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FIBONACCI SUMS MODULO 5

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Abstract. We develop closed form expressions for various finite binomial Fibonacci and Lucas sums depending on the modulo 5 nature of the upper summation limit. Our expressions are inferred from some trigonometric identities.

1. Preliminaries

As usual, the Fibonacci numbers F_n and the Lucas numbers L_n are defined, for $n \in \mathbb{Z}$, by the following recurrence relations for $n \geq 2$:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \ \ F_1 = 1,$$

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \ \ L_1 = 1.$$

For negative subscripts we have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$.

Throughout this paper, we denote the golden ratio by $\alpha = \frac{1+\sqrt{5}}{2}$ and write $\beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha}$. The Fibonacci and Lucas numbers possess the explicit formulas (Binet formulas)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z}.$$

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The sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are indexed in the On-Line Encyclopedia of Integer Sequences [17] as entries A000045 and A000032, respectively. For more information we refer to Koshy [12] and Vajda [18] who have written excellent books dealing with Fibonacci and Lucas numbers.

There exists a countless number of binomial sums involving Fibonacci and Lucas numbers. For some new articles in this field we refer to the papers [1, 5, 2, 6].

In this paper, we introduce closed form expressions for finite Fibonacci and Lucas sums involving different kinds of binomial coefficients and depending on the modulo 5 nature of the upper summation limit. Our expressions are derived from various trigonometric identities, particularly utilizing Waring formulas and Chebyshev polynomials of the first and second kinds. We also present some series involving Bernoulli polynomials.

We note that some of our results were announced without proofs in [4].

2. Fibonacci sums modulo 5 from the $\sin nx$ and $\cos nx$ expansions

We begin with a known lemma [9, 1.331(3) and 1.331(1)].

Lemma 2.1. *If n is a positive integer, then*

$$\sum_{k=1}^{\lfloor n/2\rfloor} \frac{(-1)^{k-1}n}{k}\binom{n-k-1}{k-1}2^{n-2k-1}\cos^{n-2k}x = 2^{n-1}\cos^n x - \cos nx, \tag{2.1}$$

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^k\binom{n-k-1}{k}2^{n-2k-1}\cos^{n-2k-1}x = \frac{\sin nx}{\sin x}. \tag{2.2}$$

Lemma 2.2. *If n is an integer, then*

$$\cos\left(\frac{n\pi}{5}\right) = \begin{cases} (-1)^n, & \text{if } n\equiv 0 \pmod 5,\\ (-1)^{n-1}\alpha/2, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ (-1)^{n-1}\beta/2, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases} \tag{2.3}$$

$$\cos\left(\frac{2n\pi}{5}\right) = \begin{cases} 1, & \text{if } n\equiv 0 \pmod 5,\\ -\beta/2, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ -\alpha/2, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5.\end{cases} \tag{2.4}$$

PROOF. Relations stated in (2.3) can be proved easily by elementary methods. For instance, they follow by applying the addition theorem for the cosine function

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

combined with the special values

$$\cos\left(\frac{\pi}{5}\right) = \frac{\alpha}{2}, \quad \cos\left(\frac{2\pi}{5}\right) = -\frac{\beta}{2}, \quad \cos\left(\frac{3\pi}{5}\right) = \frac{\beta}{2}, \quad \cos\left(\frac{4\pi}{5}\right) = -\frac{\alpha}{2}.$$

Relations stated in (2.4) follow directly from (2.3). $\square$

In our first main results we state Lucas (Fibonacci) identities involving binomial coefficient and additional parameter.

THEOREM 2.3. *If n is a positive integer and t is any integer, then*

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}L_{n-2k+t} = \begin{cases} L_{n+t} - (-1)^n 2L_t, & \textit{if } n \equiv 0 \pmod 5, \\ L_{n+t} + (-1)^n L_{t+1}, & \textit{if } n \equiv 1 \textit{ or } 4 \pmod 5, \\ L_{n+t} - (-1)^n L_{t-1}, & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 5, \end{cases}$$

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}F_{n-2k+t} = \begin{cases} F_{n+t} - (-1)^n 2F_t, & \textit{if } n \equiv 0 \pmod 5, \\ F_{n+t} + (-1)^n F_{t+1}, & \textit{if } n \equiv 1 \textit{ or } 4 \pmod 5, \\ F_{n+t} - (-1)^n F_{t-1}, & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 5. \end{cases}$$

PROOF. Set $x = \pi/5$ in (2.1) and use (2.3) and the fact that

$$2\alpha^r = L_r + F_r\sqrt{5}, \quad 2\beta^r = L_r - F_r\sqrt{5} \tag{2.5}$$

for any integer r. $\square$

We proceed with some corollaries.

Corollary 2.4. *If n is a positive integer, then*

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}F_{2k}=\begin{cases}-2F_n, & \text{if } n\equiv 0 \pmod 5,\\ -F_{n-1}, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ F_{n+1}, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}F_{n-2k+\delta}=F_{n+\delta},$$

where

$$\delta=\begin{cases}0, & \text{if } n\equiv 0 \pmod 5,\\ -1, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ 1, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5.\end{cases}$$

Corollary 2.5. *If n is a positive integer, then*

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}L_{n-2k-1}$$
$$=\begin{cases}L_{n-1}-(-1)^n 3, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\\ L_{n-1}+(-1)^n 2, & \text{otherwise},\end{cases}$$

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}L_{n-2k+1}$$
$$=\begin{cases}L_{n+1}+(-1)^n 3, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ L_{n+1}-(-1)^n 2, & \text{otherwise},\end{cases}$$

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}L_{n-2k}$$
$$=\begin{cases}L_{n}-(-1)^n 4, & \text{if } n\equiv 0 \pmod 5,\\ L_{n}+(-1)^n, & \text{otherwise}.\end{cases}$$

Lemma 2.6. *If n is an integer, then*

$$\frac{\sin(n\pi/5)}{\sin(\pi/5)}=\begin{cases}0, & \text{if } n\equiv 0 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}\alpha, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}\tag{2.6}$$

$$\frac{\sin(3n\pi/5)}{\sin(3\pi/5)} = \begin{cases} 0, & \textit{if } n \equiv 0 \pmod 5, \\ (-1)^{\lfloor n/5 \rfloor}, & \textit{if } n \equiv 1 \textit{ or } 4 \pmod 5, \\ (-1)^{\lfloor n/5 \rfloor}\beta, & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 5. \end{cases} \tag{2.7}$$

From Lemma 2.6 we can deduce the following Lucas and Fibonacci binomial identities modulo 5.

THEOREM 2.7. *If n is a positive integer and t is any integer, then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} L_{n-2k+t} = \begin{cases} (-1)^{\lfloor (n+1)/5 \rfloor} L_t, & \textit{if } n \equiv 0 \textit{ or } 3 \pmod 5, \\ (-1)^{\lfloor (n+1)/5 \rfloor} L_{t+1}, & \textit{if } n \equiv 1 \textit{ or } 2 \pmod 5, \\ 0, & \textit{if } n \equiv 4 \pmod 5, \end{cases} \tag{2.8}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} F_{n-2k+t} = \begin{cases} (-1)^{\lfloor (n+1)/5 \rfloor} F_t, & \textit{if } n \equiv 0 \textit{ or } 3 \pmod 5, \\ (-1)^{\lfloor n/5 \rfloor} F_{t+1}, & \textit{if } n \equiv 1 \textit{ or } 2 \pmod 5, \\ 0, & \textit{if } n \equiv 4 \pmod 5. \end{cases} \tag{2.9}$$

PROOF. Set $x = \pi/5$ in (2.2), use (2.6), (2.5) and simplify. □

A variant of the Lucas and Fibonacci sums with even subscripts is stated as the next corollary.

COROLLARY 2.8. *If n is a positive integer, then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} \binom{n-k}{k} L_{2k} = \begin{cases} (-1)^{\lfloor (n+1)/5 \rfloor} L_n, & \textit{if } n \equiv 0 \textit{ or } 3 \pmod 5, \\ (-1)^{\lfloor (n+1)/5 \rfloor + 1} L_{n-1}, & \textit{if } n \equiv 1 \textit{ or } 2 \pmod 5, \\ 0, & \textit{if } n \equiv 4 \pmod 5, \end{cases}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} \binom{n-k}{k} F_{2k} = \begin{cases} (-1)^{\lfloor (n+1)/5 \rfloor} F_n, & \textit{if } n \equiv 0 \textit{ or } 3 \pmod 5, \\ (-1)^{\lfloor (n+1)/5 \rfloor + 1} F_{n-1}, & \textit{if } n \equiv 1 \textit{ or } 2 \pmod 5, \\ 0, & \textit{if } n \equiv 4 \pmod 5. \end{cases}$$

Corollary 2.9. *If n is a positive integer, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}F_{n-2k+1}=\begin{cases}0, & \text{if } n\equiv 4 \pmod 5,\\ (-1)^{\lfloor (n+1)/5\rfloor}, & \text{otherwise},\end{cases}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}F_{n-2k-\delta}=0,$$

where

$$\delta=\begin{cases}0, & \text{if } n\equiv 0 \text{ or } 3 \pmod 5,\\ 1, & \text{if } n\equiv 1 \text{ or } 2 \pmod 5.\end{cases}$$

3. Fibonacci sums modulo 5 from Waring formulas

This section is based on utilizing the following trigonometric identities with the use of Waring formulas.

Lemma 3.1. *If n is a positive integer, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}2^{n-2k-1}\cos^{n-2k}x=\cos nx, \tag{3.1}$$

$$\sum_{k=0}^{(n-1)/2}(-1)^{(n-1)/2-k}\frac{n}{n-k}\binom{n-k}{k}2^{n-2k-1}\sin^{n-2k}x = \sin nx,\quad n \text{ odd}, \tag{3.2}$$

$$\sum_{k=0}^{n/2}(-1)^{n/2-k}\frac{n}{n-k}\binom{n-k}{k}2^{n-2k-1}\sin^{n-2k}x=\cos nx,\quad n \text{ even}. \tag{3.3}$$

Proof. Consider the Waring formula

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(x_1+x_2)^{n-2k}(x_1x_2)^k=x_1^n+x_2^n.$$

Let i be the imaginary unit. The choice $x_1 = e^{ix}/2$, $x_2 = e^{-ix}/2$ produces (3.1), while the choice $x_1 = e^{ix}/(2i)$, $x_2 = -e^{-ix}/(2i)$ gives $x_1 + x_2 = \sin x$, $x_1 x_2 = 1/4$, and

$$x_1^n + x_2^n = \begin{cases} (-1)^{(n-1)/2} 2^{1-n} \sin nx, & \text{if } n \text{ is odd,} \\ (-1)^{n/2} 2^{1-n} \cos nx, & \text{if } n \text{ is even,} \end{cases}$$

and hence (3.2) and (3.3). $\square$

LEMMA 3.2. *If n is a positive integer, then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} 2^{n-2k} \cos^{n-2k} x = \frac{\sin((n+1)x)}{\sin x}, \tag{3.4}$$

$$\sum_{k=0}^{(n-1)/2} (-1)^{(n-1)/2-k} \binom{n-k}{k} 2^{n-2k} \sin^{n-2k} x = \frac{\sin((n+1)x)}{\cos x}, \quad n \textit{ odd},$$

$$\sum_{k=0}^{n/2} (-1)^{n/2-k} \binom{n-k}{k} 2^{n-2k} \sin^{n-2k} x = \frac{\cos((n+1)x)}{\cos x}, \quad n \textit{ even}.$$

PROOF. Similar to the proof of Lemma 3.1. We use the dual to the Waring formula

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (x_1 + x_2)^{n-2k} (x_1 x_2)^k = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2}. \qquad \square$$

THEOREM 3.3. *If n is a positive integer and t is any integer, then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} \frac{n}{n-k} \binom{n-k}{k} F_{n-2k+t} = \begin{cases} 2F_t, & \textit{if } n \equiv 0 \pmod 5, \\ -F_{t+1}, & \textit{if } n \equiv 1 \textit{ or } 4 \pmod 5, \\ F_{t-1}, & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 5, \end{cases}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} \frac{n}{n-k} \binom{n-k}{k} L_{n-2k+t} = \begin{cases} 2L_t, & \textit{if } n \equiv 0 \pmod 5, \\ -L_{t+1}, & \textit{if } n \equiv 1 \textit{ or } 4 \pmod 5, \\ L_{t-1}, & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 5. \end{cases}$$

PROOF. We apply equation (3.1). Inserting $x = \pi/5$ and $x = 3\pi/5$, respectively, and keeping in mind the trigonometric identity $\cos 3x = 4\cos^3 x - 3\cos x$ we end with

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k}\alpha^{n-2k+t} = \begin{cases} 2\alpha^t, & \text{if } n \equiv 0 \pmod 5,\\ -\alpha^{t+1}, & \text{if } n \equiv 1 \text{ or } 4 \pmod 5,\\ \alpha^{t-1}, & \text{if } n \equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

and

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k}\beta^{n-2k+t}$$
$$= \begin{cases} 2\beta^t, & \text{if } n \equiv 0 \pmod 5,\\ -\beta^t(\alpha^3 - 3\alpha), & \text{if } n \equiv 1 \text{ or } 4 \pmod 5,\\ -\beta^t(\beta^3 - 3\beta), & \text{if } n \equiv 2 \text{ or } 3 \pmod 5.\end{cases}$$

To complete the proof simplify the terms in brackets and combine according the Binet formulas. □

From Theorem 3.3 we can immediately obtain the following finite binomial sums.

COROLLARY 3.4. *If n is a positive integer, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k}F_{n-2k} = \begin{cases} 0, & \textit{if } n \equiv 0 \pmod 5,\\ -1, & \textit{if } n \equiv 1 \textit{ or } 4 \pmod 5,\\ 1, & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 5,\end{cases}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k}L_{n-2k} = \begin{cases} 4, & \textit{if } n \equiv 0 \pmod 5,\\ -1, & \textit{otherwise},\end{cases}$$

and

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k}F_{n+1-2k} = \begin{cases} 2, & \textit{if } n \equiv 0 \pmod 5,\\ -1, & \textit{if } n \equiv 1 \textit{ or } 4 \pmod 5,\\ 0, & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 5,\end{cases}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k}L_{n+1-2k} = \begin{cases} 2, & \textit{if } n \equiv 0, 2 \textit{ or } 3 \pmod 5,\\ -3, & \textit{otherwise}.\end{cases}$$

Remark. Identities (2.8) and (2.9) in Theorem 2.7 can also be obtained straightforwardly by evaluating the trigonometric identity (3.4) at $x = \pi/5$ and $x = 3\pi/5$, respectively, while using (2.6) and (2.7).

4. Fibonacci sums modulo 5 from Chebyshev polynomials

For any integer $n \geq 0$, the Chebyshev polynomials $\{T_n(x)\}_{n\geq 0}$ of the first kind are defined by the second-order recurrence relation [16]

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 2, \quad T_0(x) = 1, \quad T_1(x) = x,$$

while the Chebyshev polynomials $\{U_n(x)\}_{n\geq 0}$ of the second kind are defined by

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n \geq 2, \quad U_0(x) = 1, \quad U_1(x) = 2x.$$

The Chebyshev polynomials possess the representations

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2 - 1)^k x^{n-2k},$$

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} (x^2 - 1)^k x^{n-2k},$$

and have the exact Binet-like formulas

$$T_n(x) = \frac{1}{2}\left((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right),$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left((x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}\right).$$

The properties of Chebyshev polynomials of the first and second kinds have been studied extensively in the literature. The reader can find in the recent papers [7, 8, 11, 14, 15, 19] additional information about them, especially about their products, convolutions, power sums as well as their connections to Fibonacci numbers and polynomials.

Lemma 4.1. *For all $x \in \mathbb{C}$ and a positive integer n, we have the following identities:*

$$n\sum_{k=0}^{n}(-1)^k\frac{4^k}{n+k}\binom{n+k}{n-k}\sin^{2k}\Big(\frac{x}{2}\Big)=\cos nx, \tag{4.1}$$

$$n\sum_{k=0}^{n}(-1)^{n-k}\frac{4^k}{n+k}\binom{n+k}{n-k}\cos^{2k}\Big(\frac{x}{2}\Big)=\cos nx. \tag{4.2}$$

Proof. Identities (4.1) and (4.2) are consequences of the identity

$$n\sum_{k=0}^{n}\frac{(-2)^k}{n+k}\binom{n+k}{n-k}(1\mp x)^k=(\pm 1)^n T_n(x) \tag{4.3}$$

derived in [3]. □

Lemma 4.2. *If n is a non-negative integer, then*

$$T_n\Big(-\frac{\alpha}{2}\Big)=\begin{cases}1, & \textit{if } n\equiv 0 \pmod 5,\\ -\alpha/2, & \textit{if } n\equiv 1 \textit{ or } 4 \pmod 5,\\ -\beta/2, & \textit{if } n\equiv 2 \textit{ or } 3 \pmod 5,\end{cases}$$

$$T_n\Big(-\frac{\beta}{2}\Big)=\begin{cases}1, & \textit{if } n\equiv 0 \pmod 5,\\ -\beta/2, & \textit{if } n\equiv 1 \textit{ or } 4 \pmod 5,\\ -\alpha/2, & \textit{if } n\equiv 2 \textit{ or } 3 \pmod 5.\end{cases}$$

Proof. Evaluate the identity $T_n(\cos x)=\cos nx$ at $x=4\pi/5$ and $x=2\pi/5$, in turn. □

Theorem 4.3. *If n is a positive integer and t is any integer, then*

$$\sum_{k=1}^{\lceil n/2\rceil}\frac{n}{n+2k-1}\binom{n+2k-1}{n-2k+1}5^kF_{2k+t-1} \tag{4.4}$$

$$-\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n}{n+2k}\binom{n+2k}{n-2k}5^kL_{2k+t}=\begin{cases}-L_t, & \textit{if } n\equiv 0 \pmod 5,\\ L_{t+1}/2, & \textit{if } n\equiv 1 \textit{ or } 4 \pmod 5,\\ -L_{t-1}/2, & \textit{if } n\equiv 2 \textit{ or } 3 \pmod 5,\end{cases}$$

$$(4.5)\quad \sum_{k=1}^{\lceil n/2\rceil} \frac{n}{n+2k-1}\binom{n+2k-1}{n-2k+1}5^{k-1}L_{2k+t-1}$$

$$-\sum_{k=0}^{\lfloor n/2\rfloor} \frac{n}{n+2k}\binom{n+2k}{n-2k}5^{k}F_{2k+t} = \begin{cases} -F_t, & \textit{if } n\equiv 0 \pmod 5,\\ F_{t+1}/2, & \textit{if } n\equiv 1 \textit{ or } 4 \pmod 5,\\ -F_{t-1}/2, & \textit{if } n\equiv 2 \textit{ or } 3 \pmod 5.\end{cases}$$

Proof. Using $x=-\alpha/2$ and $x=-\beta/2$, in turn, in (4.3) with the upper sign gives, in view of Lemma 4.2,

$$\sum_{k=0}^{n} \frac{n}{n+k}\binom{n+k}{n-k}(\sqrt{5})^k\big((-1)^{k+1}\lambda\alpha^{k+t}-\beta^{k+t}\big)$$

$$= \begin{cases} -(\lambda\alpha^t+\beta^t), & \text{if } n\equiv 0 \pmod 5,\\ (\lambda\alpha^{t+1}+\beta^{t+1})/2, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ -(\lambda\alpha^{t-1}+\beta^{t-1})/2, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

from which (4.4) and (4.5) now follow upon setting $\lambda=1$ and $\lambda=-1$, in turn, and using the Binet formulas and the summation identity

$$\sum_{j=0}^{n} f_j = \sum_{j=0}^{\lfloor n/2\rfloor} f_{2j} + \sum_{j=1}^{\lceil n/2\rceil} f_{2j-1}. \qquad \square$$

We observe the following special cases of the prior result.

Corollary 4.4. *If n is a positive integer, then*

$$\sum_{k=1}^{\lceil n/2\rceil} \frac{n}{n+2k-1}\binom{n+2k-1}{n-2k+1}5^{k}L_{2k+\delta-1} = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n}{n+2k}\binom{n+2k}{n-2k}5^{k+1}F_{2k+\delta},$$

where

$$\delta = \begin{cases} 0, & \textit{if } n\equiv 0 \pmod 5,\\ -1, & \textit{if } n\equiv 1 \textit{ or } 4 \pmod 5,\\ 1, & \textit{if } n\equiv 2 \textit{ or } 3 \pmod 5.\end{cases}$$

THEOREM 4.5. *If n is a positive integer and t is any integer, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}L_{2k+t}=\begin{cases}L_t, & \text{if } n=0 \pmod 5,\\ L_{t-1}/2, & \text{if } n=1 \text{ or } 4 \pmod 5,\\ -L_{t+1}/2, & \text{if } n=2 \text{ or } 3 \pmod 5,\end{cases}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}F_{2k+t}=\begin{cases}F_t, & \text{if } n=0 \pmod 5,\\ F_{t-1}/2, & \text{if } n=1 \text{ or } 4 \pmod 5,\\ -F_{t+1}/2, & \text{if } n=2 \text{ or } 3 \pmod 5.\end{cases}$$

PROOF. Set $x=\pi/5$ in (4.1) and use (2.3) and the fact that $\sin(\pi/10)=-\beta/2$ to obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}\beta^{2k+t}=\begin{cases}\beta^t, & \text{if } n=0 \pmod 5,\\ \beta^{t-1}/2, & \text{if } n=1 \text{ or } 4 \pmod 5,\\ -\beta^{t+1}/2, & \text{if } n=2 \text{ or } 3 \pmod 5,\end{cases}$$

from which the results follow by (2.5). □

Using Theorem 4.5, we have the following binomial Fibonacci identities modulo 5.

COROLLARY 4.6. *If n is a positive integer, then*

$$\sum_{k=0}^{n}\frac{(-1)^k}{n+k}\binom{n+k}{n-k}F_{2k+\delta}=0,$$

where

$$\delta=\begin{cases}0, & \text{if } n=0 \pmod 5,\\ 1, & \text{if } n=1 \text{ or } 4 \pmod 5,\\ -1, & \text{if } n=2 \text{ or } 3 \pmod 5.\end{cases}$$

LEMMA 4.7. *If x is a complex variable and n is a positive integer, then*

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{4^k k}{n+k}\binom{n+k}{n-k}\sin^{2k-2}\left(\frac{x}{2}\right)=\frac{2\sin nx}{\sin x}, \tag{4.6}$$

$$\sum_{k=1}^{n}(-1)^{n-k}\frac{4^k k}{n+k}\binom{n+k}{n-k}\cos^{2k-2}\left(\frac{x}{2}\right)=\frac{2\sin nx}{\sin x}. \tag{4.7}$$

PROOF. Identities (4.6) and (4.7) come from the following identities derived in [3]:

$$\sum_{k=1}^{n}(-1)^{n-k}\frac{2^k k}{n+k}\binom{n+k}{n-k}(1\mp x)^{k-1}=(\mp 1)^{n-1}U_{n-1}(x),$$
$$\sum_{k=1}^{n}(-1)^{n-k}\frac{4^k k}{n+k}\binom{n+k}{n-k}x^{2k-1}=U_{2n-1}(x). \qquad \square$$

THEOREM 4.8. *If n is a positive integer and n is any integer, then*

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}L_{2k+t}$$
$$=\begin{cases}0, & \text{if } n\equiv 0 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}L_{t+2}/2, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor+1}L_{t+1}/2, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}F_{2k+t}$$
$$=\begin{cases}0, & \text{if } n\equiv 0 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}F_{t+2}/2, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor+1}F_{t+1}/2, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5.\end{cases}$$

PROOF. Set $x=\pi/5$ and $x=3\pi/5$, respectively, in (4.6), and use (2.6) and (2.7). $\square$

REMARK. Theorem 4.8 can also be proved using (4.7). Using the trigonometric identities $\sin 2x=2\sin x\cos x$ and $\cos 3x=4\cos^3 x-3\cos x$ and working with $x=2\pi/5$ and $x=6\pi/5$, respectively, we end with

$$2\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}L_{2k-1+t}$$
$$=\begin{cases}0, & \text{if } n\equiv 0 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}(\alpha^{t+1}-\beta^{t-3}+3\beta^{t-1}), & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}(-\alpha^{t}+\beta^{t+4}-3\beta^{t+2}), & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

and

$$2\sqrt{5}\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}F_{2k-1+t}$$
$$=\begin{cases}0, & \text{if } n\equiv 0 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}\big(\alpha^{t+1}+\beta^{t-3}-3\beta^{t-1}\big), & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ (-1)^{\lfloor n/5\rfloor}\big(-\alpha^{t}-\beta^{t+4}+3\beta^{t+2}\big), & \text{if } n\equiv 2 \text{ or } 3 \pmod 5.\end{cases}$$

To get Theorem 4.8 simplify the terms in brackets and replace t by $t+1$.

Applying Theorem 4.8 yields the following two corollaries.

COROLLARY 4.9. *If n is a positive integer, then*

$$\sum_{k=1}^{n}(-1)^{k}\frac{k}{n+k}\binom{n+k}{n-k}F_{2k-\delta}=0,$$

where

$$\delta=\begin{cases}2, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ 1, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5.\end{cases}$$

COROLLARY 4.10. *If n is a positive integer and t is any integer, then we have:*

If $n\equiv 0 \pmod 5$, then

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^{k}\binom{n-k-1}{k}L_{n-2k+t}=2\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}L_{2k+t},$$
$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^{k}\binom{n-k-1}{k}F_{n-2k+t}=2\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}F_{2k+t},$$

if $n\equiv 1$ or $4 \pmod 5$, then

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^{k}\binom{n-k-1}{k}L_{n-2k+t}=2\sum_{k=1}^{n}(-1)^{k+1}\frac{k}{n+k}\binom{n+k}{n-k}L_{2k-1+t},$$
$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^{k}\binom{n-k-1}{k}F_{n-2k+t}=2\sum_{k=1}^{n}(-1)^{k+1}\frac{k}{n+k}\binom{n+k}{n-k}F_{2k-1+t},$$

if $n \equiv 2$ *or* $3 \pmod 5$, *then*

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^k \binom{n-k-1}{k} L_{n-2k+t} = 2\sum_{k=1}^{n} (-1)^k \frac{k}{n+k}\binom{n+k}{n-k} L_{2k+1+t},$$

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^k \binom{n-k-1}{k} F_{n-2k+t} = 2\sum_{k=1}^{n} (-1)^k \frac{k}{n+k}\binom{n+k}{n-k} F_{2k+1+t}.$$

Proof. Compare Theorem 4.8 with Theorem 2.7. □

Lemma 4.11. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n} (-1)^{n-k} 4^k \binom{n+k}{n-k} \cos^{2k} x = \frac{\sin((2n+1)x)}{\sin x}. \tag{4.8}$$

Proof. Evaluate the identity [3]

$$\sum_{k=0}^{n} (-1)^{n-k} 4^k \binom{n+k}{n-k} x^{2k} = U_{2n}(x)$$

at $x = \cos x$. □

Lemma 4.12. *If n is an integer, then*

$$\frac{\sin\big((2n+1)\pi/5\big)}{\sin(\pi/5)} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 5,\\ \alpha, & \text{if } n \equiv 1 \pmod 5,\\ 0, & \text{if } n \equiv 2 \pmod 5,\\ -\alpha, & \text{if } n \equiv 3 \pmod 5,\\ -1, & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

From Lemmas 4.11 and 4.12 we can deduce the following Fibonacci and Lucas binomial identities modulo 5.

Theorem 4.13. *If n is a non-negative integer and t is any integer, then*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} L_{2k+t} = \begin{cases} L_t, & \text{if } n \equiv 0 \pmod 5,\\ L_{t+1}, & \text{if } n \equiv 1 \pmod 5,\\ 0, & \text{if } n \equiv 2 \pmod 5,\\ -L_{t+1}, & \text{if } n \equiv 3 \pmod 5,\\ -L_t, & \text{if } n \equiv 4 \pmod 5, \end{cases}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}F_{2k+t}=\begin{cases}F_t, & \textit{if } n\equiv 0 \pmod 5,\\ F_{t+1}, & \textit{if } n\equiv 1 \pmod 5,\\ 0, & \textit{if } n\equiv 2 \pmod 5,\\ -F_{t+1}, & \textit{if } n\equiv 3 \pmod 5,\\ -F_t, & \textit{if } n\equiv 4 \pmod 5.\end{cases}$$

PROOF. Set $x=\pi/5$ in (4.8) and use Lemma 4.12. □

LEMMA 4.14 ([10, (41.2.16.1)]). *If n is a positive integer and x is any variable, then*

$$\sum_{k=1}^{n}\frac{(-1)^k}{\cos x-\cos(\pi k/n)}=\frac{1}{2}\left(\frac{1}{1-\cos x}+\frac{(-1)^n}{1+\cos x}\right)-\frac{n}{\sin x\sin nx}. \tag{4.9}$$

Further interesting identities involving Fibonacci and Lucas numbers are stated in the next theorem.

THEOREM 4.15. *If n is a positive integer and t is any integer, then*

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\big(L_{t-1}+2L_t\cos(\pi k/n)\big)}{4\cos^2(\pi k/n)-2\cos(\pi k/n)-1}$$
$$=\frac{1}{2}\big(L_{t+2}+(-1)^nF_{t-1}\big)-2(-1)^{\lfloor n/5\rfloor}n\cdot\begin{cases}0, & \textit{if } n\equiv 0 \pmod 5,\\ F_{t+1}, & \textit{if } n\equiv 1 \textit{ or } 4 \pmod 5,\\ F_t, & \textit{if } n\equiv 2 \textit{ or } 3 \pmod 5,\end{cases}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\big(F_{t-1}+2F_t\cos(\pi k/n)\big)}{4\cos^2(\pi k/n)-2\cos(\pi k/n)-1}$$
$$=\frac{1}{2}\left(F_{t+2}+\frac{(-1)^n}{5}L_{t-1}\right)-\frac{2(-1)^{\lfloor n/5\rfloor}}{5}n\cdot\begin{cases}0, & \textit{if } n\equiv 0 \pmod 5,\\ L_{t+1}, & \textit{if } n\equiv 1 \textit{ or } 4 \pmod 5,\\ L_t, & \textit{if } n\equiv 2 \textit{ or } 3 \pmod 5.\end{cases}$$

PROOF. Set $x=\pi/5$ and $x=3\pi/5$, in turn, in (4.9) to obtain

$$2\sum_{k=1}^{n}\frac{(-1)^k}{\alpha-2\cos(\pi k/n)}=\frac{1}{2-\alpha}+\frac{(-1)^n}{2+\alpha}-\frac{4n\alpha}{\sqrt{5}}\frac{\sin(\pi/5)}{\sin(n\pi/5)}$$

and

$$2\sum_{k=1}^{n}\frac{(-1)^k}{\beta-2\cos(\pi k/n)}=\frac{1}{2-\beta}+\frac{(-1)^n}{2+\beta}+\frac{4n\beta}{\sqrt{5}}\frac{\sin(3\pi/5)}{\sin(3n\pi/5)},$$

from which the identities follow. □

By setting $t = 0$ and $t = 1$ in Theorem 4.15, we obtain the following.

Corollary 4.16. *If n is a positive integer, then*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}\,(4\cos(\pi k/n)-1)}{4\cos^2(\pi k/n)-2\cos(\pi k/n)-1} = \frac{3+(-1)^n}{2} - 2(-1)^{\lfloor n/5\rfloor}\, n\cdot \begin{cases} 0, & \text{if } n\equiv 0,\,2 \text{ or } 3 \pmod 5,\\ 1, & \text{otherwise,}\end{cases}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{4\cos^2(\pi k/n)-2\cos(\pi k/n)-1} = \frac{5-(-1)^n}{10} - \frac{2(-1)^{\lfloor n/5\rfloor}}{5}\, n\cdot \begin{cases} 0, & \text{if } n\equiv 0 \pmod 5,\\ 1, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ 2, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}\cos^2(\pi k/2n)}{4\cos^2(\pi k/n)-2\cos(\pi k/n)-1} = \frac{1}{2} - \frac{(-1)^{\lfloor n/5\rfloor}}{2}\, n\cdot \begin{cases} 0, & \text{if } n\equiv 0 \pmod 5,\\ 1, & \text{otherwise,}\end{cases}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}\cos(\pi k/n)}{4\cos^2(\pi k/n)-2\cos(\pi k/n)-1} = \frac{5+(-1)^n}{10} - \frac{(-1)^{\lfloor n/5\rfloor}}{5}\, n\cdot \begin{cases} 0, & \text{if } n\equiv 0 \pmod 5,\\ 3, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ 1, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5.\end{cases}$$

5. Some additional observations

We close this paper with some additional observations leading to possibly new series representations of the constant α involving Bernoulli polynomials. Recall that Bernoulli polynomials $B_n(t)$, $n \geq 0$, may be defined by the

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} t^k,$$

where B_n is the nth Bernoulli number, defined by the power series

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!},\quad |z|<2\pi.$$

We have $B_n(1)=B_n(0)=B_n$ for all $n\geq 2$ and $B_{2n+1}=0$ for all $n\geq 1$.

THEOREM 5.1. *Let m be a non-negative integer. Then*

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}\Big)=(-1)^{m-1},$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}}{(2k+1)!}\frac{\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+\frac{1}{2}\Big)=0,$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+1\Big)=(-1)^{m},$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+\frac{3}{2}\Big)=(-1)^{m}\alpha, \tag{5.1}$$

and

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+2\Big)=(-1)^{m}\alpha. \tag{5.2}$$

PROOF. Combine (2.6) with the representation [13, Eq. (2.5)]

$$\frac{\sin xt}{\sin t}=\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}B_{2k+1}\Big(\frac{1+x}{2}\Big)t^{2k},\quad |t|<\pi. \tag{5.3}$$ □

When $m=0$ then from (5.1) and (5.2) we get the special series:

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{3}{2}\Big)=\alpha,$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k}B_{2k+1}(2)=\alpha.$$

From Raabe's formula

$$B_n(ax) = a^{n-1} \sum_{k=0}^{a-1} B_n\Big(x + \frac{k}{a}\Big)$$

we get

$$B_{2k+1}(2) = 2^{2k}\Big(B_{2k+1}(1) + B_{2k+1}\Big(\frac{3}{2}\Big)\Big)$$

and

$$\sum_{k=0}^{\infty}(-1)^k \frac{2^{2k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k} B_{2k+1}\Big(\frac{3}{2}\Big) = \alpha,$$

$$\sum_{k=1}^{\infty}(-1)^k \frac{2^{4k+1}}{(2k+1)!}\frac{\pi^{2k}}{25^k} B_{2k+1}\Big(\frac{3}{2}\Big) = \alpha - 3 = \sqrt{5}\beta.$$

But making use of $B_n(t+1) - B_n(t) = nt^{n-1}$ we see that

$$B_{2k+1}\Big(\frac{3}{2}\Big) = \frac{2k+1}{2^{2k}}$$

and thus the series turn into

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}\frac{\pi^{2k}}{25^k} = \frac{\alpha}{2} - 1 = -\frac{\beta^2}{2} \tag{5.4}$$

and

$$\sum_{k=1}^{\infty}(-1)^k \frac{2^{2k+1}}{(2k)!}\frac{\pi^{2k}}{25^k} = \sqrt{5}\beta. \tag{5.5}$$

The series (5.4) and (5.5) are essentially $\cosh(i\pi/5) = \cos(\pi/5) = \alpha/2$ and $\cosh(2i\pi/5) = \cos(2\pi/5) = -\beta/2$ which we encountered at the beginning of the paper.

Combining (2.7) with (5.3) we have the following theorem. The details of we leave to the reader.

THEOREM 5.2. *Let m be a non-negative integer. Then*

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{9^k\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}\Big)=(-1)^{m-1},$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}}{(2k+1)!}\frac{9^k\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+\frac{1}{2}\Big)=0,$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{9^k\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+1\Big)=(-1)^{m},$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{9^k\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+\frac{3}{2}\Big)=(-1)^{m}\beta, \tag{5.6}$$

and

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{9^k\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{5m}{2}+2\Big)=(-1)^{m}\beta. \tag{5.7}$$

Finally, we obtain the following special series as a consequence of (5.6) and (5.7):

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{9^k\pi^{2k}}{25^k}B_{2k+1}\Big(\frac{3}{2}\Big)=\beta,$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}\frac{9^k\pi^{2k}}{25^k}B_{2k+1}(2)=\beta.$$

6. Concluding comments

In this paper, we presented new closed forms for some types of finite Fibonacci and Lucas sums involving different kinds of binomial coefficients and depending on the modulo 5 nature of the upper summation limit. To prove our results, we applied some trigonometric identities utilizing Waring formulas and Chebyshev polynomials of the first and second kinds.

Using similar techniques, we can generalize our findings to more common number sequences. Let us give, for example, a generalization of Theorems 2.3, 3.3 and 4.5 to the case of the gibonacci (generalized Fibonacci) sequence defined by the recurrence $G_n=G_{n-1}+G_{n-2}$, $n\geq 2$, with $G_0=a$ and $G_1=b$, where a and b are arbitrary [12, 18]. Note that F_n corresponds to the case of G_n when $a = 1$ and $b = 0$, while L_n to the case when $a = 1$ and $b = 2$. The following identities modulo 5 hold for positive integer n and any integer t:

$$n\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{k}\binom{n-k-1}{k-1}G_{n-2k+t} = \begin{cases} G_{n+t}-(-1)^n 2G_t, & \text{if } n\equiv 0 \pmod 5,\\ G_{n+t}+(-1)^n G_{t+1}, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ G_{n+t}-(-1)^n G_{t-1}, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

$$n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^{n-k}}{n-k}\binom{n-k}{k}G_{n-2k+t} = \begin{cases} 2G_t, & \text{if } n\equiv 0 \pmod 5,\\ -G_{t+1}, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5,\\ G_{t-1}, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

$$n\sum_{k=0}^{n}\frac{(-1)^{n-k}}{n+k}\binom{n+k}{n-k}G_{2k+t} = \begin{cases} G_t, & \text{if } n = 0 \pmod 5,\\ G_{t-1}/2, & \text{if } n = 1 \text{ or } 4 \pmod 5,\\ -G_{t+1}/2, & \text{if } n = 2 \text{ or } 3 \pmod 5.\end{cases}$$

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