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SPEED OF LIGHT OR COMPOSITION OF VELOCITIES

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Dedicated to Professor Kazimierz Nikodem on the occasion of his 70th birthday

Abstract. We analyze in our paper questions of the theory of relativity. We approach this theory from the point of view of velocities and their composition. This is where the functional equations appear. Solving them leads to a world where velocities are bounded from above, the upper bound being exactly the "speed of light".

1. Introduction

Consider two inertial frames of reference U and U'. For simplicity we admit that points of U (resp. U') are "one dimensional", i.e. they are of the form (t_U, x_U) (resp. $(t_{U'}, x_{U'})$) where t_U denotes the time variable, and x_U denotes the space (one-dimensional) variable. Assume that the frame U' moves with the velocity v with respect to the frame U. Analyzing the relations between the variables, and taking into account the first Newton's law of motion we conclude

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that the change is linear, i.e. there exist functions a, b, c and d, depending on v and such that

$$t_U = a(v)t_{U'} + b(v)x_{U'},$$
$$x_U = c(v)t_{U'} + d(v)x_{U'}.$$

In other words we get

$$Z_U^\top = M(v)Z_{U'}^\top,$$

where $M\colon \Delta \to GL_2(\mathbb{R})$ is a matrix function, defined on an interval Δ, $Z_U = (t_U, x_U)$ and $Z_{U'} = (t_{U'}, x_{U'})$. Following A. Szymacha [10], see also R.D. Sard [8] or A. Sommerfeld [9] or W. Benz [2], we speak about "addition" or "composition" of velocities. Suppose u and v are velocities from Δ. Then we call "sum" of the two velocities an operation $\oplus$, which is commutative, but not necessarily we have

$$u \oplus v \in \Delta.$$

We assume also that $0 \oplus v = v$, whenever $0 \in \Delta$, and (cf. commutativity of $\oplus$) $u \oplus 0 = u$, for all $u, v \in \Delta$. It is noteworthy to observe that $a(0) = 1 = d(0), b(0) = c(0) = 0$, because $v = 0$ means that both frames stay at the same position.

We introduce some new notation right now. Let denote by $\Delta_\oplus$ the set $\{(u, v) \in \Delta \times \Delta : u \oplus v \in \Delta\}$. Further, denote by Δ^v the set $\{u \in \Delta : (u, v) \in \Delta_\oplus\}$.

Let us consider the implication

$$(u, v) \in \Delta_\oplus \Longrightarrow M(u)M(v) = M(u \oplus v). \tag{1.1}$$

Equation (1.1) implies that

$$a(u)a(v) + b(u)c(v) = a(u \oplus v), \tag{1.2}$$
$$a(u)b(v) + b(u)d(v) = b(u \oplus v), \tag{1.3}$$
$$c(u)a(v) + d(u)c(v) = c(u \oplus v), \tag{1.4}$$
$$c(u)b(v) + d(u)d(v) = d(u \oplus v),$$

if $(u, v) \in \Delta_\oplus$. Commutativity of $\oplus$ together with (1.2) yield

$$b(u)c(v) = c(u)b(v)$$

for all $(u, v) \in \Delta_{\oplus}$. Thus either $c = 0$ or there exists a $v_0 \in \Delta$ such that

$$b(u) = \beta c(u) \tag{1.5}$$

for some $\beta \in \mathbb{R}$ and all $u \in \Delta^{v_0}$. Similarly, commutativity of $\oplus$ together with (1.4) yield

$$a(u)c(v) + c(u)d(v) = a(v)c(u) + c(v)d(u)$$

or

$$\left(a(u) - d(u)\right) c(v) = \left(a(v) - d(v)\right) c(u),$$

for all $(u, v) \in \Delta_{\oplus}$. Thus again, either $c = 0$ or there is an $\alpha \in \mathbb{R}$ such that

$$d(u) = a(u) + \alpha c(u), \tag{1.6}$$

for all $u \in \Delta^{v_0}$, where $c(v_0) \neq 0$.

2. Functional equations

Let us consider the following cases, based on our observations from the Introduction.

I) $c = 0$.

It is enough to deal with the two following subcases:

1^o $b = 0$

2^o $b \neq 0$.

The case 1^o gives

$$\begin{cases} a(u)a(v) = a(u \oplus v), \\ d(u)d(v) = d(u \oplus v), \end{cases}$$

for all $(u, v) \in \Delta_{\oplus}$, and will not be discussed in the sequel.

The case 2^o, and the equation (1.3) allow us to calculate d with the aid of a and b for all $(u, v) \in \Delta_{\oplus}$, at present we also have $c = 0 \cdot b$.

II) $c \neq 0$.

The present case leads to the following, taking into account (1.3), (1.4), (1.5) and (1.6):

$$\begin{cases} a(u)a(v) + \beta c(u)c(v) = a(u \oplus v), \\ a(u)c(v) + c(u)a(v) + \alpha c(u)c(v) = c(u \oplus v), \end{cases} \tag{2.1}$$

if $(u, v) \in \Delta_{\oplus}$.

Let us proceed to determine the operation $\oplus$.

A) Sometimes it is easy to calculate $u \oplus v$. For instance, suppose that $a(u) \neq 0$, $u \in \Delta$, and $c(u) = ua(u)$, $u \in \Delta$ (note that this corresponds to the definition of speed as the ratio $\frac{x_U}{t_U}$ if we consider the frame U' with $x_{U'} = 0$). Then

$$\begin{cases} a(u)a(v)\,(1 + \beta uv) = a(u \oplus v), \\ a(u)a(v)\,(u + v + \alpha uv) = (u \oplus v)\, a(u \oplus v), \end{cases}$$

for all for all $(u, v) \in \Delta_{\oplus}$, whence (because $a(u) \neq 0$, $u \in \Delta$)

$$u \oplus v = \frac{u + v + \alpha uv}{1 + \beta uv}$$

whenever $u \oplus v \in \Delta$. Of course, we see that $1 + \beta uv \neq 0$ for for all $(u, v) \in \Delta_{\oplus}$ (otherwise we would have $a(u \oplus v) = 0$, contrary to our assumption).

B) On the other hand, assume that a and c are continuous, and c is invertible (then c is a homeomorphism), and reconsider (2.1). It turns out from the second equation in (2.1) that $\oplus$ is continuous. Let us assume also that $0 \in \Delta$. Since $0 = 0 \oplus 0 \in \Delta$ by continuity we get that there exists an interval $\Delta_1 \subset \Delta$ such that $\Delta_1 \times \Delta_1 \subset \Delta_{\oplus}$. Let us define

$$f := a \circ c^{-1} \colon c(\Delta_1) \to \mathbb{R}. \tag{2.2}$$

Denote $x := c(u)$, $y := c(v)$, for $(u, v) \in \Delta_1 \times \Delta_1$.

We get from (2.1)

$$f\,[f(x)y + f(y)x + \alpha xy] = f(x)f(y) + \beta xy, \quad x, y \in c(\Delta_1). \tag{2.3}$$

Substituting

$$\varphi(x) := f(x) + \frac{\alpha x}{2} \tag{2.4}$$

we transform (2.3) into

$$\varphi\left[x\varphi(y)+y\varphi(x)\right]=\varphi(x)\varphi(y)+\left(\frac{\alpha^2}{4}+\beta\right)xy,\quad x,y\in c(\Delta_1)$$

or, denoting by $K:=\frac{\alpha^2}{4}+\beta$,

$$\varphi\left[x\varphi(y)+y\varphi(x)\right]=\varphi(x)\varphi(y)+Kxy,\quad x,y\in c(\Delta_1),\tag{2.5}$$

whence with $x:=c(u)$, $y:=c(v)$ it follows from (2.1) that

$$u\oplus v=c^{-1}(x)\oplus c^{-1}(y)=c^{-1}\left(A_\varphi(x,y)\right)=c^{-1}\left(A_\varphi(c(u),c(v))\right),\tag{2.6}$$

where $A_\varphi(x,y):=x\varphi(y)+y\varphi(x)$, $x,y\in c(\Delta_1)$.

The functional equation (2.5) has been considered earlier by several authors. Let us mention e.g. P. Volkmann and H. Weigel [11], N. Brillouët and J. Dhombres [3], to some extent it is connected with associativity equation (see J. Aczél [1] and R. Craigen, Zs. Páles [4]). Moreover, it has very much in common with the celebrated Abel's equation, proposed by Hilbert to solve in his Fifth Problem (cf., eg. [5] and [7]). There is a paper by the author of the present note ([6]) where exactly the situation like present is treated, we deal with conditional associativity, which leads to (2.5). In [6] we proved a theorem on the general continuous solution of (2.5). We will not present the whole theorem, it contains 14 possible cases, and each case has its so called conjugate case. The number can be reduced to 5, if we assume that $\varphi(0)=1$. Let us observe that this is actually the case in our present situation: since obviously $M(0)=I$ then $a(0)=1$, $c(0)=0$, whence $f(0)=\varphi(0)=1$.

THEOREM 2.1. *Let $I\subset\mathbb{R}$ be an interval containing* 0 *and let $\varphi\colon I\to\mathbb{R}$ be a continuous function such that $\varphi(0)=1$. Then A_φ is locally associative*[1] *if and only if φ has one of the following forms:*

(S_1) $\varphi(x)=1+Ax$ *for $x\in I$ where I is arbitrary and $A\in\mathbb{R}$ is an arbitrary constant,*

(S_2) $\varphi(x)=|Ax-(1/2)|+(1/2)$ *for $x\in I$, where I is arbitrary and $A\in\mathbb{R}\backslash\{0\}$ is an arbitrary constant such that $1/2A\in I$,*

[1]We say that A_φ is locally associative if the following condition is satisfied:

$$(x\in I)\wedge(y\in I)\wedge(z\in I)\wedge(A_\varphi(x,y)\in I)\wedge(A_\varphi(y,z)\in I)\Longrightarrow$$
$$A_\varphi\left(x,A_\varphi(y,z)\right)=A_\varphi\left(A_\varphi(x,y),z\right).$$

(S_3) $\varphi(x) = \gamma^{-1}\left(\frac{x}{E}\right)$ *for* $x \in I$ *where* $E \neq 0$ *is an arbitrary constant,* $\gamma\colon \left[\frac{1}{e}, +\infty\right) \to \mathbb{R}$ *is defined by* $\gamma(u) = u \ln u$ *and* $I \subset E\left[-\frac{1}{2\sqrt{e}}, +\infty\right)$,

(S_4) $\varphi(x) = g_\alpha^{-1}(Ax) + Ax$ *for* $x \in I$, *where* $A \neq 0$ *and* $\alpha \neq 1$ *are arbitrary constants,* $g_\alpha\colon K_\alpha \to \mathbb{R}$ *is given by* $g_\alpha(u) = \frac{u^\alpha - u}{2}$ *with*

$$K_\alpha = \begin{cases} (0, +\infty) & \text{for } \alpha \leqslant 0, \\ \left(\alpha^{\frac{\alpha}{1-\alpha}}, +\infty\right) & \text{for } \alpha \in (0, +\infty)\backslash\{1\}, \end{cases}$$

and I *is contained in* I_α, *where*

$$I_\alpha = \begin{cases} \mathbb{R} & \text{for } \alpha < 0, \\ \left(\frac{1}{A}\right)\left(-\infty, \frac{1}{2}\right) & \text{for } \alpha = 0, \\ \left(\frac{1}{A}\right) g_\alpha\left(\alpha^{\frac{1}{2(1-\alpha)}}\right)(-\infty, 1] & \text{for } \alpha \in (0, +\infty)\backslash\{1\}, \end{cases}$$

(S_5)

$$\varphi(x) = \begin{cases} \frac{Ax}{r_D^{-1}(x)}, & x \in I\backslash\{0\}, \\ 1, & x = 0, \end{cases}$$

where $A \neq 0$ *and* $D \in \mathbb{R}$ *are arbitrary constants, and* $r_D\colon R_D \to \mathbb{R}$ *is a function given by* $r_D(u) = \left(\frac{1}{A}\right)\left(\frac{u}{\sqrt{1-u^2}}\right)\exp(D \arctan u)$ *for* $u \in R_D$ *with* $R_0 = \mathbb{R}$ *and* $R_D = \left(\frac{1}{D}\right)[-1, = \infty)$, *for* $D \neq 0$. *Moreover, the interval* $I \subset r_D\left(\left[D - \sqrt{1 + D^2}, D + \sqrt{1 + D^2}\right]\right)$.

2.1. Some examples

Taking into account Theorem 2.1, as well as formulae (2.2), (2.4), and, above all, (2.6), we get some specific formulae for composition of velocities.

Example 1. In the case (S_1) we have $A_\varphi(x, y) = x + y + 2Axy$, $x, y \in \mathbb{R}_+ = [0, +\infty)$. Taking $a(u) \equiv 1$, and $c(u) = u$, $u \in \mathbb{R}_+$, we get $c^{-1}(x) = x, x \in \mathbb{R}_+$. Thus, taking $x := c(u)$, $y := c(v)$, we get (cf. (2.6))

$$u \oplus v = A_\varphi(u, v) = u + v + 2Auv,$$

which, with $A = 0$, is just the Galilean way to add the velocities.

EXAMPLE 2. In the case (S_1), let us take $A = 1$, and let $I \subset \mathbb{R}$ be an arbitrary interval containing 0 and $\frac{1}{2}$. We have

$$\varphi(x) = \begin{cases} x & \text{for } x \geqslant \frac{1}{2}, \\ 1 - x & \text{for } x < \frac{1}{2}. \end{cases}$$

Such a definition leads to

$$A_\varphi(x, y) = \begin{cases} 2xy & \text{for } x, y \geqslant \frac{1}{2}, \\ x & \text{for } x \geqslant \frac{1}{2} > y, \\ y & \text{for } y \geqslant \frac{1}{2} > x, \\ x + y - 2xy & \text{for } x, y < \frac{1}{2}. \end{cases}$$

Now, putting $c(u) = u$, we obtain

$$u \oplus v = \begin{cases} 2uv & \text{for } u, v \geqslant \frac{1}{2}, \\ u & \text{for } u \geqslant \frac{1}{2} > v, \\ v & \text{for } v \geqslant \frac{1}{2} > u, \\ u + v - 2uv & \text{for } u, v < \frac{1}{2}. \end{cases}$$

EXAMPLE 3. In the case (S_3), let us take $E = 1$, and let $I \subset \left[-\frac{1}{2\sqrt{e}}, +\infty\right)$ be an arbitrary interval containing 0. We have

$$\varphi(x) = \gamma^{-1}(x),$$

for all $x \in I$. We see that $\gamma(1) = 0$, whence $\varphi(0) = \gamma^{-1}(0) = 1$. For any $x, y \in I$ we have (admitting that $x = \gamma(u)$, $y = \gamma(v)$ for some $u, v \in \Delta_1 \subset \left[\frac{1}{e}, +\infty\right)$)

$$\begin{aligned} A_\varphi(x, y) &= x\varphi(y) + y\varphi(x) = \gamma(u)v + \gamma(v)u \\ &= uv(\ln u + \ln v) = uv \ln(uv) = \gamma(uv) = \gamma\left(\gamma^{-1}(x)\gamma^{-1}(y)\right). \end{aligned}$$

Defining $c\colon \Delta_1 \to \mathbb{R}$ by $c(u) = u \ln u = \gamma(u)$ we obtain (cf. (2.6))

$$u \oplus v = \left(c^{-1} \circ \gamma\right)\left(\left(\gamma^{-1} \circ c\right)(u) \cdot \left(\gamma^{-1} \circ c\right)(v)\right) = uv.$$

Example 4. In the case (S_4), put $A = 1$. For any $\alpha \neq 1$ and $I \subset I_\alpha$ with $0 \in I$, we have $\varphi\colon I \to \mathbb{R}$ of the form $\varphi(x) = g_\alpha^{-1}(x) + x$. It follows that for every $x, y \in I$ the following formula holds

$$A_\varphi(x, y) = 2xy + xg_\alpha^{-1}(y) + yg_\alpha^{-1}(x),$$

which means that substituting $x := g_\alpha(u)$ and $y := g_\alpha(v)$, we get

$$A_\varphi(g_\alpha(u), g_\alpha(v)) = 2g_\alpha(u)g_\alpha(v) + vg_\alpha(u) + ug_\alpha(v).$$

Put $c := g_\alpha$, then taking into account (2.6) we get

$$u \oplus v = g_\alpha^{-1}\left((g_\alpha(u), g_\alpha(v))\right) = g_\alpha^{-1}\left(2g_\alpha(u)g_\alpha(v) + vg_\alpha(u) + ug_\alpha(v)\right).$$

Example 5. In the case (S_5), let us take $D = 0$. We have

$$\varphi(x) = \begin{cases} \frac{Ax}{r_0^{-1}(x)} & \text{for } x \in I\backslash\{0\}, \\ 1 & \text{for } x = 0, \end{cases}$$

or, since $r_0^{-1}(x) = \frac{Ax}{\sqrt{1+(Ax)^2}}$,

$$\varphi(x) = \sqrt{1 + (Ax)^2}.$$

Denote $C := A^2$. Then

$$A_\varphi(x, y) = x\sqrt{1 + Cy^2} + y\sqrt{1 + Cx^2}, \quad x, y \in I.$$

Put $a(u) := \frac{1}{\sqrt{1-Cu^2}}$, $c(u) := \frac{u}{\sqrt{1-Cu^2}}$. Using (2.6) again we obtain

$$u \oplus v = \frac{u + v}{1 + Cuv},$$

or the Einstein addition of velocities.

Remark 2.1. Actually, Theorem 2.1 yields large families of possible composition of velocities. However, even if they are mathematically correct, we do not know whether they have any physical meaning.

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