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Report of Meeting

The Twenty-third Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities Brenna (Poland), January 31 – February 3, 2024

The Twenty-third Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held in Hotel Kotarz in Brenna, Poland, from January 31 to February 3, 2024. The meeting was organized by the Institute of Mathematics of the University of Silesia.

15 participants came from the University of Debrecen (Hungary), 6 from the University of Silesia in Katowice (Poland), 3 from the University of the National Education Commission, Krakow (Poland), 3 from the University of Rzeszów (Poland), 2 from the University of Zielona Góra (Poland), 1 from the University of Miskolc (Hungary).

Professor Maciej Sablik opened the Seminar and welcomed the participants to Brenna.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iteration theory, equations on abstract algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity, characterization of premiums for insurances with the use of functional equations, fixed point theorems of Knaster–Tarski and applications to the theory of fractals, orthogonality equations, inequalities involving stochastic orderings, computer assisted solutions of different classes of functional equations, functional equations in fuzzy logic.

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Interesting discussions were generated by the talks.

There was also a Problems and remarks session and a festive dinner.

The closing address was given by Professor Zsolt Páles. His invitation to the Twenty-fourth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in 2025 in Hungary was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in section 1, problems and remarks in section 2, and the list of participants in the final section.

1. Abstracts of talks

Roman Badora: *Searching for new versions of the Kranz separation theorem*

Suppose we have two real functionals defined on a commutative semigroup (or group) and one of them lies below the other. During the talk, we will analyze which of the systems of two inequalities describing the relationship between the values of these functionals on the sum of arguments and the sum of their values on these arguments guarantees the separation of the given functionals by an additive mapping.

Mihály Bessenyei: *Existence theorems for invariance equations* (Joint work with Evelin Pénzes)

The Kuratowski measure of noncompactness provides direct approach to the Sadovskiĭ fixed point theorem or to Hutchinson's fundamental result concerning fractals. It turns out that this measure is not distinguished: Requiring quite simple properties on a set-function, we can prove analogous results. The common idea behind is an abstract domain invariance property which can be justified with the Knaster–Tarski and the Kantorovitch Fixed Point Theorems.

Zoltán Boros: *An alternative equation involving two generalized monomials* (Joint work with Rayene Menzer)

In this presentation, we consider generalized monomials or polynomials $f,\, g\colon \mathbb{R} \to \mathbb{R}$ that satisfy the additional equation $f(x)g(y) = 0$ for the pairs $(x,y) \in D$, where $D \subset \mathbb{R}^2$ is given by some algebraic condition. In the particular cases when f and g are generalized polynomials and there exist nonconstant regular polynomials p and q that fulfill

$$D = \{\, (p(t), q(t)) \mid t \in \mathbb{R} \,\}$$

or f and g are generalized monomials and there exists a non-zero rational m fulfilling

$$D = \{\,(x, y) \in \mathbb{R}^2 \,|\, x^2 - my^2 = 1\,\},$$

we prove that either f or g is identically equal to zero.

Our research is motivated by such results for $g = f$ in [1] and [2].

References

[1] Z. Boros and W. Fechner, *An alternative equation for polynomial functions*, Aequationes Math. **89** (2015), no. 1, 17–22.

[2] Z. Boros and R. Menzer, *An alternative equation for generalized monomials*, Aequationes Math. **97** (2023), no. 1, 113–120.

Jacek Chmieliński: *Alternate additivity of the Birkhoff-James orthogonality* (Joint work with Paweł Wójcik)

The Birkhoff-James orthogonality $\perp_B$ is not additive (neither on the right nor on the left) unless certain additional geometrical properties (like smoothness, strict convexity or inner product structure) are imposed on the underlying space. We establish some weaker forms of the said additivity which are true without any additional assumptions. In particular, we show that for a real normed space and arbitrary vectors x, y, z, we always have the alternative:

$$x\perp_B y \quad \text{and} \quad x\perp_B z \quad \Longrightarrow \quad x\perp_B(y+z) \quad \text{or} \quad x\perp_B(y-z).$$

For the left-additivity, the situation is more complex. If the underlying space is a two-dimensional real normed space, then we have for all x, y, z:

$$y\perp_B x \quad \text{and} \quad z\perp_B x \quad \Longrightarrow \quad (y+z)\perp_B x \quad \text{or} \quad (y-z)\perp_B x.$$

If the dimension of the considered space is greater than two, the latter condition characterizes inner product spaces among all smooth or strictly convex real normed spaces.

Jacek Chudziak: *Risk diversification with the zero utility principle* (Joint work with Paweł Pasteczka and Patryk Rela)

Let $\mathcal{X}_+$ be a family of risks, that is non-negative essentially bounded random variables on a given probability space. A zero utility premium for $X \in \mathcal{X}_+$, introduced by Bühlmann [1], is defined through the equation

$$E[u(H_u(X) - X)] = 0, \tag{1}$$

where $u:\mathbb{R}\to\mathbb{R}$ is a strictly increasing continuous function such that $u(0)=0$. In [2], characterizations of various important properties of the zero utility principle defined by (1) were established. In particular, it was proved there that the principle is convex if and only if u is concave. This result has been extended in [3], where it was shown that H_u is quasi-convex, that is

$$H_u\left(\frac{X+Y}{2}\right)\le \max\{H_u(X),H_u(Y)\}\quad \text{for } X,Y\in\mathcal{X}_+,$$

if and only if it is convex. Thus, a quasi-convexity of H_u is equivalent to concavity of u. It is known that the quasi-convexity has the following natural interpretation: a premium for a portfolio composed of two risks using the arithmetic mean does not exceed a maximum of the premiums for the individual risks. Motivated by the above results, for a given function u we are interested in functions $f\colon[0,\infty)^2\to[0,\infty)$ allowing the risk diversification, that is for which the following inequality is satisfied

$$\text{(2)}\qquad H_u(f(X,Y))\le \max\{H_u(X),H_u(Y)\}\quad \text{for } X,Y\in\mathcal{X}_+.$$

In our investigations, we apply some results concerning properties of the quasi-deviation means, proved by Páles [4].

References

[1] H. Bühlmann, *Mathematical Models in Risk Theory*, Springer-Verlag, Berlin, 1970.

[2] J. Chudziak, *On applications of inequalities for quasideviation means in actuarial mathematics*, Math. Inequal. Appl. **21** (2018), no. 3, 601–610.

[3] J. Chudziak, *On quasi-convexity of the zero utility principle*, J. Nonlinear Convex Anal. **19** (2018), no. 5, 749–758.

[4] Zs. Páles, *General inequalities for quasideviation means*, Aequationes Math. **36** (1988), no. 1, 32–56.

Attila Gilányi: *Determining types of functional equations with computer* (Joint work with Lan Nhi To)

Nowadays, computer assisted investigations play an increasingly important role connected to studies of functional equations, inequalities and related topics (cf., e.g., the papers [1], [2], [3], [4] and the references therein). In this talk, we present a package of computer programs developed in the computer algebra system MAPLE, which is able to decide about certain functional equations to which class of functional equations they belong and (if applicable) it can determine their type as well.

References

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[2] A. Gilányi and L.N. To, *Computer assisted investigation of alienness of linear functional equations*, Aequationes Math. **97** (2023), no. 5–6, 1185–1199.
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Richárd Grünwald: *Properties of the set of solutions of the global comparison problem of Gini means* (Joint work with Zsolt Páles)

Let us recall the definition of the *n-variable Gini mean corresponding to the pair of parameters* $(p,q) \in \mathbb{R}^2$:

$$G^{[n]}_{p,q}(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{x_1^p+\dots+x_n^p}{x_1^q+\dots+x_n^q}\right)^{\frac{1}{p-q}} & \text{if } p\neq q,\\[2ex] \exp\left(\dfrac{x_1^p\ln(x_1)+\dots+x_n^p\ln(x_n)}{x_1^p+\dots+x_n^p}\right) & \text{if } p=q, \end{cases}$$

$x_1,\dots,x_n \in \mathbb{R}_+$.

Let us consider the global comparison problem of Gini means with fixed number of variables in a subinterval I of $\mathbb{R}_+$, i.e., the following inequality

$$G^{[n]}_{r,s}(x_1,\dots,x_n) \le G^{[n]}_{p,q}(x_1,\dots,x_n), \tag{1}$$

where $n \in \mathbb{N}, n \ge 2$ is fixed, $(p,q),(r,s) \in \mathbb{R}^2$ and $x_1,\dots,x_n \in I$.

Given a nonempty subinterval I of $\mathbb{R}_+$ and $n \in \mathbb{N}$, we introduce the sets

$$\Gamma_n(I) := \{((r,s),(p,q)) \in \mathbb{R}^2\times\mathbb{R}^2 \mid (1) \text{ holds for all } x_1,\dots,x_n \in I\},$$
$$\Gamma_\infty(I) := \bigcap_{n=1}^{\infty} \Gamma_n(I).$$

In the talk, we investigate the properties of these sets and their relationship to each other.

References

[1] Z. Daróczy and L. Losonczi, *Über den Vergleich von Mittelwerten*, Publ. Math. Debrecen **17** (1970), 289–297.
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Eszter Gselmann: *A characterization of differential operators in the ring of complex polynomials* (Joint work with Włodzimierz Fechner)

This talk aims to provide a full characterization of all operators $T\colon \mathscr{P}(\mathbb{C}) \to \mathscr{P}(\mathbb{C})$ acting on the space of all complex polynomials that satisfy the Leibniz rule

$$T(f \cdot g) = T(f) \cdot g + f \cdot T(g)$$

for all $f, g \in \mathscr{P}(\mathbb{C})$. We do not assume the linearity of T. As we will see, contrary to the well-known theorems for function spaces there are many other solutions here, not only differential operators. From our main result, we also derive two corollaries, showing that in some special cases operators that satisfy the Leibniz rule have some particular form.

Mehak Iqbal: *Quadratic functions as solutions of polynomial equations* (Joint work with Eszter Gselmann)

Polynomial equations play a significant role in algebra and the theory of functional equations. If the unknown functions in the equation are additive, relatively many results are known. In some specific cases, according to classical results, the unknown additive functions are homomorphisms, derivations, or linear combinations of these. Now the question arises whether the solutions can be described even if the unknown functions are not assumed to be additive but to be generalized monomials. As a starting point, we will deal with generalized monomials of degree two, that is, with quadratic functions. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield of $\mathbb{K}$. Our main objective is to determine all those quadratic functions $q\colon \mathbb{F} \to \mathbb{K}$ that satisfy a Levi–Civita equation on the multiplicative structure, i.e., that can be written as

$$q(xy) = \sum_{i=1}^{k} g_i(x)h_i(y) \quad \left(x, y \in \mathbb{F}^{\times}\right)$$

with some positive integer k and with some appropriate functions g_i, h_i, $i = 1, \dots, k$. For this, those quadratic functions q that satisfy the equations

$$q(xy) = q(x)q(y) \quad \left(x, y \in \mathbb{F}^{\times}\right) \quad \text{and} \quad q(xy) = x^2 q(y) + q(x)y^2 \quad \left(x, y \in \mathbb{F}^{\times}\right),$$

respectively, must first be determined.

Justyna Jarczyk: *Characterization of complex-valued exponential functions via an iterative functional equation* (Joint work with Witold Jarczyk)

Fix a positive integer $n \geq 2$ and a number $a \in (0, +\infty]$. Let $f_1, \dots, f_n$ be selfmappings of the interval $(0, a)$ summing up to the identity function:

$$\sum_{j=1}^{n} f_j(x) = x, \quad x \in (0, a).$$

Given an $(n-1)$-th root $\omega \in \mathbb{C}$ of unity and a complex number c, and defining $\psi_{\omega,c} : (0, a) \to \mathbb{C}$ by $\psi_{\omega,c}(x) = \omega \exp(cx)$, we see that

$$\begin{aligned} \psi_{\omega,c}(x) &= \omega \exp\left(c \sum_{j=1}^{n} f_j(x)\right) = \omega \prod_{j=1}^{n} \exp\left(c f_j(x)\right) \\ &= \frac{\omega}{\omega^n} \prod_{j=1}^{n} \psi_{\omega,c}\left(f_j(x)\right) = \prod_{j=1}^{n} \psi_{\omega,c}\left(f_j(x)\right) \end{aligned}$$

for all $x \in (0, a)$. Therefore $\psi_{\omega,c}$ satisfies the functional equation

$$\psi(x) = \prod_{j=1}^{n} \psi\left(f_j(x)\right).$$

During the talk we prove that under some assumptions also the converse is true.

Witold Jarczyk: *Extension theorem for simultaneous q-difference equations and some its consequences* (Joint work with Paweł Pasteczka)

Given a set $T \subset (0, +\infty)$, intervals $I \subset (0, +\infty)$ and $J \subset \mathbb{R}$, as well as functions $g_t \colon I \times J \to J$ with t's running through the set

$$T^* := T \cup \left\{t^{-1} \colon t \in T\right\} \cup \{1\}$$

we study the simultaneous q-difference equations

$$\varphi(tx) = g_t\left(x, \varphi(x)\right), \quad t \in T^*,$$

postulated for $x \in I \cap t^{-1} I$; here the unknown function φ is assumed to map I into J. We present an extension theorem stating that if φ is continuous [analytic] on a nontrivial subinterval of I, then φ is continuous [analytic] provided $g_t, t \in T^*$, are continuous [analytic]. The crucial assumption of the extension theorem is formulated with the help of the so-called limit ratio R_T

which is a uniquely determined number from $[1,+\infty]$, characterising some density property of the set T^*. As an application of the extension theorem we find the form of all continuous on a subinterval of I solutions $\varphi\colon I \to \mathbb{R}$ of the simultaneous equations

$$\varphi(tx) = \varphi(x) + c(t)x^p, \quad t \in T,$$

where $c\colon T \to \mathbb{R}$ is an arbitrary function, p is a given real number and $\sup I > R_T \inf I$.

Tibor Kiss: *On a non-symmetric version of the drop theorem*

As is widely known, the convex hull of the union of a convex subset and a point of a linear space equals to the union of the segments starting at the given point and ending in the set in question. This result is called the *drop theorem*. In the talk, we restrict ourselves to the real number line and deal with a variant of this result.

For a fixed parameter $t \in [0,1]$, we say that a subset $D \subseteq \mathbb{R}$ is *non-symmetrically t-convex* if $tx + (1-t)y \in D$ whenever $x, y \in D$ with $y \leq x$. To avoid the trivial cases, we also assume that $t \notin \{0, \frac{1}{2}, 1\}$.

In the talk, we give a sufficient condition under which the non-symmetric t-convex hull of a non-symmetric t-convex segment and a point outside it can be represented in the way detailed above.

References

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[2] K. Nikodem and Zs. Páles, *Note on t-quasiaffine functions*, Ann. Univ. Sci. Budapest. Sect. Comput. **29** (2008), 127–139.

Radosław Łukasik: *Definition and properties of a fuzzy Xor*

In this talk, we show that the fuzzy Xor defined in [1] cannot have some properties presented in that paper. We also provide new constructions of fuzzy Xor based on the composition of other fuzzy connectives.

Reference

[1] B.C. Bedregal, R.H.S. Reiser, and G.P. Dimuro, *Revisiting Xor-implications: classes of fuzzy (co)implications based on f-Xor (f-XNor) connectives*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems **21** (2013), no. 6, 899–925.

Rayene Menzer: *An alternative equation for polynomial functions on locally compact Abelian groups* (Joint work with Zoltán Boros)

In our presentation we establish the following result:

Theorem. *Let G be a locally compact Abelian group which is generated by any neighborhood of zero. Let μ denote the Haar measure on G, and let us assume that μ is σ-finite. Let $f\colon G \to \mathbb{C}$ be a generalized polynomial fulfilling*

$$f(x)f(y) = 0 \tag{1}$$

for all $(x, y) \in D$, where $D \subseteq G^2$ is a $\mu \times \mu$ measurable subset with positive measure. Then $f(x) = 0$ for every $x \in G$.

This research is motivated by the particular case in [3] when $G = \mathbb{R}^k$ for some natural number k and f is additive, as well as by similar investigations in [1] for real generalized polynomials with a particular algebraic constraint (namely, when D is the unit circle). The main tool is provided by Székelyhidi's results [4] on the zeros of generalized polynomials in an abstract setting. A particular case of our main theorem is in [2].

References

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Gábor Marcell Molnár: *On approximate convexity* (Joint work with Zsolt Páles)

Let X be a real linear space, $D \subseteq X$ nonempty, convex and $D_\Delta := \{x - y : x, y \in D\}$. Let $\varphi\colon \frac{1}{2}D_\Delta \to \mathbb{R}$ be a given function, called an *error function.* We say that a function $f\colon D \to \mathbb{R}$ is *φ-Jensen convex on D* (or *φ-midconvex on D*) if

$$f\Big(\frac{x+y}{2}\Big) \le \frac{1}{2}f(x) + \frac{1}{2}f(y) + \varphi\Big(\frac{x-y}{2}\Big) \quad (x, y \in D).$$

The basic problem related to a φ-Jensen convex function $f\colon D \to \mathbb{R}$ is to deduce further approximate convexity properties.

In the talk, we present an approach to finding approximate convexity properties of a φ-Jensen convex function.

Reference

[1] J. Makó and Zs. Páles, *On φ-convexity*, Publ. Math. Debrecen **80** (2012), no. 1–2, 107–126.

Gergő Nagy: *Points of operator convexity of functions on operator algebras*

In 2010, Silvestrov, Osaka and Tomiyama verified that a C^*-algebra $\mathcal{A}$ is commutative exactly when there exists a continuous function $f\colon [0,\infty[\to \mathbb{R}$ which is not convex on the set of all positive semidefinite 2×2 matrices but convex on the collection of all positive elements in $\mathcal{A}$, i.e. $\mathcal{A}$-convex. As a local version of this theorem, Virosztek showed that in certain cases, the "points of operator convexity" of convex, but not $\mathcal{A}$-convex functions are precisely the central elements of the algebra. In the talk, after a brief overview of some related former results, we present the following generalization of this statement. If $D\subset\mathbb{R}$ is an open interval and $f\in\mathcal{C}^2(D)$ is a convex function satisfying a certain technical condition, and $a\in\mathcal{A}$ is a self-adjoint element, then a is central if and only if it is a point of operator convexity of f.

Andrzej Olbryś: *On approximate convexity*

Let D be a convex subset of a real linear space X. In this talk, we examine the properties of functions $f: D\to\mathbb{R}$ satisfying the inequality

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)+\phi(t(x-y))-t\phi(x-y),$$

for all $x,y\in D,\ t\in[0,1]$, where $\phi\colon X\to\mathbb{R}$ is a given function.

Zsolt Páles: *Taylor-type theorems with respect to Chebyshev systems*

The aim of the talk is to present an exact error formula for the Taylor-type interpolation of smooth functions in terms of Chebyshev systems. The main tool to achieve this goal is the following easy-to-prove result:

Theorem. *Let $I\subseteq\mathbb{R}$ be a nondegenerate interval and let $A\colon I\to\mathbb{R}^{n\times n}$ be a continuous matrix-valued function. Assume that $Y\colon I\to\mathbb{R}^{n\times n}$ is a matrix-valued solution of the linear differential equation*

$$Y'(x)=A(x)Y(x)\quad (x\in I)$$

such that $Y(x)$ is nonsingular for all $x \in I$. Then, for all continuously differentiable functions $f\colon I \to \mathbb{R}^n$ and for all $a, x \in I$, the equality

$$f(x) = Y(x)\Big(Y^{-1}(a)f(a) + \int_a^x Y^{-1}(t)\big(f'(t) - A(t)f(t)\big)dt\Big)$$

holds.

Paweł Pasteczka: *Multivariable generalizations of bivariate means via invariance*

For a given p-variable mean $M\colon I^p \to I$ (I is a subinterval of $\mathbb{R}$), following Horwitz and Lawson–Lim, we can define (under certain assumption) its $(p+1)$-variable β-invariant extension as the unique solution $K\colon I^{p+1} \to I$ of the functional equation

$$K\big(M(x_2,\dots,x_{p+1}), M(x_1,x_3,\dots,x_{p+1}),\dots,M(x_1,\dots,x_p)\big) = K(x_1,\dots,x_{p+1}), \quad \text{for all } x_1,\dots,x_{p+1} \in I$$

in the family of means.

Applying this procedure iteratively we can obtain a mean which is defined for vectors of arbitrary lengths starting from the bivariate one. The aim of this talk is to study the properties of such extensions.

References

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Patryk Rela: *The Orlicz premium principle under uncertainty* (Joint work with Jacek Chudziak)

Under the expected utility model the Orlicz premium principle for a risk X, represented by a non-negative essentially bounded random variable on a given probability space, is defined implicitly, as a unique solution $H_{\alpha,\Phi}(X)$ of the equation

$$E\left[\Phi\left(\frac{X}{H_{\alpha,\Phi}(X)}\right)\right] = 1 - \alpha, \tag{1}$$

where $\alpha \in [0,1)$ is a given parameter and $\Phi\colon [0,\infty) \to [0,\infty)$ is a normalized Young function, that is a strictly increasing, convex function $\Phi\colon [0,\infty) \to [0,\infty)$ satisfying $\Phi(0) = 0$, $\Phi(1) = 1$ and $\lim_{x\to\infty} \Phi(x) = \infty$. The Orlicz premium in this setting has been introduced by [2]. Several details concerning properties of the premium defined by (1) can by found in [1].

In order to define the Orlicz premium principle under uncertainty, assume that $(\Omega, \mathcal{F})$ is a measurable space and $\mu\colon \mathcal{F} \to [0,1]$ is a capacity, that is a monotone set function satisfying $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Let $\mathcal{X}_+$ be a family of all $\mathcal{F}$-measurable functions $X\colon \Omega \to [0,\infty)$ such that $\mu(\{X > t\}) = 0$ for some $t \in \mathbb{R}$. The premium for $X \in \mathcal{X}_+$ is defined through the equation

$$E_\mu \left[\Phi \left(\frac{X}{H_{\mu,\alpha,\Phi}(X)} \right) \right] = 1 - \alpha, \tag{2}$$

where $\alpha \in [0,1)$, $\Phi\colon [0,\infty) \to [0,\infty)$ is a normalized Young function and

$$E_\mu [X] = \int_0^\infty \mu(\{X > x\}) dx \quad \text{for } X \in \mathcal{X}_+,$$

is the Choquet integral with respect to the capacity μ.

The aim of this talk is to prove the existence and uniqueness of the Orlicz premium defined by (2) and to characterize its several important properties.

References

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Maciej Sablik: *Generalized discount factors*

We consider the so called generalized discount factors, i.e. nonincreasing functions $\phi\colon \mathbb{N} \to [0,1]$, satisfying $\phi(0) = 1$ and $\phi(n+k) \geq \phi(n)\phi(k)$, for $n,k \in \mathbb{N}$. Typical example is a generalized *hyperbolic discount factor* given by $\phi(i) = (1+hi)^{-\frac{r}{h}}$ with $h > 0$, $r > 0$ and $\frac{r}{h} \leq 1$. The discount factors appear in the problems of long run stochastic control (see e.g. Łukasz Stettner [1]).

Reference

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Justyna Sikorska: *On a characterization of the logarithmic mean* (Joint work with Timothy Nadhomi and Maciej Sablik)

Let $f\colon I \to \mathbb{R}$ and φ be an increasing function defined on the range of f.

The function f is said to be φ-convex whenever $\varphi \circ f$ is convex, that is, for all $x, y \in I$, $t \in [0,1]$,

$$\varphi(f(tx + (1-t)y)) \le t\varphi(f(x)) + (1-t)\varphi(f(y)),$$

and if φ is one-to-one,

$$f(tx + (1-t)y) \le \varphi^{-1}(t\varphi(f(x)) + (1-t)\varphi(f(y))).$$

Starting from the celebrated Hermite–Hadamard inequality for φ-convex functions, we give some characterization of the logarithmic mean.

László Székelyhidi: *On the Spectral Synthesis Theorem of Laurent Schwartz*

In this talk, we present a short proof for L. Schwartz's fundamental spectral synthesis theorem on the reals. The proof is based on our localization method and on the spectral analysis result proved by J. P. Kahane using the Carleman transform.

References

[1] J.-P. Kahane, *Lectures on Mean Periodic Functions*, Tata Institute of Fundamental Research, Bombay, 1959.
[2] L. Schwartz, *Théorie générale des fonctions moyenne-périodiques*, Ann. of Math. (2) **48** (1947), 857–929.

Patrícia Szokol: *Some results and open questions on quasi-arithmetic means* (Joint work with Pál Burai and Gergely Kiss)

In this presentation, we focus the characterization theorem of János Aczél on quasi-arithmetic means. In his proof, continuity is used essentially but a little bit furtively. We show that every bisymmetric, symmetric, reflexive, strictly monotonic binary map on a proper interval is continuous, in particular it is a quasi-arithmetic mean. Furthermore, we present some remarkable consequences of the previous result. We demonstrate that this result can be refined in the way that the symmetry condition can be weakened by assuming symmetry only for a pair of distinct points of an interval. Finally, concerning the obtained results we present some open questions.

Tomasz Szostok: *Inequalities for 2-convex functions involving signed measures* (Joint work with Constantin P. Niculescu)

We present some remarks concerning problems posed in [1].

Reference

[1] D.-Ş. Marinescu and C.P. Niculescu, *Old and new on the* 3*-convex functions*, arXiv preprint, 2023. Available at arXiv: 2305.04353v1.

Lan Nhi To: *Computer assisted investigation of Levi–Civita type functional equations* (Joint work with Attila Gilányi)

We consider Levi–Civita type functional equations

$$f(x+y)=\sum_{i=1}^{n} g_i(x)h_i(y), \tag{1}$$

where n is a positive integer, G is an Abelian group and $f, g_i, h_i\colon G \to \mathbb{C}$ $(i = 1, 2, \dots, n)$ are unknown functions.

Based on results by László Székelyhidi ([1]), we developed a computer program (written in the computer algebra system Maple) for determining the solution of functional equations of type (1).

In this talk, we present the Maple function with some demo examples of well-known Levi–Civita type functional equations and also in some cases when the right-hand side of the input functional equation contains many terms.

Reference

[1] L. Székelyhidi, *On the Levi–Civita functional equation*, Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum, 301, Forschungszentrum Graz, Mathematisch-Statistische Sektion, Graz, 1988, 23 pp.

Norbert Tóth: *The coincidence set of generalized monotone functions* (Joint work with Mihály Bessenyei)

In a recent paper, Fu and Solow prove that the set of zeros of a convex function is either an interval or a finite set of at most two elements. Motivated by their result, we investigate the coincidence set of generalized lines and generalized convex functions, when the underlying notion is induced by an n-parameter Beckenbach family. It turns out that the situation in the extended context is quite similar to that of Fu and Solow: The coincidence set is either an interval or a finite set of at most n elements. Moreover, we show that the coincidence set can have k elements if $k \in [1, n] \cap \mathbb{N}$ and the family is an extended and complete Chebyshev-system.

Reference

[1] D. Solow and F. Fu, *On the roots of convex functions*, J. Convex Anal. **30** (2023), no. 1, 143–157.

Péter Tóth: *On measurable solutions of an alternative functional equation*

Let I_1, I_2 be nonempty open intervals of the real line, and let $J := \frac{1}{2}(I_1 + I_2)$. The solutions of the functional equation

$$\varphi\left(\frac{x+y}{2}\right)(\psi_1(x) - \psi_2(y)) = 0 \quad (\text{for all } x \in I_1 \text{ and } y \in I_2) \tag{1}$$

where the functions $\psi_1 \colon I_1 \longrightarrow \mathbb{R}$, $\psi_2 \colon I_2 \longrightarrow \mathbb{R}$ and $\varphi \colon J \longrightarrow \mathbb{R}$ are unknown, were described by T. Kiss [1]. It has been established that if $\varphi^{-1}(0)$ is closed then the nontrivial solutions of (1) are constant on some open subintervals of their domain.

During the Problems and Remarks session of the 59th International Symposium on Functional Equations, Kiss proposed the following question (see [2]). Does the mentioned characterization of the solutions of (1) remain valid when the Darboux property is assumed for φ, instead of the closedness of $\varphi^{-1}(0)$? This is motivated by the fact that in certain applications (such as the invariance problem of generalized weighted quasi-arithmetic means) the functions appearing in (1) are derivatives, for which the set of zeros might not be closed.

In our talk, we present that unfortunately (1) has such nontrivial solutions $(\psi_1, \psi_2, \varphi)$ which are Darboux, yet neither function is constant on any open subinterval. On the other hand, we will show that if φ is measurable then an analogous version of the known characterization theorem for the solutions holds. Hence, if φ is supposed to be the derivative of a differentiable function, then (1) has exactly the same solutions as described in [1, Theorem 6], which was desired for the applications.

References

[1] T. Kiss, *A Pexider equation containing the arithmetic mean*, Aequationes Math. (2023). DOI: 10.1007/s00010-023-00966-x.

[2] *Report of Meeting. The 59th International Symposium on Functional Equations, Hotel Aurum, Hajdúszoboszló (Hungary), June 18–25, 2023*, Aequationes Math. **97** (2023), no. 5–6, 1259–1290.

Paweł Wójcik: *On an orthogonality equation in finite-dimensional normed spaces* (Joint work with Karol Gryszka)

Let X, Y be real normed spaces and let ρ'_+, ρ'_- be norm derivatives. In this talk we consider a system of functional equations

$$\forall_{x,y\in X} \quad \begin{cases} \rho'_+\big(f(x), f(y)\big) = g(x)\rho'_+(x,y), \\ \rho'_-\big(f(x), f(y)\big) = g(x)\rho'_-(x,y), \end{cases}$$

with unknown functions $f\colon X\to Y$, $g\colon X\to\mathbb{R}$. As a consequence, we present partial answer to open problem posed in [2].

References

[1] C. Alsina, J. Sikorska, and M.S. Tomás, *Norm Derivatives and Characterizations of Inner Product Spaces*, World Scientific, Hackensack, NJ, 2010.
[2] K. Gryszka and P. Wójcik, *Generalized orthogonality equations in finite-dimensional normed spaces*, Ann. Funct. Anal. **14** (2023), no. 2, Paper No. 41, 13 pp.

Sebastian Wójcik: *Comonotonic additivity of the zero utility principle under uncertainty* (Joint work with Jacek Chudziak)

In a process of insurance contracts pricing, the insurance company assigns to any risk a non-negative real number, being a premium for the risk. There are various methods of insurance contracts pricing. In this talk we deal with a method, called the zero utility principle, introduced by H. Bühlmann (1970). This method presents the problem from the point of view of an insurance company, assuming that the premium for a given risk is determined in such a way that the company is indifferent between entering into contract and rejecting it.

We study the zero utility principle in the cumulative prospect theory (Tversky, Kahneman (1992)) under uncertainty. In this setting, the risks are represented by measurable functions defined on a given measurable space $(S,\mathbb{F})$. A premium for a risk X is defined as a unique real number $H_{(u,\mu,\nu)}(X)$ satisfying equation

$$E_{\mu\nu}[u(H_{(u,\mu,\nu)}(X)-X)]=0,$$

where $u\colon\mathbb{R}\to\mathbb{R}$ is a strictly increasing continuous function such that $u(0)=0$ and $E_{\mu\nu}$ is the Choquet integral with respect to a pair of capacities (μ,ν). Our main aim is to characterize comonotonic additivity of the principle. Recall that risks X and Y are comonotonic provided

$$(X(s_1)-X(s_2))(Y(s_1)-Y(s_2))\geq 0 \quad \text{for } s_1,s_2\in S.$$

The premium is called additive for comonotonic risks if

$$H_{(u,\mu,\nu)}(X+Y)=H_{(u,\mu,\nu)}(X)+H_{(u,\mu,\nu)}(Y)$$

for any pair of comonotonic risks X and Y.

References

[1] H. Bühlmann, *Mathematical Models in Risk Theory*, Springer-Verlag, Berlin, 1970.
[2] J. Dhaene, M. Denuit, M.J. Goovaerts, R. Kaas, and D. Vyncke, *The concept of comonotonicity in actuarial science and finance: theory*, Insur. Math. Econ. **31** (2002), no. 1, 3–33.
[3] A. Tversky and D. Kahneman, *Advances in prospect theory: Cumulative representation of uncertainty*, J. Risk Uncertain. **5** (1992), no. 4, 297–323.

2. Problems and Remarks

Remark about bivariate means. Recently Tomasz Małolepszy, from the University of Zielona Góra, has answered the following question in the negative:

Is it true that any bivariate mean separately continuous, is continuous?

The problem was posed by Justyna Jarczyk and Witold Jarczyk a couple of years ago (during the Katowice–Debrecen Winter Seminar).

Witold Jarczyk

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