homomorphisms on IL-algebras. The obtained results extend some of the existing ones on some subclasses of IL-algebras such as BL-algebras.

Definition 4.1 ([20]). Let $\mathbf{L}$ be an IL-algebra, a map $c\colon L \to L$ is called a *semi-prime closure operation* if for all $x, y \in L$ it satisfies the following conditions:
(co_1) $x \leq c(x)$ (extensivity),
(co_2) if $x \leq y$, then $c(x) \leq c(y)$ (order-preserving),
(co_3) $c(c(x)) = c(x)$ (idempotence),
(co_4) $c(x) * c(y) \leq c(x * y)$ (semi-prime).

Proposition 4.2. *Let $c\colon L \to L$ be a semi-prime closure operation on $\boldsymbol{L}$. Then the following assertions hold:*
(1) *$c(c(x) * c(y)) = c(x * y)$, and $c(L) = \{c(x) \mid x \in L\}$ is closed under $\wedge$ and $\to$.*
(2) *For any $x \in L$, $c(x) * c(e) = c(x)$.*
(3) *The subset $D = \{x \in L \mid c(x) \geq c(e)\}$ of L is a deductive system. In addition, if $x \leq y$ and $x \in D$, then $y \in D$ and D is stable with $*$.*
(4) *The subset $A = \{x \in L : c(x \wedge e) \geq c(e)\}$ is a filter of $\boldsymbol{L}$.*

Proof. (1) See [20, Lemma 8.7].
(2) By (co_1), $e \leq c(e)$, using (c_8) we have $c(x) * e = c(x) \leq c(x) * c(e) \leq c(x * e) = c(x)$, therefore $c(x) = c(e) * c(x)$.
(3) We have $c(e) \leq c(e)$, so $e \in D$. Let $x, y \in L$ be such that $x, x \to y \in D$, we will show that $y \in D$. By (c_{10}) we have $x * (x \to y) \leq y$, by (co_4) we have $c(x) * c(x \to y) \leq c(x * (x \to y)) \leq c(y)$; since $c(x), c(x \to y) \geq c(e)$ and $c(e) * c(e) = c(e)$, we deduce that $c(e) \leq c(y)$. Hence $y \in D$. Assume that $x \in D$ and $x \leq y$, then $c(e) \leq c(x) \leq c(y)$. Therefore $y \in D$. Assume that $x, y \in D$,

then $c(e) \leq c(x), c(y)$, using (co_4) we obtain $c(e) * c(e) \leq c(x) * c(y) \leq c(x * y)$; since $c(e) * c(e) = c(e)$ we have $x * y \in D$.

(4) Clearly, $e \in A$. Assume that $x, y \in A$, then $c(x \wedge e), c(y \wedge e) \geq c(e)$. We have

$$\begin{aligned} c(e) = c(e) * c(e) &\leq c(x \wedge e) * c(y \wedge e) \quad \text{by } (\mathrm{c}_4) \\ &\leq c((x \wedge e) * (y \wedge e)) \quad \text{by } (\mathrm{co}_4) \\ &\leq c((x \wedge e) \wedge (x \wedge e)) \quad \text{by using order-preservation and } (\mathrm{c}_4). \end{aligned}$$

Since $(x \wedge e) * (y \wedge e) \leq (x * y) \wedge e, (x \wedge y) \wedge e, ((x \wedge e) * (y \wedge e)) \wedge e = ((x \wedge e) * (y \wedge e))$ and $((x \wedge e) \wedge (y \wedge e) = ((x \wedge e) \wedge (y \wedge e)) \wedge e$, we deduce that $x \wedge y, x * y \in D$.

Now assume that $x \leq y$ and $x \in A$, then $c(e) \leq c(x \wedge e) \leq c(y \wedge e)$, hence $y \in A$. Thus A is a filter of $\mathbf{L}$. □

EXAMPLE 4.3. Considering the IL-algebra of Example 2.2, we define $\phi \colon L_7 \to L_7$,

$$\phi(x) = \begin{cases} \top & \text{if } x \in \{c, \top\}, \\ e & \text{if } x \in \{a, b, d, e\}, \\ \bot & \text{if } x = \bot. \end{cases}$$

It is easy to check that ϕ verifies the conditions of Definition 4.1, so ϕ is a semi-prime closure operation on $\mathbf{L}$.

One can also check that $A = \{x \in L_7 : c(x \wedge e) \geq c(e)\} = \{d, a, b, e, \top\}$ which is a maximal filter of $\mathbf{L}_7$. One can see that $\bot, c \notin A$. Furthermore, we have $D = \{x \in L : c(x) \geq c(e) = e\} = \{\top, e, a, b, d, c\}$ which is a deductive system but not a filter (because $a, c \in D$ and $a \wedge c = \bot \notin D$).

One can check that $\frac{L}{A} = \{\{e, a, b, d, \top\}, \{c\}, \{\bot\}\}$ and $c(L) = \{\bot, e, \top\}$. Clearly, ϕ is not injective.

It follows from Example 4.3 that a deductive system in an IL-algebra is not always a filter.

PROPOSITION 4.4 ([20, Proposition 8.8]). *Let $\boldsymbol{L}$ be an IL-algebra, $c \colon \boldsymbol{L} \to \boldsymbol{L}$ a semi-prime closure operation and $c(L) = \{x \in L : c(x) = x\}$. Then $(c(L), \wedge_c, \vee_c, *_c, \to_c, c(\bot), c(e), c(\top))$ is an IL-algebra defined as follows: for any $x, y \in c(L)$: $x \wedge_c y = c(x \wedge y), x \vee_c y = c(x \vee y), x *_c y = c(x * y), x \to_c y = c(x \to y)$, furthermore $\wedge_c = \wedge, \to_c = \to$.*

THEOREM 4.5. *Let $\boldsymbol{L}$ be an IL-algebra, $c \colon L \to L$ be a semi-prime closure operation and $(c(L), \wedge, \vee_c, *_c, \to_c, c(\bot), c(e), \top)$ as above. Then the following properties hold:*

(1) *$c\colon L \to c(L)$ is an IL-epimorphism.*
(2) *If $D = \{a \in L \mid c(a \wedge e) \geq c(e)\}$, then the map $\overline{c}\colon L/D \to c(L)$ defined by*

$$\overline{c}([a]_{\Theta_D}) = c(a)$$

is an IL-epimorphism. It is an isomorphism if $e = \top$.

Proof. (1) follows from $c(x*y) = c(x)*_c c(y)$ and $c(x \to y) = c(x) \to c(y)$.

(2) follows from (4) of Lemma 4.2, D is a filter of $\mathbf{L}$. It follows that $\frac{\mathbf{L}}{D}$ is an IL-algebra.

Let us show that $\overline{c}$ is well defined. Let $a\theta_D b$, then $a \to b, b \to a \in D$, that is

$$c((a \to b) \wedge e) \geq c(e) \quad \text{and} \quad c((b \to a) \wedge e) \geq c(e).$$

From $a * (a \to b) \leq b$ we have $c(a) * c(a \to b) \leq c(a * (a \to b)) \leq c(b)$, hence $c(a \to b) \leq c(a) \to c(b)$ and $a \to b \leq c(a \to b)$. Since $a \to b \in D$, we get that $c(a) \to c(b) \in D$. Similarly one has $c(b) \to c(a) \in D$, therefore $c(a) \to c(b), c(b) \to c(a) \in D$, hence $c(a)\theta_D c(b)$. It follows that $\overline{c}([a]) = \overline{c}([b])$. Thus $\overline{c}$ is well defined.

Since $c\colon L \to c(L)$ is an IL-homomorphism, $\overline{c}\colon L/D \to c(L)$ is an IL-homomorphism from the following statements:

$$\overline{c}([x] * [y]) = \overline{c}([x * y]) = c(x * y) = c(x) *_c c(y) = \overline{c}([x]) *_c \overline{c}([y]),$$

$$\overline{c}([x] \to [y]) = \overline{c}([x \to y]) = c(x \to y) = c(x) \to_c c(y) = \overline{c}([x]) \to_c \overline{c}([y])$$

$$\overline{c}([x] \wedge [y]) = \overline{c}([x \wedge y]) = c(x \wedge y) = c(x) \wedge c(y) = c(x) \wedge_c c(y).$$

The map $\overline{c}$ is surjective by definition. In Example 4.3, we get that $\overline{c}$ is an isomorphism.

In the case $e = \top$, the isomorphism follows from [14, Theorem 2.6] letting $D = \{x \in L : c(x) = e\}$. □

Theorem 4.6. *Let $\boldsymbol{L}$, $\boldsymbol{K}$ be two IL-algebras and $h\colon L \to K$ be an IL-homomorphism. Let $c_1\colon L \to L$, $c_2\colon K \to K$ be two closure operations. Set*

$$D_1 = \{x \in L \mid c_1(x \wedge e_L) \geq c(e_L)\} \quad \textit{and} \quad D_2 = \{y \in K \mid c_2(y \wedge e_K) \geq c_2(e_K)\}.$$

Then the following statements hold:

(1) *If $h(c_1(x)) \leq c_2(h(x))$ for all $x \in L$, which is called a* closed map, *then the map*

$$\overline{h}\colon L/D_1 \to K/D_2$$

defined by $\overline{h}([x]_{D_1}) = [h(x)]_{D_2}$ is an IL-homomorphism.

(2) *If h is surjective and such that $h(c_1(x)) = c_2(h(x))$ for all $x \in L$ and if $h(c_1(x))) \geq c_2(e_K)$ implies $c_1(x) \geq c_1(e_L)$, then the map $\overline{h}\colon L/D_1 \to K/D_2$ is an IL-isomorphism.*

Proof. (1) First let us show that $\overline{h}$ is well defined. Assume that $a\theta_{D_1}b$, then $c_1((a \to b) \wedge e_L) \geq c_1(e_L)$ and $c_1(e_L) \leq c_1((b \to a) \wedge e_L)$. Compatibility property yields

$$\begin{aligned} e_K = h(e_L) \leq h(c_1(e_L)) &\leq h(c_1((a \to b) \wedge e_L)) \\ &\leq c_2(h((a \to b) \wedge e_L)) \quad \text{(by assumption)} \\ &= c_2((h(a) \to h(b)) \wedge e_K) \quad \text{(due to } h \text{ is a homomorphism).} \end{aligned}$$

It follows that $e_K \leq c_2((h(a) \to h(b)) \wedge e_K)$. Since $c_2 \circ c_2 = c_2$ we deduce that $c_2(e_K) \leq c_2((h(a) \to h(b)) \wedge e_K)$, hence $h(a) \to h(b) \in D_2$. A similar argument shows that $h(b) \to h(a) \in D_2$, therefore $(h(a), h(b)) \in \theta_{D_2}$. Thus $\overline{h}$ is well defined.

Since $[x \to y]_{D_1} = [x]_{D_1} \to [y]_{D_1}$ and $[h(a) \to h(b)]_{D_2} = [h(a)]_{D_1} \to [h(b)]_{D_2}$, we have $\overline{h}([a \to b]) = \overline{h}([a] \to [b]) = [h(a) \to h(b)] = [h(a)] \to [h(b)] = \overline{h}([a]) \to \overline{h}([b])$. Similarly $\overline{h}([a] * [b]) = \overline{h}([a]) * \overline{h}([b])$. Thus $\overline{h}$ is an IL-homomorphism.

(2) Assume that h is surjective and $h(c_1(x)) = c_2(h(x))$ for all $x \in L$ and $h(c_1(x))) \geq c_2(e_K)$, implies $c_1(x) \geq c_1(e_L)$. From (1), $\overline{h}$ is an IL-homomorphism and it is surjective; since h is surjective. It remains to show that $\overline{h}$ is injective. Let $\overline{h}([a]_{D_1}) = \overline{h}([b]_{D_1})$. Let us show that $[a]_{D_1} = [b]_{D_1}$, that is $c_1((a \to b) \wedge e) \geq c_1(e_L)$ and $c_1((b \to a) \wedge e) \geq c_1(e_L)$. We have $h(a) \to h(b), h(b) \to h(a) \in D_2$, that is $c_2((h(a) \to h(b)) \wedge e_K) \geq c_2(e_K))$ and $c_2((h(b) \to h(a)) \wedge e_K) \geq c_2(e_K)$. Furthermore,

$$\begin{aligned} c_2(e_K) &\leq c_2((h(a) \to h(b)) \wedge e_K) \\ &= c_2(h(a \to b) \wedge e_K) \\ &= c_2(h((a \to b) \wedge e_L)) \quad \text{(due to } h \text{ compatible with } \to, \wedge) \\ &= h(c_1((a \to b) \wedge e_L)) \quad \text{(by assumption on } h). \end{aligned}$$

We deduce that $c_1((a \to b) \wedge e_L) \geq c_1(e_L)$ and $a \to b \in D_1$. A similar argument shows that $b \to a \in D_1$. Therefore $[a]_{D_1} = [b]_{D_1}$. Thus $\overline{h}$ is injective. Hence $\overline{h}$ is an isomorphism. $\square$

Example 4.7. Consider the IL-algebra $\mathbf{L}_7$, the isomorphism h in Example 2.20 and the closure c in Example 4.3. We have $D_1 = \{x \in L : c(x \wedge e) \geq c(e)\} = \{d, a, b, e, \top\}$, $L_7/D_1 = \{[\top], [c], [\bot]\}$ which forms a residuated lattice and $\overline{h}\colon L_7/D_1 \to L_7/D_1$ is an isomorphism, which is the identity map.

Proposition 4.8. *Let $c\colon L \to L, x \mapsto c(x)$ be a semi-prime closure operation on $\boldsymbol{L}$.*

(1) *If $F \in F(\boldsymbol{L})$, then $c(F)$ is a filter in $c(\boldsymbol{L})$ and $c(F) \subseteq F$.*
(2) *If F is a maximal filter in $\boldsymbol{L}$ such that $c(F) \neq L$, then $c(F)$ is a maximal filter in $c(\boldsymbol{L})$.*
(3) *If P is a prime filter in $\boldsymbol{L}$, then $c(P)$ is a prime filter in $c(\boldsymbol{L})$.*

Proof. (1) Since $c\colon L \to c(L)$ is a surjective homomorphism of IL-algebras we are done.

(2) Assume that $F \in max(\mathbf{L})$. Let G be a filter of $c(\mathbf{L})$ such that $c(F) \subseteq G \subseteq c(L)$ and $c(F) \neq c(G)$. We will show that $G = c(L)$. Since $c(F) \neq c(G)$, there exists $x = c(u) \in C(G) \setminus c(F)$; so $u \notin F$ and $\langle F \cup \{u\} \rangle = L$. Using (iv) of Proposition 2.15 there are $n \geq 1, f \in F$ such that $\bot \geq (f \wedge e) * (u \wedge e)^n$, and using order preservation we have

$$\begin{aligned} c(\bot) &\geq c((f \wedge e) * (u \wedge e)^n)) \\ &\geq (c(f \wedge e) * (c(u \wedge e)^n) \in c(G) \end{aligned}$$

and $c(\bot) \in c(G)$, so $c(G) = c(L)$. Therefore $c(F)$ is maximal.

(3) Assume that P is a prime filter in $\mathbf{L}$. Let us show that $c(P)$ is a prime filter of $c(\mathbf{L})$. Since c is a surjective homomorphism, $c(P)$ is a filter. Let $x = c(a)$, $y = c(b) \in c(L)$ be such that $(x \wedge c(e)) \vee (y \wedge c(e)) \in c(P)$, that is $c((a \wedge e) \vee (b \wedge e)) \in c(P)$, by definition of $c(P)$, we have $(a \wedge e) \vee (b \wedge e) \in P$; since P is a prime filter, we have $a \in P$ or $b \in P$; hence $c(a) \in c(P)$ or $c(b) \in c(P)$. Thus $c(P)$ is a prime filter. $\square$

We end this work by exploring some relationships between closure operations on $\mathbf{L}$ and those of it powers $\mathbf{L}^X$. Giving a closure operation c on $\mathbf{L}$, we define ϕ_c on $\mathbf{L}^X$ as follows:

$$\phi_c\colon L^X \to L^X, f \mapsto \phi_c(f) = c \circ f.$$

Proposition 4.9. *ϕ_c is a closure operation on $\boldsymbol{L}^X$.*

PROOF. Extensive and monotone properties are clear. Let $f \in L^X$, then $\phi_c(\phi_c(f)) = c \circ c \circ f$ and for any $x \in X$, we have $c \circ c \circ f(x) = c \circ c(f(x)) = c(f(x)) = c \circ f(x)$, hence $c \circ c \circ f = c \circ f$, that is $\phi_c \circ \phi_c = \phi_c$. To end the proof, we show that $\phi_c(f)\tilde{*}\phi_c(g) \leq \phi_c(f\tilde{*}g)$ for any $f, g \in L^X$. Clearly $\phi_c(f) = c \circ f, \phi_c(g) = c \circ g$ and $\phi_c(f\tilde{*}g) = c \circ (f\tilde{*}g) \in L^X$.

Let $x \in X$, then we have

$$\begin{aligned}\phi_c(f)\tilde{*}\phi_c(g)(x) &= \phi_c(f)(x) * \phi_c(g)(x) = c(f(x)) * c(g(x)) \\ &\leq c(f(x) * g(x)) = c \circ (f\tilde{*}g)(x) = \phi_c(f\tilde{*}g)(x).\end{aligned}$$

Hence $\phi_c(f)\tilde{*}\phi_c(g) \leq \phi_c(f\tilde{*}g)$. Thus ϕ_c is a closure operation on $\mathbf{L}^X$. □

PROPOSITION 4.10. *Let $h\colon L \to K$ be an IL-homomorphism and X a nonempty set. Set $\phi_h\colon L^X \to K^X, f \mapsto \phi_h(f) = h \circ f$. Then the following statements hold:*

(i) *ϕ_h is an IL-homomorphism.*
(ii) *If h is injective, then ϕ_h is also injective.*
(iii) *If h is an isomorphism, then ϕ is also an isomorphism.*

PROOF. (i) Let $f, g \in L^X$, we will show that $\phi_h(f\tilde{*}g) = \phi_h(f)\tilde{*}\phi_h(g)$ and $\phi_h(f\tilde{\to}g) = \phi_h(f)\tilde{\to}\phi_h(g)$. Let $x \in X$, then we have

$$\begin{aligned}\phi_h(f\tilde{\to}g)(x) &= [h \circ (f\tilde{\to}g)](x) = h[f(x) \to g(x)] \\ &= h(f(x)) \to h(g(x)) = h \circ f(x) \to h \circ g(x) \\ &= ((h \circ f) \to h \circ g))(x) = (\phi_h(f)\tilde{\to}\phi_h(g))(x).\end{aligned}$$

Hence $\phi_h(f\tilde{\to}g) = \phi_h(f)\tilde{\to}\phi(g)$ and ϕ_h is compatible with $\tilde{\to}$. A similar argument shows that ϕ_h is compatible with $\tilde{*}, \tilde{\vee}$ and $\tilde{\wedge}$. Hence ϕ_h is an homomorphism.

(ii) Let $f, g \in L^X$ be such that $\phi_h(f) = \phi_h(g)$. Then for any $x \in X$ we have $\phi_h(f)(x) = \phi_h(g)(x)$, that is $h(f(x)) = h(g(x))$, since h is injective, we deduce that $f = g$. Hence ϕ_h is also injective.

(iii) Assume that h is bijective. Then by (ii), ϕ_h is injective. Let $g \in K^X$. Then $f = h^{-1} \circ g \in L^X$ and $\phi_h(f) = g$. Thus ϕ_h is surjective and therefore bijective. □

5. Conclusion

In this paper, we introduce the Radical of filters and extended filters in IL-algebras as generalization of the case of commutative residuated lattice and investigate some of their properties. Closure operations on IL-algebras are introduced and several examples are presented. We obtain that Radical and extended filters yield examples of closure operations on IL-algebras having interesting properties. Special classes of closure operations are considered (stable closure, proper closure) and a preliminary investigation was carried out.

As an introductory paper on closure operations on IL-algebras, we focused on producing a good variety of examples and studying their first properties. Some techniques of construction of closure operations on IL-algebras are investigated. Using stable closure operations, we show that the collection of closed filters via stable closure operations forms a complete Heyting algebra. Hyperarchimedean IL-algebras are introduced and characterized in terms of proper closure operations. A concrete characterization of maximal filters in IL-algebras is obtained and that characterization allows us to show that if an IL-algebra is hyperarchimedean, then any prime filter of $\mathbf{L}$ is a maximal filter and all filters are rad-closed. We finish the paper with the study of semi-prime closure operations on IL-algebras and their relationships with homomorphisms. The obtained results generalized some of the existing results in [14]. We show that semi-prime closure operation c induces an IL-epimorphism and a quotient IL-algebra, first isomorphism theorem is investigated. Finally, it is shown that each semi-prime closure operation on an IL-algebra $\mathbf{L}$ induces a semi-prime closure operation on the power $\mathbf{L}^X$ of $\mathbf{L}$ for an arbitrary set $X \neq \emptyset$.

In our future work, we will explore more characterizations of IL-algebras using closure operations and study topology derived from closure operations in IL-algebras.

Declaration of competing interest. The authors declare that they have unknown competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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