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ON THE ALIENATION OF MULTIPLICATIVE AND ADDITIVE FUNCTIONS

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Abstract. Given S a semigroup. We study two Pexider-type functional equations

$$f(xy) + g(xy) = f(x) + f(y) + g(x)g(y), \quad x, y \in S,$$

and

$$\int_S f(xyt)d\mu(t) + \int_S g(xyt)d\mu(t) = f(x) + f(y) + g(x)g(y), \quad x, y \in S,$$

for unknown functions f and g mapping S into $\mathbb{C}$, where μ is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$ for some fixed elements $(z_i)_{i\in I}$ contained in S such that $\int_S d\mu(t) = 1$.

The main goal of this paper is to solve the above two functional equations and examine whether or not they are equivalent to the systems of equations

$$\begin{cases} f(xy) = f(x) + f(y), \\ g(xy) = g(x)g(y), \quad x, y \in S, \end{cases}$$

and

$$\begin{cases} \int_S f(xyt)d\mu(t) = f(x) + f(y), \\ \int_S g(xyt)d\mu(t) = g(x)g(y), \quad x, y \in S, \end{cases}$$

respectively.

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1. Notation and terminology

Throughout this paper S denotes an arbitrary semigroup, i.e. a set equipped with an associative binary operation and $\mu := \sum_{i\in I}\alpha_i\delta_{z_i}$ is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, where $(z_i)_{i\in I}$ are fixed elements in S and $(\alpha_i)_{i\in I}\in\mathbb{C}$ are such that $\int_S d\mu(t) = \sum_{i\in I}\alpha_i = 1$ and $\{\alpha_i|\alpha_i\neq 0,\, i\in I\}$ is a finite set. A function $A\colon S\to\mathbb{C}$ is additive if $A(xy) = A(x)+A(y)$ for all $x,y\in S$. A function $\chi\colon S\to\mathbb{C}$ is multiplicative if $\chi(xy)=\chi(x)\chi(y)$ for all $x,y\in S$.

Two complex-valued functions f and g are quadratically equivalent if for some nonvanishing constants $c_1,c_2\in\mathbb{C}$ we have

$$\Delta_y^3(c_1f+c_2g)(x)=0$$

for all $x\in S$, where Δ_y denotes the difference operator with span y given by

$$\Delta_y f(x) := f(xy)-f(x)$$

and the iterates Δ_y^n are defined respectively $\Delta_y^0 f := f$, $\Delta_y^{n+1}f := \Delta_y(\Delta_y^n f)$, $n = 1,2,\ldots$. In particular, for $n=3$:

$$\Delta_y^3 f(x) = f(xy^3)-3f(xy^2)+3f(xy)-f(x)$$

or, equivalently,

$$\Delta_y^3 = f_{y^3}-3f_{y^2}+3f_y-f,$$

where $f_y(x)=f(xy)$ for all $x,y\in S$.

2. Introduction

Let A and m be an additive and a multiplicative functions respectively. Then

$$A(xy) = A(x)+A(y),\quad x,y\in S, \tag{2.1}$$

and

$$m(xy) = m(x)m(y),\quad x,y\in S. \tag{2.2}$$

Summing up these two functional equations side by side we obtain

$$A(xy) + m(xy) = A(x) + A(y) + m(x)m(y), \quad x, y \in S.$$

Conversely, a natural question is whether given two functions $f\colon S \to \mathbb{C}$ and $g\colon S \to \mathbb{C}$, the corresponding equation

$$f(xy) + g(xy) = f(x) + f(y) + g(x)g(y), \quad x, y \in S, \tag{2.3}$$

implies the additivity of f and hence the multiplicativity of g (alienation phenomenon). For the more detailed study of the history of this phenomenon see the survey article [10].

During their investigations of the alienation of Cauchy's functional equations, Dhombres [2] (where the alienation idea comes from) and Ger [6, 7, 9] obtained interesting results about the generalized ring homomorphisms equation

$$af(xy) + bf(x + y) + cf(x)f(y) + df(x) + df(y) = 0.$$

By algebraic methods Ger [8] proved that the solutions $f, g\colon M \to R$ of the Pexider functional equation

$$f(x + y) + g(x + y) = f(x) + f(y) + g(x)g(y), \quad x, y \in M, \tag{2.4}$$

on a commutative monoid M mapping into an integral domain R are closely related to the solutions of the trigonometric functional equation

$$h(x + y) = \varphi(x)h(y) + h(x), \quad x, y \in M. \tag{2.5}$$

The solutions of equation (2.5) are known on commutative monoids, and then the solutions of (2.4) are described by means of additive and multiplicative functions. Furthermore, Ger [8] establishes the alienation of additivity and exponentiality up to quadratic equivalence.

By using Stetkær result [13] about the solutions of (2.5) on groups, Ger main result can be extended to the more general case, when M is a group not necessarily commutative.

The purpose of the present paper is to show how Ger's work [8] on commutative monoids extends to the much wider framework of semigroups not necessarily commutative. Our main goal is to give the general solutions of (2.3) on all semigroups by proving a similar result as [8, Theorem 3]. In the proof we use the solutions of cosine addition law:

$$g(xy) = g(x)g(y) + f(x)f(y), \quad x, y \in S,$$

instead of equation (2.5). A secondary goal is to obtain the solutions of the functional equation

$$\int_S f(xyt)d\mu(t) + \int_S g(xyt)d\mu(t) = f(x) + f(y) + g(x)g(y), \quad x, y \in S, \tag{2.6}$$

and to prove that the system of equations

$$\begin{cases} \int_S f(xyt)d\mu(t) = f(x) + f(y), \\ \int_S g(xyt)d\mu(t) = g(x)g(y), \quad x, y \in S, \end{cases}$$

is equivalent to (2.6) when f and g are assumed not to be quadratically equivalent.

The solutions of the functional equation

$$\int_S f(xyt)d\mu(t) = f(x) + f(y), \quad x, y \in S, \tag{2.7}$$

are of the form $f = A + \int_S A(t)d\mu(t)$ where A is an additive function (see Lemma 4.1 below), while the functional equation

$$\int_S g(xyt)d\mu(t) = g(x)g(y), \quad x, y \in S, \tag{2.8}$$

has been solved by E. Elqorachi and A. Redouani [5, Corollary 2.5]. The solutions are of the form $g = \chi \int_S \chi(t)d\mu(t)$, where χ is a multiplicative function.

3. Solutions of the functional equations (2.3) and alienation of equations (2.1) and (2.2)

The present section is dedicated to solve the functional equation (2.3). The solutions are expressed (in Theorem 3.1) in terms of multiplicative and additive functions. In addition, by help of these results, we will show in Theorem 3.2 that the functional equations (2.1) and (2.2) are alien to each other for non-quadratically equivalent functions.

Theorem 3.1. *The solutions $f, g\colon S \to \mathbb{C}$ of the functional equation*

$$f(xy) + g(xy) = f(x) + f(y) + g(x)g(y), \quad x, y \in S,$$

are the following pairs:

(a) $g = \frac{\chi+\alpha^2}{\alpha^2+1}$, $f = A - \frac{\alpha^2}{(\alpha^2+1)^2}(\chi - 1)$, *where* $\alpha \in \mathbb{C}\setminus\{i,-i\}$ *and* A *is additive and* χ *is multiplicative;*
(b) $g = A + 1$, $f = B + \frac{1}{2}A^2$, *where* A, B *are additive.*

Proof. Let the pair (f, g) be a solution of equation (2.3). If $g = 1$ or $g = 0$, then f is additive. This occurs in case (a) with $\chi = 1$ or with $\chi = 0$ and $\alpha = 0$. From now on we assume that $g \neq 1$ and $g \neq 0$ and we follow the proof given by Ger [8] in the commutative case. It is well known that the Cauchy difference $C_f(x, y) := f(xy) - f(x) - f(y)$, $x, y \in S$ satisfies the cocycle equation

$$C_f(xy, z) + C_f(x, y) = C_f(x, yz) + C_f(y, z), \quad x, y, z \in S. \tag{3.1}$$

From (2.3) we find that

$$C_f(x, y) = g(x)g(y) - g(xy), \quad x, y \in S.$$

Inserting the last identity into (3.1), we obtain the following functional equation

$$\begin{aligned} g(xy)g(z) - g(y)g(z) + g(yz) \\ = g(x)g(yz) - g(x)g(y) + g(xy), \quad x, y, z \in S. \end{aligned} \tag{3.2}$$

Now, the rest of the proof takes another way, we simply notice that g satisfies the cosine addition law. Indeed, by subtracting $g(x)g(y)g(z)$ from both sides of (3.2), we get

$$(g(z) - 1)\left[g(xy) - g(x)g(y)\right] = (g(x) - 1)\left[g(yz) - g(y)g(z)\right], \quad x, y, z \in S.$$

Since $g \neq 1$, there exists $x_0 \in S$ such that

$$\begin{aligned} g(xy) - g(x)g(y) \\ = (g(x_0) - 1)^{-1}(g(x) - 1)\left[g(yx_0) - g(y)g(x_0)\right], \quad y \in S. \end{aligned} \tag{3.3}$$

Using (3.3), the term $g(yx_0) - g(y)g(x_0)$ can be written as

$$g(yx_0) - g(y)g(x_0) = (g(x_0) - 1)^{-1}(g(y) - 1)\left[g(x_0^2) - \big(g(x_0)\big)^2\right], \quad x, y \in S.$$

It follows from (3.3) that g satisfies the cosine functional equation

$$g(xy) = g(x)g(y) + \alpha^2(g(x) - 1)(g(y) - 1), \quad x, y \in S, \tag{3.4}$$

where $\alpha^2 = (g(x_0) - 1))^{-2}\left[g(x_0^2) - \big(g(x_0)\big)^2\right]$.

If $\alpha = 0$, then g is multiplicative and, consequently, f is additive. This occurs in case (a). Otherwise we have to deal with a particular case of the cosine addition law (3.4) with $\alpha \neq 0$. According to [3, Theorem 3.2] (see [1, Theorem 2.2] for simpler formulas), the pair $(g, \alpha(g-1))$ has one of the forms included in the following list:

(i) $g = 0$ on S^2, where $S^2 = \{xy \mid x, y \in S\}$ and $\alpha(g-1) = \pm ig$.
(ii) $g = \frac{1}{1+c^2}m$ and $\alpha(g-1) = \frac{c}{1+c^2}m$, where $c \in \mathbb{C}\setminus\{0, i, -i\}$ and m is a nonzero multiplicative function.
(iii) $g = \frac{\delta^{-1}\chi_1 + \delta\chi_2}{\delta^{-1}+\delta}$ and $\alpha(g-1) = \frac{\chi_2 - \chi_1}{\delta^{-1}+\delta}$, where χ_1 and χ_2 are two multiplicative functions such that $\chi_1 \neq \chi_2$ and $\delta \in \mathbb{C}\setminus\{0, i, -i\}$.
(iv) $g = \chi \pm \phi$ and $\alpha(g-1) = -i\phi$, where χ is a nonzero multiplicative function and ϕ is a solution of the special sine addition law

$$\phi(xy) = \phi(x)\chi(y) + \phi(y)\chi(x),\ x, y \in S. \tag{3.5}$$

First case: the pair $(g, \alpha(g-1))$ has the form (i). This case is omitted because the identities $\alpha(g-1) = \pm ig$ and $g = 0$ on S^2 imply that $\alpha = 0$.

Second case: the pair $(g, \alpha(g-1))$ has the form (ii). In this case we have $\alpha \neq c$ and $g = \frac{\alpha}{\alpha - c}$. Since g is a solution of (3.4), we check by elementary computations that $\alpha c = -1$ and hence $g = \frac{\alpha^2}{\alpha^2+1}$. Using this formulas in (2.3), we get that $f - \frac{\alpha^2}{(\alpha^2+1)^2}$ is an additive function. That is $f = A + \frac{\alpha^2}{(\alpha^2+1)^2}$, where A is additive. This occurs in case (a) with $\chi = 0$.

Third case: the pair $(g, \alpha(g-1))$ has the form (iii). In this case we deduce that

$$\left[\frac{\delta^{-1} + \alpha^{-1}}{\delta^{-1} + \delta}\right]\chi_1 + \left[\frac{\delta - \alpha^{-1}}{\delta^{-1} + \delta}\right]\chi_2 - 1 = 0. \tag{3.6}$$

Using [11, Theorem 3.18], equation (3.6) shows that $\chi_1 = 1$ and $\delta = \alpha^{-1}$ or $\chi_2 = 1$ and $\delta = \alpha$ since $\chi_1 \neq \chi_2$, then $g = \frac{\chi + \alpha^2}{\alpha^2+1}$, where χ is a multiplicative function. Using this form of g in equation (2.3) we get after some rearrangements that $f = A - \frac{\alpha^2}{(\alpha^2+1)^2}(\chi - 1)$, where A is an additive function. This occurs in case (a).

Fourth case: the pair $(g, \alpha(g-1))$ has the form (iv). In this case we have

$$\chi + (i\alpha^{-1} \pm 1)\phi - 1 = 0. \tag{3.7}$$

Since $\chi \neq 0$, we deduce that $\phi \neq \chi$ because otherwise, we will have $\chi(xy) = 2\chi(x)\chi(y)$ for all $x, y \in S$ which implies χ=0. In view of [4, Lemma 4.2], the identity (3.7) shows that $\phi = 1$ or $\chi = 1$.

The case $\phi = 1$ does not occur because, otherwise, we will have $\chi = 1$, which contradicts the fact that $\phi \neq \chi$. So we have to take $\chi = 1$. It follows that ϕ is additive, consequently $g = A + 1$, where A is an additive function. Then, from (2.3) we deduce that f and g satisfy $f(xy) = f(x) + f(y) - g(xy) + g(x)g(y)$, $x, y \in S$, or equivalently

$$f(xy) = f(x) + f(y) + A(x)A(y), \quad x, y \in S. \tag{3.8}$$

Consider the particular Levi–Civita equation

$$\psi(xy) = \psi(x)\chi(y) + \chi(x)\psi(y) + \phi(x)\phi(y), \quad x, y \in S, \tag{3.9}$$

for unknown $\psi\colon S \to \mathbb{C}$, where χ is a nonzero multiplicative function and ϕ is a nonzero solution of the special sine addition law (3.5). Taking $\chi = 1$ and $\phi = A$, one can observe that the functional equation (3.8) is a particular instance of (3.9). According to [4, Theorem 4.1] with $\chi = 1$, we have $f = B + \frac{1}{2}A^2$, where B is an additive function. This is case (b), and this completes the proof. □

Theorem 3.2. *Let S be a semigroup. If functions $f\colon S \to \mathbb{C}$ and $g\colon S \to \mathbb{C}$ are not quadratically equivalent then they satisfy the functional equation*

$$f(xy) + g(xy) = f(x) + f(y) + g(x)g(y), \quad x, y \in S,$$

if and only if g is multiplicative and f is additive.

Proof. The "if" part of the proof is obvious.

Let the pair (f, g) be a solution of (2.3). If the pair (f, g) has the form as in Theorem 3.1(b), then $\Delta_y^3(f + g) = 0$. This is a contradiction. If (f, g) has the form as in Theorem 3.1(a) such that $\alpha \neq 0$, then we find that $\Delta_y^3[f - (c-1)g] = \Delta_y^3[A - (c-1)] = 0$, where $c = \frac{1}{\alpha^2+1}$, which contradicts the fact that f and g are not quadratically equivalent. So, necessarily we have $\alpha = 0$. Then g is multiplicative and f is additive. This completes the proof. □

4. Solutions of the functional equations (2.6) and alienation of equations (2.7) and (2.8)

In this section we give (in Theorem 4.2) an exhaustive list of solutions of the functional equation (2.6). We express them in terms of multiplicative and additive functions. Furthermore, on the basis of these results, we prove in Theorem 4.3 that the functional equations (2.1) and (2.2) are alien to each other for non-quadratically equivalent functions.

To prove the next theorem we will need the following:

Lemma 4.1. *Let $f\colon S \to \mathbb{C}$ be a solution of the functional equation* (2.7). *Then f is of the form $f = A + \int_S A(t)d\mu(t)$ where A is an additive function.*

Proof. By replacing x by xy and y by ks in (2.7) and integrating the result obtained with respect to k and s we get

$$(4.1)\quad \int_S \int_S \int_S f(xykst)d\mu(k)d\mu(s)d\mu(t) = f(xy)\left(\int_S d\mu(t)\right)^2 + \int_S \int_S f(ks)d\mu(k)d\mu(s).$$

By replacing x by xyk and y by s in (2.7) and integrating the result obtained with respect to k and s we obtain

$$(4.2)\quad \int_S \int_S \int_S f(xykst)d\mu(k)d\mu(s)d\mu(t) = \int_S f(xyk)d\mu(k)\int_S d\mu(t) + \int_S f(s)d\mu(s)\int_S d\mu(t) = \int_S d\mu(t)\left[f(x) + f(y) + \int_S f(s)d\mu(s)\right].$$

By comparing (4.1) and (4.2) and taking into account that $\int_S d\mu(t) = 1$ we deduce that $f + \int_S f(s)d\mu(s) - \int_S \int_S f(ks)d\mu(k)d\mu(s)$ is additive. Consequently $f = A - \int_S f(s)d\mu(s) + \int_S \int_S f(ks)d\mu(k)d\mu(s)$, where A is an additive function. Putting this back into (2.7) we find that $-\int_S f(s)d\mu(s) + \int_S \int_S f(ks)d\mu(k)d\mu(s) = \int_S A(s)d\mu(s)$. This completes the proof. □

Theorem 4.2. *The solutions $f, g\colon S \to \mathbb{C}$ of the functional equation*

$$\int_S f(xyt)d\mu(t) + \int_S g(xyt)d\mu(t) = f(x) + f(y) + g(x)g(y), \quad x, y \in S,$$

are the following pairs:

(a) $g = c$, $f = A + \int_S A(t)d\mu(t) + c - c^2$, *where $c \in \mathbb{C}$ and A is additive;*

(b) $g = A$, $f = B + \int_S B(t)d\mu(t) + \frac{1}{2}\left[A^2 + \int_S A^2(t)d\mu(t)\right] - 1$, *where A and B are additive, and $\int_S A(t)d\mu(t) = -1$;*

(c) $g = c(\varphi - 1)$, $f = A + \int_S A(t)d\mu(t) + c^2(\varphi - 1) - c$, *where* $c \in \mathbb{C}\setminus\{-1, 0\}$, φ *is a multiplicative function such that* $\varphi \neq 1$, $\int_S \varphi(t)d\mu(t) = \frac{c}{c+1}$ *and* A *is additive;*

(d) $g = \chi \int_S \chi(t)d\mu(t)$, $f = A + \int_S A(t)d\mu(t)$, *where* χ *is an nonzero multiplicative function such that* $\chi \neq 1$ *and* A *is additive;*

(e) $g = r\frac{\chi+\alpha^2}{\alpha^2+1}$, $f = r^2\left[A + \int_S A(t)d\mu(t) - \frac{\alpha^2}{(\alpha^2+1)^2}(\chi - 1)\right] + r(1-r)\frac{\alpha^2}{\alpha^2+1}$, *where* $\alpha \in \mathbb{C}\setminus\{0, i, -i\}$, $r \in \mathbb{C}\setminus\{0\}$ *and* $r \neq \frac{\alpha^2+1}{\alpha^2}$, A *is additive and* χ *is a multiplicative function such that* $\chi \neq 1$ *and* $\int_S \chi(t)d\mu(t) = \frac{r}{\alpha^2(1-r)+1}$*;*

(f) $g = r(A+1)$, $f = r^2\left[B + \int_S B(t)d\mu(t) + \frac{1}{2}A^2 + \frac{1}{2}\int_S A^2(t)d\mu(t)\right] - (r-1)^2$, *where* $r \in \mathbb{C}\setminus\{0\}$ *and* A, B *are additive such that* $A \neq 0$ *and* $\int_S A(t)d\mu(t) = \frac{r-1}{r}$.

Proof. Let the pair (f, g) be a solution of (2.6). If g is constant, that is $g = c$, where $c \in \mathbb{C}$, we see from (2.6) that $f - c + c^2$ is a solution of equation (2.7). Then, using Lemma 4.1, we conclude that $f - c + c^2 = A + \int_S A(t)d\mu(t)$, where A is additive. This proves (a). From now on we assume that g is not constant.

Using the associative property of the semigroup operation, we compute the term $\int_S f(xyzt)d\mu(t) + \int_S g(xyzt)d\mu(t)$ first as $\int_S f((xy)zt)d\mu(t) + \int_S g((xy)zt)d\mu(t)$, then as $\int_S f(x(yz)t)d\mu(t) + \int_S g(x(yz)t)d\mu(t)$. Comparing the results, we obtain the following

$$f(xy) + f(z) + g(xy)g(z) = f(x) + f(yz) + g(x)g(yz), \quad x, y, z \in S. \tag{4.3}$$

Subtracting $g(x)g(y)g(z) + f(y)$ from both sides of (4.3), we get

$$\begin{aligned} f(xy) - f(x) - f(y) + g(z)\left[g(xy) - g(x)g(y)\right] \\ = f(yz) - f(y) - f(z) + g(x)\left[g(yz) - g(y)g(z)\right], \quad x, y, z \in S. \end{aligned} \tag{4.4}$$

Since g is not constant, there exist $z_1, z_2 \in S$ such that $g(z_1) \neq g(z_2)$. So, from (4.4) we get

$$f(xy) - f(x) - f(y) + g(z_1)\left[g(xy) - g(x)g(y)\right] = k_1(y) + g(x)k_2(y), \quad x, y \in S,$$

and

$$f(xy) - f(x) - f(y) + g(z_2)\left[g(xy) - g(x)g(y)\right] = k_3(y) + g(x)k_4(y), \quad x, y \in S,$$

for some functions k_j, $j = 1, 2, 3, 4$, or equivalently

$$\begin{bmatrix} 1 & g(z_1) \\ 1 & g(z_2) \end{bmatrix} \begin{bmatrix} f(xy) - f(x) - f(y) \\ g(xy) - g(x)g(y) \end{bmatrix} = \begin{bmatrix} k_1(y) & k_2(y) \\ k_3(y) & k_4(y) \end{bmatrix} \begin{bmatrix} 1 \\ g(x) \end{bmatrix}, \quad x, y \in S.$$

Since $g(z_1) \neq g(z_2)$ we have

$$f(xy) - f(x) - f(y) = \rho_1(y) + g(x)\rho_2(y), \quad x, y \in S, \tag{4.5}$$

and

$$g(xy) - g(x)g(y) = \rho_3(y) + g(x)\rho_4(y), \quad x, y \in S, \tag{4.6}$$

for some functions ρ_j, $j = 1, 2, 3, 4$.

Putting this back into (4.4) we get after some rearrangements that

$$\begin{aligned} g(x)\left[\rho_2(y) + g(z)\rho_4(y) - \rho_3(z) - g(y)\rho_4(z)\right] \\ = \rho_1(z) + g(y)\rho_2(z) - \rho_1(y) - g(z)\rho_3(y), \quad x, y, z \in S. \end{aligned} \tag{4.7}$$

Taking into account that g is not constant, (4.7) shows that

$$\rho_1(z) + g(y)\rho_2(z) = \rho_1(y) + g(z)\rho_3(y), \quad y, z \in S, \tag{4.8}$$

and

$$\rho_2(y) + g(z)\rho_4(y) = \rho_3(z) + g(y)\rho_4(z), \quad y, z \in S. \tag{4.9}$$

Consequently, for the same reason, (4.8) and (4.9) show that

$$\rho_j = a_j + b_j g, \quad j = 1, 2, 3, 4, \tag{4.10}$$

where a_j, b_j are constants.

Substituting (4.10) into (4.8) and (4.9) we infer that

$$b_1 = a_2 = a_3 \quad \text{and} \quad b_3 = b_2 = a_4.$$

Then, from (4.5) and (4.6) we deduce that f and g satisfy the functional equations

$$f(xy) = f(x) + f(y) + b_1 g(x) + b_1 g(y) + b_2 g(x)g(y) + a_1, \quad x, y \in S, \tag{4.11}$$

and

$$g(xy) = (b_4 + 1)g(y)g(x) + b_2 g(x) + b_2 g(y) + b_1, \quad x, y \in S. \tag{4.12}$$

By replacing y by t in (4.11) and (4.12) and integrating the results obtained with respect to t we get

$$\int_S f(xt)d\mu(t) = f(x) + a_0 g(x) + b_0, \quad x \in S, \tag{4.13}$$

and

$$\int_S g(xt)d\mu(t) = c_0 g(x) + d_0, \quad x \in S, \tag{4.14}$$

where $a_0 := b_1 + b_2 \int_S g(t)d\mu(t)$, $b_0 := \int_S f(t)d\mu(t) + b_1 \int_S g(t)d\mu(t) + a_1$, $c_0 := (b_4+1)\int_S g(t)d\mu(t) + b_2$ and $d_0 := b_2 \int_S g(t)d\mu(t) + b_1$. Now, by using the relations (4.13) and (4.14) in (2.6) we get

$$f(xy) + rg(xy) + q = f(x) + f(y) + g(x)g(y), \quad x, y \in S, \tag{4.15}$$

where $r := a_0 + c_0$ and $q = b_0 + d_0$.

First case: $r = 0$. In this case (4.15) becomes

$$f(xy) + q = f(x) + f(y) + g(x)g(y), \quad x, y \in S. \tag{4.16}$$

Equation (4.16) shows that the Cauchy difference C_f has the form

$$C_f(x, y) = g(x)g(y) - q, \quad x, y \in S.$$

Using again the cocycle equation (3.1) leads to

$$g(z)\left[g(xy) - g(y)\right] = g(x)\left[g(yz) - g(y)\right], \quad x, y, z \in S.$$

Since $g \neq 0$, we deduce that

$$g(xy) = g(x)\varphi(y) + g(y), \quad x, y \in S, \tag{4.17}$$

where $\varphi(y) := (g(s_0))^{-1}\left[g(ys_0) - g(y)\right]$, for some $s_0 \in S$ such that $g(s_0) \neq 0$.

Now, from [12, Proposition 3], the identity (4.17) shows that φ is multiplicative. In view of (4.12) g is central: $g(xy) = g(yx)$, $x, y \in S$. Using the centrality of g, (4.17) shows that

$$g(x)\left[\varphi(y) - 1\right] = g(y)\left[\varphi(x) - 1\right], \quad x, y \in S.$$

We may distinguish two subcases here.

If $\varphi = 1$, then (4.17) shows that $g = A$, where A is additive, and since f and g satisfy equation (4.16), we conclude that

$$f(xy) - q = [f(x) - q] + [f(y) - q] + A(x)A(y), \quad x, y \in S.$$

Then from [4, Theorem 4.1] we have $f = B + \frac{1}{2}A^2 + q$, where B is an additive function. Since the pair (f, g) satisfies (2.6), by taking $x = y$ we have the following relation

$$(4.18)\quad 2\left[\int_S A(t)d\mu(t) + 1\right] A(x)$$
$$= q - \int_S A(t)d\mu(t) - \frac{1}{2}\int_S A^2(t)d\mu(t) - \int_S B(t)d\mu(t), \quad x \in S.$$

Recall that g is a nonconstant function. So, from (4.18) we deduce that $\int_S A(t)d\mu(t) = -1$ and $q = \int_S B(t)d\mu(t) + \frac{1}{2}\int_S A^2(t)d\mu(t) - 1$. This proves (b).

If $\varphi \neq 1$, then

$$(4.19)\quad g(x) = c(\varphi(x) - 1), \quad x \in S,$$

where $c \in \mathbb{C}\setminus\{0\}$. Inserting (4.19) into (4.16), we deduce that $f - c^2(\varphi - 1) - q$ is an additive function, that is $f = A + c^2(\varphi - 1) + q$, where A is additive. Putting back the new forms of f and g into (2.6) yields

$$(4.20)\quad c\left[\int_S \varphi(t)d\mu(t) + c\int_S \varphi(t)d\mu(t) - c\right]\varphi(x)\varphi(y)$$
$$= q - \int_S A(t)d\mu(t) + c, \quad x, y \in S.$$

From (4.19) we deduce that φ can not be constant. So (4.20) shows that $\int_S \varphi(t)d\mu(t) + c\int_S \varphi(t)d\mu(t) - c = 0$ and $q - \int_S A(t)d\mu(t) + c = 0$. Then $f(x) = A(x) + \int_S A(t)d\mu(t) + c^2(\varphi - 1) - c, \quad x \in S$ and $\int_S \varphi(t)d\mu(t) = \frac{c}{c+1}$. This proves (c).

Second case: $r \neq 0$. From equation (4.15) one can easily verify that $F := r^{-2}(f - q)$ and $G := r^{-1}g$ solve the functional equation (2.3). So, we know from Theorem 3.1 that there are only the following two possibilities:

(i) $g = r\frac{\chi + \alpha^2}{\alpha^2 + 1}$ and $f = r^2\left[A - \frac{\alpha^2}{(\alpha^2+1)^2}(\chi - 1)\right] + q$, where $\alpha \in \mathbb{C}\setminus\{i, -i\}$, A is additive and χ is multiplicative.

(ii) $g = r(A + 1)$ and $f = r^2\left[B + \frac{1}{2}A^2\right] + q$, where A, B are additive functions.

Subcase 1: If the pair (f,g) has the form (i). If $\alpha=0$, then f and g satisfy the following equations

$$f(xy)-q=[f(x)-q]+[f(y)-q], \quad x,y\in S, \tag{4.21}$$

and

$$r^{-1}g(xy)=\left[r^{-1}g(x)\right]\left[r^{-1}g(y)\right], \quad x,y\in S, \tag{4.22}$$

respectively.

From (4.21) and (4.22) we deduce that

$$\int_S f(xyt)d\mu(t)=f(x)+f(y)+\int_S f(t)d\mu(t)-2q, \quad x,y\in S \tag{4.23}$$

and

$$\int_S g(xyt)d\mu(t)=r^{-2}g(x)g(y)\int_S g(t)d\mu(t), \quad x,y\in S. \tag{4.24}$$

Now, by inserting (4.23) and (4.24) into (2.6), we see that

$$\left[r^{-2}\int_S g(t)d\mu(t)-1\right]g(x)g(y)=2q-\int_S f(t)d\mu(t), \quad x,y\in S.$$

Since g is not constant, the last identity implies that $r^{-2}\int_S g(t)d\mu(t)=1$ and $2q-\int_S f(t)d\mu(t)=0$. It follows that f and g are solutions of (2.7) and (2.8) respectively. Then, from Lemma 4.1 and [5, Corollary 2.5], we deduce the desired forms of f and g. This proves (d).

From now on we assume that $\alpha\neq 0$. Using the notation $c:=\frac{1}{\alpha^2+1}$, f and g can be written as $f=r^2\left[A-c(1-c)(\chi-1)\right]+q$ and $g=r\left[c\chi+(1-c)\right]$ respectively. Putting this back into (2.6), one gets

$$\begin{aligned} rc\left[(1-r(1-c))\int_S \chi(t)d\mu(t)-rc\right]\chi(x)\chi(y) \\ =q-r^2\int_S A(t)d\mu(t)-r(1-r)(1-c), \quad x,y\in S. \end{aligned} \tag{4.25}$$

As χ is not constant, (4.25) implies that $1-r(1-c)\neq 0$ and $(1-r(1-c))\int_S \chi(t)d\mu(t)-rc=0$. Consequently we have $r\neq\frac{1}{1-c}$, $\int_S \chi(t)d\mu(t)=\frac{rc}{1-r(1-c)}$ and $q-r^2\int_S A(t)d\mu(t)-r(1-c)(1-r)=0$. This proves (e).

Subcase 2: If the pair (f, g) has the form (ii). Then, taking $x = y$, (2.6) implies that

$$2r\left[r\int_S A(t)d\mu(t) + 1 - r\right]A(x) = q - r + r^2 - \frac{r^2}{2}\int_S A^2(t)d\mu(t) - r\int_S A(t)d\mu(t) - r^2\int_S B(t)d\mu(t), \quad x \in S. \tag{4.26}$$

Since A is not constant, we deduce from (4.26) that $r\int_S A(t)d\mu(t)+1-r = 0$. Consequently we have $\int_S A(t)d\mu(t) = \frac{r-1}{r}$ and $q = \frac{r^2}{2}\int_S A^2(t)d\mu(t) + r^2\int_S B(t)d\mu(t) - (r-1)^2$. This proves case (f). □

THEOREM 4.3. *Let S be a semigroup and μ is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$ for some fixed elements $(z_i)_{i\in I}$ contained in S such that $\int_S d\mu = 1$. If functions $f\colon S \to \mathbb{C}$ and $g\colon S \to \mathbb{C}$ are not quadratically equivalent then they satisfy the equation*

$$\int_S f(xyt)d\mu(t) + \int_S g(xyt)d\mu(t) = f(x) + f(y) + g(x)g(y), \quad x, y \in S,$$

if and only if f and g solve the system

$$\begin{cases} \int_S f(xyt)d\mu(t) = f(x) + f(y), \\ \int_S g(xyt)d\mu(t) = g(x)g(y), \quad x, y \in S. \end{cases}$$

PROOF. The "if"" part of the proof is obvious.

Let the pair (f, g) be a solution of (2.6). We can easily check that if (f, g) has one of the forms (a), (b) and (f) of Theorem 4.2, then we have $\Delta_y^3(f + g) = 0$, which contradicts our hypothesis about f and g. In the case when (f, g) has the form (c) we obtain $\Delta_y^3(f - cg) = 0$ and when (f, g) has the form (e) we have $\Delta_y^3(f + r_0\frac{\alpha^2}{\alpha^2+1}g) = 0$. For the same reason, the two last cases does not occur. It follows that the only possible case is when (f, g) has the form (d). This completes the proof. □

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