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NEW RESULTS ABOUT QUADRATIC FUNCTIONAL EQUATION ON SEMIGROUPS

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Abstract. Let S be a semigroup, let $(H,+)$ be a uniquely 2-divisible, abelian group and let φ,ψ be two endomorphisms of S that need not be involutive. In this paper, we express the solutions $f\colon S\to H$ of the following quadratic functional equation

$$f(x\varphi(y))+f(\psi(y)x)=2f(x)+2f(y),\quad x,y\in S,$$

in terms of bi-additive maps and solutions of the symmetrized additive Cauchy equation. Some applications of this result are presented.

1. Set up and notation

Throughout our paper the setup, notation and terminology are as follows: S is a semigroup (a set equipped with an associative composition rule $(x,y)\mapsto xy$), φ and ψ are two endomorphisms of S, and $(H,+)$ denotes a 2-torsion free, abelian group with neutral element 0. The group $(H,+)$ is 2-torsion free means that

$$2h=0\implies h=0\quad\text{for all } h\in H.$$

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We say that $(H,+)$ is uniquely 2-divisible, if the map $h \mapsto 2h$ is a bijection of H onto H. The endomorphism φ is said to be involutive if $\varphi^2 = id$.

Let $f\colon S \to H$ be a function. We say that:

(i) f is φ-even if $f \circ \varphi = f$,
(ii) f is additive if $f(xy) = f(x) + f(y)$ for all $x, y \in S$, and
(iii) f is central if $f(xy) = f(yx)$ for all $x, y \in S$.

Let $B\colon S \times S \to H$ be a map. We say that:

(i) B is bi-additive if it is additive in each variable,
(ii) B is symmetric if $B(x,y) = B(y,x)$ for all $x, y \in S$, and
(iii) B is φ-invariant if it verifies $B(\varphi(x), \varphi(y)) = B(x,y)$ for all $x, y \in S$.

We recall that the Cauchy difference $Cf\colon S \times S \to H$ of a function $f\colon S \to H$ is defined by

$$Cf(x,y) := f(xy) - f(x) - f(y), \quad x, y \in S,$$

while its second order Cauchy difference $C^2 f\colon S \times S \times S \to H$ is defined by

$$C^2 f(x,y,z) := f(xyz) - f(xy) - f(yz) - f(xz) + f(x) + f(y) + f(z).$$

The functional equation $C^2 f = 0$ is called Whitehead's functional equation by Faĭziev and Sahoo [10]. It is equivalent to

$$f(xyz) + f(x) + f(y) + f(z) = f(xy) + f(yz) + f(xz), \quad x, y, z \in S. \tag{1.1}$$

This equation was studied and solved on semigroups by Stetkær in [17].

The functional equation

$$f(xy) + f(yx) = 2f(x) + 2f(y), \quad x, y \in S, \tag{1.2}$$

is called the symmetrized additive Cauchy equation. Some results about (1.2) can be found in [16, Chapter 2].

We say that the semigroup S is a topological semigroup, if S is equipped with a topology such that the map $(x,y) \mapsto xy$ from $S \times S$ to S is continuous, with the product topology considered in $S \times S$.

2. Introduction

Jordan and von Neumann [12] proved that the norm of a normed vector space $(V, \|.\|)$ is pre-Hilbertian if and only if it satisfies the following parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in V.$$

By setting $f(x) := \|x\|^2$, we obtain the following quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in V, \tag{2.1}$$

which was generalized to a group G by considering the following functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \quad x, y \in G.$$

This equation was investigated by several authors in different algebraic structures (see, e.g., [5–9, 11, 13–15] and [16, Chapter 13]).

In [9], Fadli et al. studied the solutions $f\colon S \to H$ of the quadratic functional equation

$$f(xy) + f(\tau(y)x) = 2f(x) + 2f(y), \quad x, y \in S,$$

where τ is an involutive automorphism of S. In [1], Aissi et al. studied the following generalization of quadratic functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x) + 2f(y), \quad x, y \in S,$$

where σ and τ are two involutive automorphisms on S, by determining its solutions $f\colon S \to H$ in terms of additive and bi-additive maps, and solutions of the symmetrized additive Cauchy equation.

Note that the last equation was studied on abelian semigroups by Fadli et al. in [8]. Also this equation was studied on abelian semigroups in the case where $\sigma = id$ and τ is an arbitrary endomorphism by Sabour and Kabbaj in [13], and was studied on abelian semigroups by Akkaoui et al. in [3] in the case where σ and τ are two arbitrary endomorphisms such that at least one of them is surjective.

Motivated by these recent results, we will solve in this work a generalized variant of quadratic functional equation on a semigroup equipped with endomorphisms, which need not be involutive. More exactly, we find the solutions $f\colon S \to H$ of the functional equation

$$f(x\varphi(y)) + f(\psi(y)x) = 2f(x) + 2f(y), \quad x, y \in S, \tag{2.2}$$

in terms of bi-additive maps and solutions of the symmetrized additive Cauchy equation. Since S is not necessarily abelian, and φ and ψ are not necessarily involutive automorphisms, equation (2.2) is a natural generalization of the previous functional equations.

We note that the equations (1.1) and (1.2) play important roles in finding the solutions of the functional equation (2.2).

Similar functional equations that have also been studied are

$$f(x\varphi(y)) + f(\psi(y)x) = 2f(x)f(y), \quad x, y \in S, \tag{2.3}$$

$$f(x\varphi(y)) + f(\psi(y)x) = 2f(x), \quad x, y \in S. \tag{2.4}$$

The complex-valued solutions of (2.3) were determined in [4], while the solutions $f\colon S \to H$ of (2.4) are determined in [2].

Our main contribution to the knowledge about the quadratic functional equation is to solve (2.2) on semigroups (not necessarily abelian) in the case where $(H, +)$ is uniquely 2-divisible and φ, ψ are arbitrary endomorphisms.

3. Main result

To establish our main result Theorem 3.3 we present two important lemmas:

LEMMA 3.1. *Let $f\colon S \to H$ be a solution of* (2.2). *Then*

(i) *the function $f \circ \varphi + f \circ \psi - 2f$ is a constant, and*

(ii) *the function $f \circ \varphi - f \circ \psi$ is additive.*

PROOF. (i) Let $f\colon S \to H$ be a solution of (2.2). Making the substitutions $(\varphi(x), y)$ and $(\psi(y), x)$ in (2.2), we obtain respectively

$$(f \circ \varphi)(xy) + f(\psi(y)\varphi(x)) = 2(f \circ \varphi)(x) + 2f(y),$$

$$f(\psi(y)\varphi(x)) + (f \circ \psi)(xy) = 2(f \circ \psi)(y) + 2f(x),$$

for all $x, y \in S$. By subtracting the second identity from the first, we arrive at

$$(f \circ \varphi - f \circ \psi)(xy) = 2(f \circ \varphi)(x) - 2(f \circ \psi)(y) + 2f(y) - 2f(x) \quad \text{for all } x, y \in S. \tag{3.1}$$

Now, making the substitutions (x, yz) and (xy, z) in (3.1), we get respectively for all $x, y, z \in S$

$$(f\circ\varphi - f\circ\psi)(xyz) = 2(f\circ\varphi)(x) - 2(f\circ\psi)(yz) + 2f(yz) - 2f(x),$$
$$(f\circ\varphi - f\circ\psi)(xyz) = 2(f\circ\varphi)(xy) - 2(f\circ\psi)(z) + 2f(z) - 2f(xy).$$

Subtracting the second identity from the first, we get that

$$\begin{aligned} &2(f\circ\varphi)(x) - 2(f\circ\varphi)(xy) + 2f(xy) - 2f(x) \\ &\qquad = 2(f\circ\psi)(yz) - 2(f\circ\psi)(z) + 2f(z) - 2f(yz), \end{aligned}$$

for all $x, y \in S$. Since H is 2-torsion free, we can deduce that the expressions $(f\circ\varphi)(x) - (f\circ\varphi)(xy) + f(xy) - f(x)$ and $(f\circ\psi)(yz) - (f\circ\psi)(z) + f(z) - f(yz)$ depend only on y. Let h be a function on y such that

$$h(y) = (f\circ\varphi)(x) - (f\circ\varphi)(xy) + f(xy) - f(x), \quad y \in S, \tag{3.2}$$

$$h(y) = (f\circ\psi)(yz) - (f\circ\psi)(z) + f(z) - f(yz), \quad y \in S, \tag{3.3}$$

for $x, z \in S$. Replacing y by x and z by y in (3.3) and adding the new identity with (3.2), we arrive at

$$h(x) + h(y) = (f\circ\psi - f\circ\varphi)(xy) + (f\circ\varphi)(x) - (f\circ\psi)(y) + f(y) - f(x),$$

for all $x, y \in S$.

According to (3.1), we conclude for all $x, y \in S$

$$\begin{aligned} h(x) + h(y) &= 2(f\circ\psi)(y) - 2(f\circ\varphi)(x) + 2f(x) - 2f(y) \\ &\quad + (f\circ\varphi)(x) - (f\circ\psi)(y) + f(y) - f(x) \\ &= (f\circ\psi)(y) - (f\circ\varphi)(x) + f(x) - f(y). \end{aligned}$$

So

$$h(x) + (f\circ\varphi)(x) - f(x) = (f\circ\psi)(y) - f(y) - h(y), \tag{3.4}$$

for all $x, y \in S$. For $y = x$ in (3.4), we obtain $2h = f\circ\psi - f\circ\varphi$. Multiplying (3.4) by 2 and replacing $2h$ by its expression, we get after a reduction

$$(f\circ\varphi)(x) + (f\circ\psi)(x) - 2f(x) = (f\circ\varphi)(y) + (f\circ\psi)(y) - 2f(y),$$

for all $x, y \in S$. That signifies that $f\circ\varphi + f\circ\psi - 2f$ is constant.

(ii) Let $f\colon S \to H$ be a solution of (2.2), and let $x, y \in S$ be arbitrary. From (i), $f \circ \varphi + f \circ \psi - 2f$ is constant. Let $c \in H$ be such that $2f = f \circ \varphi + f \circ \psi + c$. Then identity (3.1) gives

$$\begin{aligned}(f \circ \varphi - f \circ \psi)(xy) &= 2(f \circ \varphi)(x) - 2(f \circ \psi)(y) + (f \circ \varphi)(y) \\ &\quad + (f \circ \psi)(y) + c - (f \circ \varphi)(x) - (f \circ \psi)(x) - c \\ &= (f \circ \varphi - f \circ \psi)(x) + (f \circ \varphi - f \circ \psi)(y),\end{aligned}$$

which means that $f \circ \varphi - f \circ \psi$ is additive. □

The following lemma gives some important results about (2.2).

LEMMA 3.2. *If $f\colon S \to H$ is a solution of* (2.2), *then*

(i) *f satisfies Whitehead's functional equation* (1.1), *i.e.,* $C^2 f = 0$,
(ii) $f \circ \varphi + f \circ \psi = 2f$,
(iii) *$Cf\colon S \times S \to H$ is a bi-additive φ-invariant function satisfying $Cf(\psi(y), x) = -Cf(x, \varphi(y))$ for all $x, y \in S$, and*
(iv) *the function $j\colon x \longmapsto 2f(x) - Cf(x, x)$ is a solution of the symmetrized additive Cauchy equation satisfying $Cj(x, y) = Cf(x, y) - Cf(y, x)$ for all $x, y \in S$.*

PROOF. (i) Let $f\colon S \to H$ be a solution of (2.2) and let $x, y, z \in S$. Making the substitutions $(\varphi(x), yz)$, $(\psi(z)\varphi(x), y)$ and $(\varphi(xy), z)$ in (2.2), we get

$$(f \circ \varphi)(xyz) + f(\psi(yz)\varphi(x)) = 2(f \circ \varphi)(x) + 2f(yz),$$

$$f(\psi(z)\varphi(xy)) + f(\psi(yz)\varphi(x)) = 2f(\psi(z)\varphi(x)) + 2f(y),$$

$$(f \circ \varphi)(xyz) + f(\psi(z)\varphi(xy)) = 2(f \circ \varphi)(xy) + 2f(z).$$

Subtracting the middle identity from the sum of the other two we get

$$\begin{aligned}2(f \circ \varphi)(xyz) &= 2(f \circ \varphi)(xy) + 2(f \circ \varphi)(x) - 2f(\psi(z)\varphi(x)) \\ &\quad + 2f(yz) + 2f(z) - 2f(y).\end{aligned} \tag{3.5}$$

From (2.2), we see that

$$f(\psi(z)\varphi(x)) = 2(f \circ \varphi)(x) + 2f(z) - (f \circ \varphi)(xz).$$

Then (3.5) becomes

$$\begin{aligned}2(f \circ \varphi)(xyz) &= 2(f \circ \varphi)(xy) + 2(f \circ \varphi)(xz) - 2(f \circ \varphi)(x) \\ &\quad + 2f(yz) - 2f(y) - 2f(z).\end{aligned} \tag{3.6}$$

From Lemma 3.1(ii) we have $f\circ\varphi+f\circ\psi=2f\circ\varphi-a$, where $a:=f\circ\varphi-f\circ\psi$ is an additive map, and from Lemma 3.1(i) there exists a constant $c\in H$ such that $2f=f\circ\varphi+f\circ\psi+c$. Hence $2f\circ\varphi=2f+a-c$, so (3.6) reads

$$\begin{aligned}2f(xyz)+a(xyz)-c&=2f(xy)+a(xy)-c+2f(xz)+a(xz)-c\\&\quad-2f(x)-a(x)+c+2f(yz)-2f(y)-2f(z).\end{aligned}$$

Then

$$f(xyz)=f(xy)+f(xz)+f(yz)-f(x)-f(y)-f(z),$$

for all $x,y,z\in S$, because H is 2-torsion free, which means that $C^2f=0$.

(ii) According to (i), we have $C^2f=0$, then $C^2(f\circ\varphi)=C^2(f\circ\psi)=0$, because φ,ψ are endomorphisms. But $2f=f\circ\varphi+f\circ\psi+c$, then we get $C^2(c)=0$, because the operator C^2 is linear. So $c=0$ and hence $2f=f\circ\varphi+f\circ\psi$.

(iii) That $Cf\colon S\times S\to H$ is a bi-additive map follows immediately from (i) and [17, Lemma 3], and by using (ii) and (2.2), we obtain readily $Cf(\psi(y),x)=-Cf(x,\varphi(y))$ for all $x,y\in S$. Also we have

$$Cf(\varphi(x),\varphi(y))=-Cf(\psi(y),\varphi(x))=Cf(\psi(x),\psi(y))\quad\text{for all } x,y\in S,$$

which implies that $C(f\circ\varphi)=C(f\circ\psi)$. Since $f\circ\varphi+f\circ\psi=2f$, we obtain $C(f\circ\varphi)=C(f\circ\psi)=Cf$. Then Cf is φ-invariant (also ψ-invariant).

The result (iv) follows from (i) and [17, Lemma 4], and we have

$$Cj(x,y)=2Cf(x,y)-[Cf(x,y)+Cf(y,x)]=Cf(x,y)-Cf(y,x),$$

for all $x,y\in S$, where $j\colon S\to H$ is defined by $j\colon x\longmapsto 2f(x)-Cf(x,x)$. This finishes the proof. □

In the previous results of this section, we considered $(H,+)$ as a 2-torsion free group. In what follows, we assume that $(H,+)$ is uniquely 2-divisible, and we begin by presenting our main result, which allows us to solve equation (2.2). Note that any uniquely 2-divisible group is 2-torsion free.

THEOREM 3.3. *The solutions $f\colon S\to H$ of the functional equation* (2.2) *are the functions of the form*

$$f(x)=J(x)+B(x,x),\quad x\in S, \tag{3.7}$$

where $J\colon S \to H$ *is a solution of the symmetrized additive Cauchy equation such that* $J \circ \varphi + J \circ \psi = 2J$, *and where* $B\colon S \times S \to H$ *is a bi-additive and* φ*-invariant map such that for all* $x, y \in S$ *we have*

(i) $B(x,y) - B(y,x) = CJ(x,y)$, *and*
(ii) $B(\psi(y),x) = -B(x,\varphi(y))$.

Moreover, if S *is a topological semigroup,* H *a topological vector space over* $\mathbb{R}$ *or* $\mathbb{C}$, *and* $f\colon S \to H$ *a continuous solution of* (2.2), *then the components* J *and* B *in decomposition* (3.7) *are continuous.*

Proof. Let $f\colon S \to H$ be a solution of (2.2). From Lemma 3.2(i) we see that $C^2 f = 0$. Since $(H,+)$ is uniquely 2-divisible we can define the functions $B(x,y) := \frac{1}{2}Cf(x,y)$ and $J(x) := f(x) - \frac{1}{2}Cf(x,x)$, where B is bi-additive and φ-invariant satisfying $B(\psi(y),x) = -B(x,\varphi(y))$ for all $x, y \in S$ (Lemma 3.2(iii)), and J is a solution of the symmetrized additive Cauchy equation satisfying $CJ(x,y) = B(x,y) - B(y,x)$ for all $x, y \in S$ (Lemma 3.2(iv)). So $f(x) = J(x) + B(x,x)$.

Since $2f = f \circ \varphi + f \circ \psi$ (Lemma 3.2(ii)) and B is φ-invariant (Lemma 3.2(iii)), then $J \circ \varphi + J \circ \psi = 2J$.

Conversely, let $f\colon S \to H$ be a function of the form stated in Theorem 3.3, i.e., there exist a solution of the symmetrized additive Cauchy equation $J\colon S \to H$ with $J \circ \varphi + J \circ \psi = 2J$ and a bi-additive and φ-invariant map $B\colon S \times S \to H$ with $B(\psi(y),x) = -B(x,\varphi(y))$ and $CJ(x,y) = B(x,y) - B(y,x)$ for all $x, y \in S$ such that $f(x) = J(x) + B(x,x)$. With this conditions, we have

$$
\begin{aligned}
f(x\varphi(y)) &+ f(\psi(y)x)\\
&= J(x\varphi(y)) + B(x\varphi(y), x\varphi(y)) + J(\psi(y)x) + B(\psi(y)x, \psi(y)x)\\
&= [CJ(x,\varphi(y)) + J(x) + J(\varphi(y))] + [B(x,\varphi(y)) + B(\varphi(y),x)\\
&\quad + B(x,x) + B(\varphi(y),\varphi(y))] + [CJ(\psi(y),x) + J(\psi(y)) + J(x)]\\
&\quad + [B(\psi(y),x) + B(x,\psi(y)) + B(x,x) + B(\psi(y),\psi(y))]\\
&= 2f(x) + 2f(y) + CJ(x,\varphi(y)) + CJ(\psi(y),x)\\
&\quad + B(\varphi(y),x) + B(x,\psi(y))\\
&= 2f(x) + 2f(y) + [B(x,\varphi(y)) - B(\varphi(y),x)]\\
&\quad + [B(\psi(y),x) - B(x,\psi(y))] + B(\varphi(y),x) + B(x,\psi(y))\\
&= 2f(x) + 2f(y).
\end{aligned}
$$

That signifies that f is a solution of (2.2). The continuity statements follow from [17, Theorem 7]. □

4. Applications

As consequences of our main result we have the following corollaries. The first gives the central solutions of (2.2) in terms of additive and symmetric, bi-additive maps.

Corollary 4.1. *The central solutions* $f\colon S \to H$ *of* (2.2) *are the functions of the form*

$$f(x) = J(x) + B(x,x), \quad x \in S, \tag{4.1}$$

where $J\colon S \to H$ *is an additive map such that* $J \circ \varphi + J \circ \psi = 2J$ *and where* $B\colon S \times S \to H$ *is a symmetric, bi-additive and* φ*-invariant map such that*

$$B(x, \psi(y)) = -B(x, \varphi(y)) \quad \text{for all } x, y \in S.$$

Moreover, if S *is a topological semigroup,* H *a topological vector space over* $\mathbb{R}$ *or* $\mathbb{C}$*, and* $f\colon S \to H$ *a continuous solution of* (2.2)*, then the components* J *and* B *in decomposition* (4.1) *are continuous.*

Proof. It suffices to observe that if f is central and has the form (3.7) then J is additive and B is symmetric. $\square$

With $\varphi = id$ in Theorem 3.3 we obtain the following corollary that generalizes the main result studied in [9].

Corollary 4.2. *The solutions* $f\colon S \to H$ *of the functional equation*

$$f(xy) + f(\psi(y)x) = 2f(x) + 2f(y), \quad x, y \in S,$$

are the functions of the form

$$f(x) = J(x) + B(x,x), \quad x \in S,$$

where $J\colon S \to H$ *is a solution of the symmetrized additive Cauchy equation such that* $J \circ \psi = J$*, and where* $B\colon S \times S \to H$ *is a bi-additive map such that for all* $x, y \in S$ *we have*
(i) $B(x,y) - B(y,x) = CJ(x,y)$*, and*
(ii) $B(\psi(y), x) = -B(x,y)$*.*

With $\psi = id$ in Theorem 3.3 we obtain the following corollary.

Corollary 4.3. *The solutions* $f\colon S \to H$ *of the functional equation*

$$f(x\varphi(y)) + f(yx) = 2f(x) + 2f(y), \quad x, y \in S, \tag{4.2}$$

are the functions of the form

$$f(x) = J(x) + B(x, x), \quad x \in S,$$

where $J\colon S \to H$ *is a solution of the symmetrized additive Cauchy equation such that* $J \circ \varphi = J$*, and where* $B\colon S \times S \to H$ *is a bi-additive map such that for all* $x, y \in S$ *we have*

(i) $B(x, y) - B(y, x) = CJ(x, y)$*, and*
(ii) $B(x, \varphi(y)) = -B(y, x)$*.*

The equation (4.2) is a new generalization of the quadratic functional equation (2.1).

With $\psi = \varphi$ in Theorem 3.3, we obtain the following corollary.

Corollary 4.4. *The solutions* $f\colon S \to H$ *of the functional equation*

$$f(x\varphi(y)) + f(\varphi(y)x) = 2f(x) + 2f(y), \quad x, y \in S, \tag{4.3}$$

are the φ*-even solutions of the symmetrized additive Cauchy equation.*

Proof. Let $f\colon S \to H$ be a solution of (4.3). Using Theorem 3.3, there exist a solution of the symmetrized additive Cauchy equation $J\colon S \to H$ such that $J \circ \varphi = J$, and a bi-additive, φ-invariant map $B\colon S \times S \to H$ such that $f(x) = J(x) + B(x, x)$ with the conditions

(i) $B(x, y) - B(y, x) = CJ(x, y)$, and
(ii) $B(\varphi(y), x) = -B(x, \varphi(y))$,

for all $x, y \in S$. The condition (ii) implies that $B(\varphi(x), \varphi(x)) = -B(\varphi(x), \varphi(x))$ for all $x \in S$. So $B(\varphi(x), \varphi(x)) = B(x, x) = 0$ for all $x \in S$, because H is uniquely 2-divisible and B is φ-invariant. That gives that $f = J$. The other direction is easy to verify. $\square$

5. Some examples

Example 5.1. Let G be the $(ax+b)$-group defined by

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a > 0,\, b \in \mathbb{R} \right\}$$

and let us consider the following endomorphisms on G:

$$\varphi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \psi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

for all $a > 0$ and $b \in \mathbb{R}$. We determine here the corresponding continuous central solutions $f\colon G \to \mathbb{C}$ of (2.2), which is

$$f\begin{pmatrix} ac^{-1} & b \\ 0 & 1 \end{pmatrix} + f\begin{pmatrix} ac & bc \\ 0 & 1 \end{pmatrix} = 2f\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} + 2f\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix},$$

for all $a, c > 0$ and $b, d \in \mathbb{R}$. Note that φ and ψ are not involutive automorphisms and not surjective.

The continuous additive maps $J\colon G \to \mathbb{C}$ are known in the literature and have the form (see, e.g., [16, Example 2.10])

$$J\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \gamma \log(a),$$

for all $a > 0$ and $b \in \mathbb{R}$, where $\gamma \in \mathbb{C}$ and log is the natural logarithm. The function J satisfies the condition $J \circ \varphi + J \circ \psi = 2J$ if and only if $\gamma = 0$, i.e., $J = 0$.

Also, the continuous bi-additive maps $B\colon G \times G \to \mathbb{C}$ are known in the literature and have the form (see, e.g., [16, Exercise 2.27])

$$B\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right) = \alpha \log(a) \log(c),$$

for all $a, c > 0$ and $b, d \in \mathbb{R}$, where $\alpha \in \mathbb{C}$. We observe that B is symmetric,

$$B\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \psi\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right) = -B\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \varphi\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right),$$

and

$$B\left(\varphi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \varphi\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right) = B\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right),$$

for all $a, c > 0$ and $b, d \in \mathbb{R}$.

According to Corollary 4.1, the continuous central solutions of the above equation are the functions

$$f\colon \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \longmapsto \alpha \log^2(a),$$

where α ranges over $\mathbb{C}$.

EXAMPLE 5.2. Let $S := H_3$ be the Heisenberg group (under matrix multiplication) defined by

$$H_3 := \left\{ [x,y,z] := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{R} \right\}$$

and let us consider the following endomorphisms on S

$$\varphi\left([x,y,z]\right) := [2x,0,0] \quad \text{ and } \quad \psi\left([x,y,z]\right) := [0,2y,0].$$

Note that φ and ψ are not involutive automorphisms and not surjective. In this example we determine the corresponding central continuous solutions $f\colon S \to \mathbb{C}$ of (2.2), which is

$$f([x_1+2x_2,y_1,z_1]) + f([x_1,y_1+2y_2,z_1]) = 2f([x_1,y_1,z_1]) + 2f([x_2,y_2,z_2]),$$

for all $x_1,y_1,z_1,x_2,y_2,z_2 \in \mathbb{R}$.

According to [16, Example 2.11], the continuous additive functions $J\colon S \to \mathbb{C}$ are:

$$J\colon [x,y,z] \mapsto \beta x + \gamma y,$$

where $\beta,\gamma \in \mathbb{C}$. We observe that J satisfies the condition $J\circ\varphi + J\circ\psi = 2J$.

According to [16, Exercise 2.28], the continuous symmetric and bi-additive functions $B\colon S\times S \to \mathbb{C}$ are:

$$B\colon\ ([x_1,y_1,z_1],[x_2,y_2,z_2]) \mapsto ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1),$$

where $a,b,c \in \mathbb{C}$. The function B satisfies

$$B(\varphi([x_1,y_1,z_1]),\varphi([x_2,y_2,z_2])) = B([x_1,y_1,z_1],[x_2,y_2,z_2])$$

for all $x_1,y_1,z_1,x_2,y_2,z_2 \in \mathbb{R}$ if and only if $a=b=c=0$, i.e., $B=0$.

Consequently, from Corollary 4.1, the continuous central solutions $f\colon S \to \mathbb{C}$ of the above equation are the functions

$$f\colon \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \beta x + \gamma y,$$

where $\beta, \gamma \in \mathbb{C}$.

Example 5.3. Let $S := (\mathbb{N}, +)$ (where $\mathbb{N} := \{1, 2, 3, \cdots\}$), $H := (\mathbb{Q}, +)$ and let us consider the following endomorphisms on S

$$\varphi(n) := pn \quad \text{and} \quad \psi(n) := qn \quad \text{for all}\, n \in S,$$

where $p, q \in \mathbb{N} \setminus \{1\}$. Note that S is a semigroup but is not a group, H is a group uniquely 2-divisible and φ, ψ are not surjective. In this example we show that the only corresponding solution $f\colon S \to H$ of (2.2), which is

$$f(n + pm) + f(n + qm) = 2f(n) + 2f(m), \quad n, m \in S,$$

is the null function. By applying Corollary 4.1, there exist an additive map $J\colon S \to H$ and a symmetric bi-additive map $B\colon S \times S \to H$ such that $f(n) = J(n) + B(n, n)$ with the conditions

(i) $(p + q - 2)J(n) = 0$,
(ii) $(p^2 - 1)B(n, m) = 0$, and
(iii) $(p + q)B(n, m) = 0$,

for all $n, m \in S$. Since $p, q > 1$, then $J = 0$ and $B = 0$. So $f = 0$.

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