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A KANNAPPAN-COSINE FUNCTIONAL EQUATION ON SEMIGROUPS

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Abstract. In this paper we determine the complex-valued solutions of the Kannappan-cosine functional equation $g(xyz_0) = g(x)g(y) - f(x)f(y)$, $x, y \in S$, where S is a semigroup and z_0 is a fixed element in S.

1. Introduction

The addition law for cosine is

$$\cos(x + y) = \cos(x)\,\cos(y) - \sin(x)\,\sin(y), \quad x, y \in \mathbb{R}.$$

This gives the origin of the following functional equation on any semigroup S:

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in S, \tag{1.1}$$

for the unknown functions $f, g\colon S \to \mathbb{C}$, which is called the cosine addition law. In Aczél's monograph [1, Section 3.2.3] we find continuous real valued solutions of (1.1) in case $S = \mathbb{R}$.

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The functional equation (1.1) has been solved on groups by Poulsen and Stetkær [10], on semigroups generated by their squares by Ajebbar and Elqorachi [3], and recently by Ebanks [5] on semigroups.

In [12, Theorem 3.1], Stetkær solved the following functional equation

$$g(xy) = g(x)g(y) - f(y)f(x) + \alpha f(xy), \quad x, y \in S, \tag{1.2}$$

where α is a fixed constant in $\mathbb{C}$. He expressed the solutions in terms of multiplicative functions and the solution of the special case of the sine addition law. In [13, Proposition 16], he solved the functional equation

$$f(xyz_0) = f(x)f(y), \quad x, y \in S, \tag{1.3}$$

on semigroups, and where z_0 is a fixed element in S. We shall use these results in our computations.

In this paper, we deal with the following Kannappan-cosine addition law

$$g(xyz_0) = g(x)g(y) - f(x)f(y), \quad x, y \in S, \tag{1.4}$$

on a semigroup S. The functional equation (1.4) is called Kannappan functional equation because it brings up a fixed element z_0 in S as in the paper of Kannappan [9].

In the special case, where $\{f, g\}$ is linearly dependent and $g \neq 0$, we get that there exists a constant $\lambda \in \mathbb{C}$ such that the function $(1 - \lambda^2)g$ satisfies the functional equation (1.3).

If S is a monoid with an identity element e, and $f(e) = 0$ and $g(e) \neq 0$, or $g(e) = 0$ and $f(e) \neq 0$, the last functional equation is the cosine addition law which was solved recently on general semigroups by Ebanks [5].

Now, if $\alpha := f(e) \neq 0$ and $\beta := g(e) \neq 0$ we get that the pair $\left(\frac{g}{\beta}, \frac{f}{\beta}\right)$ satisfies the following functional equation

$$\frac{g}{\beta}(xy) = \frac{g}{\beta}(x)\frac{g}{\beta}(y) - \frac{f}{\beta}(x)\frac{f}{\beta}(y) + \frac{\alpha}{\beta}\frac{f}{\beta}(xy),$$

which is of the form (1.2), and then explicit formulas for f and g on groups exist in the literature (see for example [8, Corollary 3.2.]).

The natural general setting of the functional equation (1.4) is for S being a semigroup, because the formulation of (1.4) requires only an associative composition in S, not an identity element and inverses. Thus we study in the present paper Kannappan-cosine functional equation (1.4) on semigroups S, generalizing previous works in which S is a group. So, the result of the present paper is a natural continuation of results contained in the literature.

The purpose of the present paper is to show how the relations between (1.4) and (1.2)–(1.3) on monoids extend to much wider framework, in which S is a semigroup. We find explicit formulas for the solutions, expressing them in terms of homomorphisms and additive maps from a semigroup into $\mathbb{C}$ (Theorem 4.1). The continuous solutions on topological semigroups are also found.

2. Set up, notations and terminology

Throughout this paper, S is a semigroup (a set with an associative composition) and z_0 is a fixed element in S. If S is topological, we denote by $\mathcal{C}(S)$ the algebra of continuous functions from S to the field of complex numbers $\mathbb{C}$.

Let $f\colon S \to \mathbb{C}$ be a function. We say that f is central if $f(xy) = f(yx)$ for all $x, y \in S$, and that f is abelian if $f(x_1x_2, \dots, x_n) = f(x_{\sigma(1)}x_{\sigma(2)}, \dots, x_{\sigma(n)})$ for all $x_1, x_2, \dots, x_n \in S$, all permutations σ of n elements and all $n \in \mathbb{N}$. A map $A\colon S \to \mathbb{C}$ is said to be additive if $A(xy) = A(x) + A(y)$, for all $x, y \in S$ and a map $\chi\colon S \to \mathbb{C}$ is multiplicative if $\chi(xy) = \chi(x)\chi(y)$, for all $x, y \in S$. If $\chi \neq 0$, then the nullspace $I_\chi := \{x \in S \,|\, \chi(x) = 0\}$ is either empty or a proper subset of S and I_χ is a two sided ideal in S if not empty and $S \setminus I_\chi$ is a subsemigroup of S. Note that additive and multiplicative functions are abelian.

For any subset $T \subseteq S$ let $T^2 := \{xy \,|\, x, y \in T\}$ and for any fixed element z_0 in S we let $T^2z_0 := \{xyz_0 \,|\, x, y \in T\}$.

To express solutions of our functional equations studied in this paper we will use the set $P_\chi := \{p \in I_\chi \setminus I_\chi^2 \mid up, pv, upv \in I_\chi \setminus I_\chi^2 \text{ for all } u, v \in S \setminus I_\chi\}$. For more details about P_χ we refer the reader to [4], [5] and [6].

3. Preliminaries

In this section, we give useful results to solve the functional equation (1.4).

Lemma 3.1. *Let S be a semigroup, $n \in \mathbb{N}$, and $\chi, \chi_1, \chi_2, \dots, \chi_n\colon S \to \mathbb{C}$ be different non-zero multiplicative functions. Then*

(a) *$\{\chi_1, \chi_2, \cdots, \chi_n\}$ is linearly independent.*

(b) *If $A\colon S \setminus I_\chi \to \mathbb{C}$ is a non-zero additive function, then the set $\{\chi A, \chi\}$ is linearly independent on $S \setminus I_\chi$.*

Proof. (a) See [11, Theorem 3.18]. (b) See [2, Lemma 4.4]. □

The proposition below gives the solutions of the functional equation

$$f(xyz_0) = \chi(z_0)f(x)\chi(y) + \chi(z_0)f(y)\chi(x), \quad x, y \in S. \tag{3.1}$$

PROPOSITION 3.2. *Let S be a semigroup, and $\chi\colon S \to \mathbb{C}$ be a multiplicative function such that $\chi(z_0) \neq 0$. If $f\colon S \to \mathbb{C}$ is a solution of (3.1), then*

$$f(x) = \begin{cases} \chi(x)(A(x) + A(z_0)) & \text{for } x \in S \backslash I_\chi, \\ \rho(x) & \text{for } x \in P_\chi, \\ 0 & \text{for } x \in I_\chi \setminus P_\chi, \end{cases} \tag{3.2}$$

where $A\colon S\setminus I_\chi \to \mathbb{C}$ is additive and $\rho\colon P_\chi \to \mathbb{C}$ is the restriction of f to P_χ. In addition, f is abelian and satisfies the following conditions:

(I) *$f(xy) = f(yx) = 0$ for all $x \in I_\chi \setminus P_\chi$ and $y \in S \setminus I_\chi$.*

(II) *If $x \in \{up, pv, upv\}$ with $p \in P_\chi$ and $u, v \in S \setminus I_\chi$, then $x \in P_\chi$ and we have respectively $\rho(x) = \rho(p)\chi(u)$, $\rho(x) = \rho(p)\chi(v)$ or $\rho(x) = \rho(p)\chi(uv)$.*

Conversely, the function f of the form (3.2) define a solution of (3.1). Moreover, if S is a topological semigroup and $f \in \mathcal{C}(S)$, then $\chi \in \mathcal{C}(S)$, $A \in \mathcal{C}(S \setminus I_\chi)$ and $\rho \in \mathcal{C}(P_\chi)$.

PROOF. See [7, Proposition 4.3]. □

To shorten the way to finding the solutions of functional equation (1.4), we prove the following lemma that contains some key properties.

LEMMA 3.3. *Let S be a semigroup and let $f, g\colon S \to \mathbb{C}$ be the solutions of the functional equation (1.4) with $g \neq 0$. Then*

(i) *If $f(z_0) = 0$ then*

(1) *for all $x, y \in S$,*

$$g(z_0^2)g(xy) = g(z_0)[g(x)g(y) - f(x)f(y)] + f(z_0^2)f(xy), \tag{3.3}$$

(2) *$g(z_0^2)^2 = g(z_0)^3 + f(z_0^2)^2$.*

(3) *If f and g are linearly independent then $g(z_0) \neq 0$.*

(ii) *If $f(z_0) \neq 0$, then there exists $\mu \in \mathbb{C}$ such that*

$$f(xyz_0) = f(x)g(y) + f(y)g(x) + \mu f(x)f(y), \quad x, y \in S. \tag{3.4}$$

Proof. (i) Suppose that $f(z_0)=0$.

(1) Making the substitutions (xy, z_0^2) and (xyz_0, z_0) in (1.4) we get $g(xyz_0^3)=g(xy)g(z_0^2)-f(xy)f(z_0^2)$ and $g(xyz_0^3)=g(xyz_0)g(z_0)-f(xyz_0)f(z_0)$ $=g(z_0)g(x)g(y)-g(z_0)f(x)f(y)$, respectively. Comparing these expressions, we deduce that $g(xy)g(z_0^2)-f(z_0^2)f(xy)=g(z_0)g(x)g(y)-g(z_0)f(y)f(x)$. This proves the desired identity.

(2) It follows directly by putting $x=y=z_0$ in the equation (3.3).

(3) For a contradiction we suppose that $g(z_0)=0$. Then using (1.4), we get $g(xyz_0^2)=g(x)g(yz_0)-f(x)f(yz_0)=g(xy)g(z_0)-f(xy)f(z_0)=0$ since $f(z_0)=g(z_0)=0$. Then we deduce that

$$g(x)g(yz_0)=f(x)f(yz_0), \quad x,y\in S. \tag{3.5}$$

If $g(yz_0)=0$ for all $y\in S$ then $0=g(xyz_0)=g(x)g(y)-f(x)f(y)$, $x,y\in S$. So, $g(x)g(y)=f(x)f(y)$, $x,y\in S$. Hence, $f=g$ or $f=-g$, which contradicts the fact that f and g are linearly independent. So $g\neq 0$ on Sz_0, and from (3.5) we get that $g=c_1 f$ with $c_1:=f(az_0)/g(az_0)$ for some $a\in S$ such that $g(az_0)\neq 0$. This is also a contradiction, since f and g are linearly independent. So we conclude that $g(z_0)\neq 0$.

(ii) Suppose that $f(z_0)\neq 0$. By the substitutions (x, yz_0^2) and (xyz_0, z_0) in (1.4) we get $g(xyz_0^3)=g(x)g(yz_0^2)-f(x)f(yz_0^2)=g(z_0)g(x)g(y)-g(x)f(z_0)f(y)-f(x)f(yz_0^2)$ and $g(xyz_0^3)=g(xyz_0)g(z_0)-f(xyz_0)f(z_0)=g(z_0)g(x)g(y)-g(z_0)f(x)f(y)-f(xyz_0)f(z_0)$, respectively. Then, by the associativity of the operation in S we obtain

$$\begin{aligned} f(z_0)[f(xyz_0)-f(x)g(y)-f(y)g(x)] \\ = f(x)[f(yz_0^2)-f(y)g(z_0)-f(z_0)g(y)]. \end{aligned} \tag{3.6}$$

Since $f(z_0)\neq 0$, dividing (3.6) by $f(z_0)$ we get

$$f(xyz_0)=f(x)g(y)+f(y)g(x)+f(x)\psi(y), \tag{3.7}$$

where $\psi(y):=f(z_0)^{-1}[f(yz_0^2)-f(y)g(z_0)-f(z_0)g(y)]$. Substituting (3.7) back into (3.6), we find out that $f(z_0)f(x)\psi(y)=f(x)f(y)\psi(z_0)$, which implies that $\psi(y)=\mu f(y)$ with $\mu:=\psi(z_0)/f(z_0)$. Therefore, (3.7) becomes $f(xyz_0)=f(x)g(y)+f(y)g(x)+\mu f(x)f(y)$. This completes the proof of Lemma 3.3. □

4. Main results

Now, we are ready to describe the solutions of the functional equation (1.4).

Let $\Psi_{A\chi,\rho}\colon S \to \mathbb{C}$ denote the function of the form in [6, Theorem 3.1 (B)], i.e.,

$$\Psi_{A\chi,\rho}(x) = \begin{cases} \chi(x)A(x) & \text{for } x \in S\backslash I_\chi, \\ \rho(x) & \text{for } x \in P_\chi, \\ 0 & \text{for } x \in I_\chi \setminus P_\chi, \end{cases}$$

where $\chi\colon S \to \mathbb{C}$ is a non-zero multiplicative function, $A\colon S \setminus I_\chi \to \mathbb{C}$ is additive, $\rho\colon P_\chi \to \mathbb{C}$ is the restriction of $\Psi_{A\chi,\rho}$, and the following conditions hold.

(i) $\Psi_{A\chi,\rho}(qt) = \Psi_{A\chi,\rho}(tq) = 0$ for all $q \in I_\chi$ and $t \in S \setminus I_\chi$.

(ii) If $x \in \{up, pv, upv\}$ for $p \in P_\chi$ and $u, v \in S\backslash I_\chi$, then $x \in P_\chi$ and we have $\rho(x) = \rho(p)\chi(u)$, $\rho(x) = \rho(p)\chi(v)$, or $\rho(x) = \rho(p)\chi(uv)$, respectively.

THEOREM 4.1. *The solutions $f, g\colon S \to \mathbb{C}$ of the functional equation* (1.4) *are the following pairs of functions.*

(1) *$f = g = 0$.*

(2) *$S \neq S^2 z_0$ and we have*

$$f = \pm g \quad \text{and} \quad g(x) = \begin{cases} g_{z_0}(x) & \text{for } x \in S \setminus S^2 z_0, \\ 0 & \text{for } x \in S^2 z_0, \end{cases}$$

where $g_{z_0}\colon S \setminus S^2 z_0 \to \mathbb{C}$ is an arbitrary non-zero function.

(3) *There exist a constant $d \in \mathbb{C} \setminus \{\pm 1\}$ and a multiplicative function χ on S with $\chi(z_0) \neq 0$, such that*

$$f = \frac{d\,\chi(z_0)}{1-d^2}\chi \quad \text{and} \quad g = \frac{\chi(z_0)}{1-d^2}\chi.$$

(4) *There exist a constant $c \in \mathbb{C}^* \setminus \{\pm i\}$ and two different multiplicative functions χ_1 and χ_2 on S, with $\chi_1(z_0) \neq 0$ and $\chi_2(z_0) \neq 0$ such that*

$$f = -\frac{\chi_1(z_0)\chi_1 - \chi_2(z_0)\chi_2}{i(c^{-1}+c)} \quad \text{and} \quad g = \frac{c^{-1}\chi_1(z_0)\chi_1 + c\chi_2(z_0)\chi_2}{c^{-1}+c}.$$

(5) *There exist constants* $q, \gamma \in \mathbb{C}^*$ *and two different non-zero multiplicative functions* χ_1 *and* χ_2 *on* S, *with*

$$\chi_1(z_0) = \frac{q^2 - (1+\xi)^2}{2\gamma q}, \quad \chi_2(z_0) = -\frac{q^2 - (1-\xi)^2}{2\gamma q},$$

and $\xi := \pm\sqrt{1+q^2}$ *such that*

$$f = \frac{\chi_1 + \chi_2}{2\gamma} + \xi\frac{\chi_1 - \chi_2}{2\gamma} \quad \textit{and} \quad g = q\frac{\chi_1 - \chi_2}{2\gamma}.$$

(6) *There exist constants* $q \in \mathbb{C} \setminus \{\pm\alpha\}$, $\gamma \in \mathbb{C}^* \setminus \{\pm\alpha\}$ *and* $\delta \in \mathbb{C} \setminus \{\pm 1\}$, *and two different non-zero multiplicative functions* χ_1 *and* χ_2 *on* S, *with*

$$\chi_1(z_0) = \frac{(1+\delta)^2 - (\alpha+q)^2}{2\gamma(1+\delta)}, \quad \chi_2(z_0) = \frac{(1-\delta)^2 - (\alpha-q)^2}{2\gamma(1-\delta)},$$

and $\delta := \pm\sqrt{1+q^2-\alpha^2}$ *such that*

$$f = \alpha\frac{\chi_1 + \chi_2}{2\gamma} + q\frac{\chi_1 - \chi_2}{2\gamma} \quad \textit{and} \quad g = \frac{\chi_1 + \chi_2}{2\gamma} + \delta\frac{\chi_1 - \chi_2}{2\gamma}.$$

(7) *There exist a constant* $\beta \in \mathbb{C}^*$, *a non-zero multiplicative function* χ *on* S, *an additive function* $A\colon S \setminus I_\chi \to \mathbb{C}$ *and a function* $\rho\colon P_\chi \to \mathbb{C}$ *with* $\chi(z_0) = 1/\beta$ *and* $A(z_0) = 0$ *such that*

$$f = \frac{1}{\beta}\Psi_{A\chi,\rho} \quad \textit{and} \quad g = \frac{1}{\beta}\left(\chi \pm \Psi_{A\chi,\rho}\right).$$

(8) *There exist a multiplicative function* χ *on* S *with* $\chi(z_0) \neq 0$, *an additive function* $A\colon S \setminus I_\chi \to \mathbb{C}$ *and a function* $\rho\colon P_\chi \to \mathbb{C}$ *such that*

$$f = A(z_0)\chi + \Psi_{A\chi,\rho} \quad \textit{and} \quad g = (\chi(z_0) \pm A(z_0))\,\chi + \Psi_{A\chi,\rho}.$$

Moreover, if S *is a topological semigroup and* $f \in \mathcal{C}(S)$ *then* $g \in \mathcal{C}(S)$ *in cases* (1), (2), (4)–(8), *and if* $d \neq 0$ *then also in* (3).

Proof. If $g = 0$, then (1.4) reduces to $f(x)f(y) = 0$ for all $x, y \in S$. This implies that $f = 0$, so we get the first part of solutions. From now we may assume that $g \neq 0$.

If f and g are linearly dependent, then there exists $d \in \mathbb{C}$ such that $f = dg$. Substituting this into (1.4) we get the following functional equation

$$g(xyz_0) = (1-d^2)g(x)g(y), \quad x, y \in S.$$

If $d^2 = 1$, then $g(xyz_0) = 0$ for all $x, y \in S$. Therefore, $S \neq S^2 z_0$ because $g \neq 0$. So, we are in solution family (2) with g_{z_0} an arbitrary non-zero function.

If $d^2 \neq 1$, then by [13, Proposition 16] there exists a multiplicative function χ on S such that $\chi(z_0)\chi := (1-d^2)g$ and $\chi(z_0) \neq 0$. Then we deduce that $g = \dfrac{\chi(z_0)}{1-d^2}\chi$ and $f = d\,g = \dfrac{d\,\chi(z_0)}{1-d^2}\chi$, so we have the solution family (3).

For the rest of the proof, we assume that f and g are linearly independent. We split the proof into two cases according to whether $f(z_0) = 0$ or $f(z_0) \neq 0$.

Case I. Suppose $f(z_0) = 0$. Then by Lemma 3.3 (i)-(3) and (i)-(1), we have $g(z_0) \neq 0$ and

$$g(z_0^2)g(xy) = g(z_0)g(x)g(y) - g(z_0)f(x)f(y) + f(z_0^2)f(xy), \quad x, y \in S, \tag{4.1}$$

respectively.

Subcase I.1. Assume that $g(z_0^2) = 0$. Then by Lemma 3.3 (i)-(2) and (i)-(3), we get $f(z_0^2) \neq 0$ since f and g are linearly independent and then (4.1) can be rewritten as $f(xy) = \gamma f(x)f(y) - \gamma g(x)g(y)$, $x, y \in S$, where $\gamma := \dfrac{g(z_0)}{f(z_0^2)} \neq 0$. Consequently, the pair $(\gamma f, \gamma g)$ satisfies the cosine addition formula (1.1). So, according to [12, Theorem 6.1] and taking into account that f and g are linearly independent, we know that there are only the following possibilities.

(I.1.i) There exist a constant $q \in \mathbb{C}^*$ and two different non-zero multiplicative functions χ_1 and χ_2 on S such that $\gamma g = q\dfrac{\chi_1 - \chi_2}{2}$ and $\gamma f = \dfrac{\chi_1 + \chi_2}{2} \pm \left(\sqrt{1+q^2}\right)\dfrac{\chi_1 - \chi_2}{2}$, which gives $f = \dfrac{\chi_1 + \chi_2}{2\gamma} \pm \left(\sqrt{1+q^2}\right)\dfrac{\chi_1 - \chi_2}{2\gamma}$ and $g = q\dfrac{\chi_1 - \chi_2}{2\gamma}$. By putting $\xi := \pm\sqrt{1+q^2}$ and using (1.4) we get

$$\begin{aligned}\frac{1}{4\gamma^2}\left(q^2 - (1+\xi)^2\right)\chi_1(xy) &+ \frac{1}{4\gamma^2}\left(q^2 - (1-\xi)^2\right)\chi_2(xy) \\ &= \frac{q}{2\gamma}\chi_1(z_0)\chi_1(xy) - \frac{q}{2\gamma}\chi_2(z_0)\chi_2(xy),\end{aligned}$$

which implies by Lemma 3.1 (i) that $\dfrac{q}{2\gamma}\chi_1(z_0) = \dfrac{1}{4\gamma^2}\left(q^2 - (1+\xi)^2\right)$ and $\dfrac{q}{2\gamma}\chi_2(z_0) = -\dfrac{1}{4\gamma^2}\left(q^2 - (1-\xi)^2\right)$, since χ_1 and χ_2 are different and non-zero. Then we deduce that

$$\chi_1(z_0) = \frac{1}{2\gamma q}\left(q^2 - (1+\xi)^2\right) \quad \text{and} \quad \chi_2(z_0) = -\frac{1}{2\gamma q}\left(q^2 - (1-\xi)^2\right).$$

So, we are in part (5).

(I.1.ii) There exist a non-zero multiplicative function χ on S, an additive function A on $S \setminus I_\chi$ and a function ρ on P_χ such that $\gamma g = \Psi_{A\chi,\rho}$ and $\gamma f = \chi \pm \Psi_{A\chi,\rho}$.

If $z_0 \in I_\chi \setminus P_\chi$ we have $\gamma g(z_0) = \Psi_{A\chi,\rho}(z_0) = 0$ by definition of $\Psi_{A\chi,\rho}$. If $z_0 \in P_\chi$ we have $\chi(z_0) = 0$ and $|\gamma g(z_0)|=|\rho(z_0)|=|\chi(z_0) \pm \rho(z_0)|=|\gamma f(z_0)|= 0$. So, if $z_0 \in I_\chi$ we get that $\gamma g(z_0) = 0$, which is a contradiction because $g(z_0) \neq 0$ and $\gamma = \dfrac{g(z_0)}{f(z_0^2)}$.

Hence, $z_0 \in S \setminus I_\chi$ and we have $\chi(z_0) \neq 0$. Since $f(z_0) = 0$, by the assumption, we get $f(z_0) = \dfrac{1}{\gamma}[\chi(z_0) \pm A(z_0)\chi(z_0)] = 0$, which implies that $A(z_0) = -1$. Now for all $x, y \in S \setminus I_\chi$, we have $xyz_0 \in S \setminus I_\chi$, then by using (1.4) we get $\left(\dfrac{1}{\gamma} - \chi(z_0)\right)\chi(xy) + \left(\dfrac{1}{\gamma} + \chi(z_0)\right)\chi(xy)A(xy) = 0$, which implies according to Lemma 3.1(i), that $\dfrac{1}{\gamma} - \chi(z_0) = 0$ and $\dfrac{1}{\gamma} + \chi(z_0) = 0$, since $A \neq 0$. Therefore, $\chi(z_0) = \dfrac{1}{\gamma} = -\dfrac{1}{\gamma}$, which is a contradiction because $\dfrac{1}{\gamma} \neq 0$ by the assumption. So we do not have a solution corresponding to this possibility.

Subcase I.2. Suppose that $g(z_0^2) \neq 0$, then (4.1) can be rewritten as follows $\beta g(xy) = \beta^2 g(x)g(y) - \beta^2 f(x)f(y) + \alpha\beta f(xy)$, $x, y \in S$ with $\beta := \dfrac{g(z_0)}{g(z_0^2)} \neq 0$ and $\alpha := \dfrac{f(z_0^2)}{g(z_0^2)}$. This shows that the pair $(\beta g, \beta f)$ satisfies the functional equation (1.2). So, according to [12, Theorem 3.1], and taking into account that f and g are linearly independent, there are only the following possibilities.

(I.2.i) There exist a constant $q \in \mathbb{C}\setminus\{\pm\alpha\}$ and two different non-zero multiplicative functions χ_1 and χ_2 on S such that $\beta f = \alpha\dfrac{\chi_1 + \chi_2}{2} + q\dfrac{\chi_1 - \chi_2}{2}$ and $\beta g = \dfrac{\chi_1 + \chi_2}{2} \pm \sqrt{1 + q^2 - \alpha^2}\dfrac{\chi_1 - \chi_2}{2}$. Introducing $\delta := \pm\sqrt{1 + q^2 - \alpha^2}$ we find that $f = \alpha\dfrac{\chi_1 + \chi_2}{2\beta} + q\dfrac{\chi_1 - \chi_2}{2\beta}$ and $g = \dfrac{\chi_1 + \chi_2}{2\beta} + \delta\dfrac{\chi_1 - \chi_2}{2\beta}$. By using (1.4), we get

$$\frac{1}{4\beta^2}\left((1+\delta)^2 - (\alpha+q)^2\right)\chi_1(xy) + \frac{1}{4\beta^2}\left((1-\delta)^2 - (\alpha-q)^2\right)\chi_2(xy)$$
$$= \frac{1}{2\beta}(1+\delta)\chi_1(z_0)\chi_1(xy) + \frac{1}{2\beta}(1-\delta)\chi_2(z_0)\chi_2(xy).$$

So, by Lemma 3.1(i) we obtain $\dfrac{1}{2\beta}(1+\delta)\chi_1(z_0) = \dfrac{1}{4\beta^2}\left((1+\delta)^2 - (\alpha+q)^2\right)$ and $\dfrac{1}{2\beta}(1-\delta)\chi_2(z_0) = \dfrac{1}{4\beta^2}\left((1-\delta)^2 - (\alpha-q)^2\right)$, since χ_1 and χ_2 are different non-zero multiplicative functions. Notice that $\delta \neq \pm 1$ because $q \neq$

$\pm\alpha$. Therefore we deduce that $\chi_1(z_0) = \dfrac{(1+\delta)^2-(\alpha+q)^2}{2\beta(1+\delta)}$ and $\chi_2(z_0) = \dfrac{(1-\delta)^2-(\alpha-q)^2}{2\beta(1-\delta)}$. Hence, by writing γ instead of β we get part (6).

(I.2.ii) $\alpha \neq 0$ and there exist two different non-zero multiplicative functions χ_1 and χ_2 on S such that $\beta f = \alpha\chi_1$ and $\beta g = \chi_2$. By using (1.4) again we get $\dfrac{1}{\beta}\Big(\chi_2(z_0) - \dfrac{1}{\beta}\Big)\chi_2(xy) + \dfrac{\alpha^2}{\beta^2}\chi_1(xy) = 0$, which gives $\chi_2(z_0) = \dfrac{1}{\beta}$ and $\alpha = 0$, since χ_1 and χ_2 are different. This possibility is excluded because $\alpha \neq 0$.

(I.2.iii) There exist a non-zero multiplicative function χ on S, an additive function A on $S \setminus I_\chi$ and a function ρ on P_χ such that $\beta f = \alpha\chi + \Psi_{A\chi,\rho}$ and $\beta g = \chi \pm \Psi_{A\chi,\rho}$, which gives $f = \dfrac{1}{\beta}(\alpha\chi + \Psi_{A\chi,\rho})$ and $g = \dfrac{1}{\beta}(\chi \pm \Psi_{A\chi,\rho})$.

If $g = \dfrac{1}{\beta}(\chi+\Psi_{A\chi,\rho})$ then $z_0 \notin I_\chi \setminus P_\chi$. Indeed, otherwise we have $\chi(z_0) = 0$ and $\Psi_{A\chi,\rho}(z_0) = 0$. Then $\beta g(z_0) = \chi(z_0) + \Psi_{A\chi,\rho}(z_0) = 0$. This contradicts the fact that $g(z_0) \neq 0$.

On the other hand $z_0 \notin P_\chi$. Indeed, otherwise we have $\chi(z_0) = 0$. Then $\beta g(z_0) = \Psi_{A\chi,\rho}(z_0) = \beta f(z_0) = 0$, which is a contradiction because $g(z_0) \neq 0$. So, $z_0 \in S \setminus I_\chi$ and then $\chi(z_0) \neq 0$. Since $f(z_0) = 0$ we get that $f(z_0) = \dfrac{\chi(z_0)}{\beta}[\alpha + A(z_0)] = 0$, which implies that $A(z_0) = -\alpha$. Now, let $x, y \in S \setminus I_\chi$ be arbitrary. We have $xyz_0 \in S \setminus I_\chi$. By using (1.4), we get

$$\Big(\frac{1-\alpha^2}{\beta^2} + \frac{\alpha-1}{\beta}\chi(z_0)\Big)\chi(xy) + \Big(\frac{1-\alpha}{\beta^2} - \frac{1}{\beta}\chi(z_0)\Big)\chi(xy)A(xy) = 0. \tag{4.2}$$

If $A = 0$ then $\rho \neq 0$ because $\Psi_{A\chi,\rho} \neq 0$, $\alpha = 0$, and $\chi(z_0) = \dfrac{1}{\beta}$ by (4.2). This is a special case of solution part (7).

If $A \neq 0$ then by Lemma 3.1 (ii) we get from (4.2) that

$$\frac{1-\alpha^2}{\beta^2} + \frac{\alpha-1}{\beta}\chi(z_0) = 0 \quad \text{and} \quad \frac{1-\alpha}{\beta^2} - \frac{1}{\beta}\chi(z_0) = 0.$$

As $\alpha \neq 1$, because $\chi(z_0) \neq 0$, we deduce that $\chi(z_0) = \dfrac{1-\alpha}{\beta}$ and $\chi(z_0) = \dfrac{1+\alpha}{\beta}$. So, we obtain that $\alpha = 0$ and $\chi(z_0) = \dfrac{1}{\beta}$, and the form of f reduces to $f = \dfrac{1}{\beta}\Psi_{A\chi,\rho}$. So we are in part (7).

If $g = \dfrac{1}{\beta}(\chi - \Psi_{A\chi,\rho})$, by using a similar computation as above, we show that we are also in part (7).

Case II. Suppose $f(z_0) \neq 0$. By using system (1.4) and (3.4), we deduce by an elementary computation that for any $\lambda \in \mathbb{C}$

$$(4.3)\quad (g-\lambda f)(xyz_0) = (g-\lambda f)(x)(g-\lambda f)(y) - (\lambda^2+\mu\lambda+1)f(x)f(y), \quad x,y \in S.$$

Let λ_1 and λ_2 be the two roots of the equation $\lambda^2+\mu\lambda+1=0$. Then $\lambda_1\lambda_2 = 1$ which gives $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. According to [13, Proposition 16] we deduce from (4.3) that $g-\lambda_1 f := \chi_1(z_0)\chi_1$ and $g-\lambda_2 f := \chi_2(z_0)\chi_2$, where χ_1 and χ_2 are two multiplicative functions such that $\chi_1(z_0) \neq 0$ and $\chi_2(z_0) \neq 0$, because f and g are linearly independent.

If $\lambda_1 \neq \lambda_2$, then $\chi_1 \neq \chi_2$ and we get $g = \dfrac{\lambda_2\chi_1(z_0)\chi_1 - \lambda_1\chi_2(z_0)\chi_2}{\lambda_2-\lambda_1}$ and $f = \dfrac{\chi_1(z_0)\chi_1 - \chi_2(z_0)\chi_2}{\lambda_2-\lambda_1}$. By putting $\lambda_1 = ic$, we get the solution of category (4).

If $\lambda_1 = \lambda_2 =: \lambda$, then $g - \lambda f =: \chi(z_0)\chi$ where χ is a multiplicative function on S such that $\chi(z_0) \neq 0$, because f and g are linearly independent. Hence,

$$(4.4)\qquad g = \chi(z_0)\chi + \lambda f.$$

Substituting this in (3.4), an elementary computation shows that

$$f(xyz_0) = \chi(z_0)f(x)\chi(y) + \chi(z_0)f(y)\chi(x) + (2\lambda+\mu)f(x)f(y),$$

for all $x,y \in S$.

Moreover $\lambda = 1$ or $\lambda = -1$ because $\lambda_1\lambda_2 = 1$. Hence, $(\lambda,\mu) = (1,-2)$ or $(\lambda,\mu) = (-1,2)$ since $\lambda^2+\mu\lambda+1 = 0$ and $\lambda \in \{-1,1\}$. So, the functional equation above reduces to

$$f(xyz_0) = \chi(z_0)f(x)\chi(y) + \chi(z_0)f(y)\chi(x),$$

for all $x,y \in S$. Thus, the function f satisfies (3.1). Hence, in view of Proposition 3.2, we get $f = A(z_0)\chi + \Psi_{A\chi,\rho}$. Then, by (4.4), we derive that $g = \chi(z_0)\chi + \lambda f = (\chi(z_0) + \lambda A(z_0))\chi + \lambda^2\Psi_{A\chi,\rho} = (\chi(z_0) \pm A(z_0))\chi + \Psi_{A\chi,\rho}$. This is part (8).

Conversely, it is easy to check that the formulas for f and g listed in Theorem 4.1 define solutions of (1.4).

Finally, suppose that S is a topological semigroup. The continuity of the solutions of the forms (1)–(6) follows directly from [11, Theorem 3.18], and for the ones of the forms (7) and (8) it is parallel to the proof used in [5, Theorem 2.1] for categories (7) and (8). This completes the proof of Theorem 4.1. □

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