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Philosophical and Mathematical Correspondence between Gottlob Frege and Bertrand Russell in the years 1902—1904 Some Uninvestigated Topics[*]

Abstract: Although the connections between Frege's and Russell's investigations are commonly known (Hylton 2010), there are some topics in their letters which do not seem to have been analysed until now:
1. Paradoxes formulated by Russell on the basis of Frege's rules: a) „»ξ can never take the place of a proper name« is a false proposition when ξ is a proposition"; b) "A function never takes the place of a subject." A solution of this problem was based on the reference/sense theory and on the distinction between the first- and second-level names (Frege).
2. The inconsistency in Frege's system may be avoided by the introduction of: a) a new kind of objects called quasi-objects (Frege); b) logical types (Frege and Russell); c) mathematics without classes (Russell); d) some restrictions on the domain of function (Frege).
3. Since the inconsistency is connected with a class, what is class? In one of the letters, Frege compared a class to a chair composed of atoms. This approach seems to be similar to the collective understanding of a set (Stanisław Leśniewski).
4. Russell doubted that the difference between sense and reference of expressions was essential. Hence, Frege found some additional reasons to distinguish between them: semiotic, epistemological, from identity, and from mathematical practice. This discussion can be seen as a next step in developing the theory of descriptions by Bertrand Russell.

Keywords: Gottlob Frege, Bertrand Russell, objects, logical types, classes, function, sense, references

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[*] A previous version of this paper was published in Polish: G. Besler: *Tematyka korespondencji naukowej Gottloba Fregego z Bertrandem Russellem w latach*

> Yet there are ideas, to which an approximation has place. That is the case with the mathematical ideas.
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> I. Kant: *Logic*, p. 128.

Introduction

In the year 2013 Felix Meiner Publishing House resumed a German publication of the so far discovered letters written by and to Gottlob Frege.[1] These include, for example, his correspondence with Bertrand Russell, David Hilbert, Edmund Husserl, Giuseppe Peano, Philip Jourdain and Ludwig Wittgenstein. It seems to me that the most interesting part is the collection of letters exchanged between Russell and Frege. In this article I will present some uninvestigated topics of their correspondence.

One hundred years have passed since Frege and Russell exchanged their last letters. This circumstance leads to careful consideration of this correspondence, known mainly from the fact that it related to the problem of antinomy. These letters show a subtle intermingling of mathematics, logic and philosophy, which cannot (at least not always) be kept apart. Frege tries to convince Russell of his proposed solution and unconvinced Russell argues for his position. As a result, we are presented with a continuous attempt to achieve clarity of expression and a frank assessment of these efforts. The motto of this paper well defines the spirit of this correspondence: an at-

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1902—1904. W: *Filozofia matematyki i informatyki*. Red. R. Murawski. Kraków 2015, 91—106.

[1] G. Frege: *Nachgelassene Schriften und wissenschaftlicher Briefwechsel*. Bd. 2 *Wissenschaftlicher Briefwechsel*. Hrsg., bearb., eingel. und mit Anm. versehen von G. Gabriel, H. Hermes, F. Kambartel, Ch. Thiel und A. Veraart. Hamburg, Felix Meiner Verlag (1976) 2013. A selection of this correspondence is also available: *Gottlob Freges Briefwechsel mit D. Hilbert, E. Husserl, B. Russell, sowie ausgewählte Einzelbriefe Freges.*. Einl., Anm., Reg. G. Gabriel, F. Kambartel, Ch. Thiel. Hamburg 1980. Frege and Russell wrote the letters in German. Frege, in fact, was only a logicist concerned with arithmetic; Russell held the stronger thesis: "The Nature of number, of infinity, of space, time and motion, and of mathematical inference itself, are all questions to which […] an answer professing itself demonstrable with mathematical certainty will be given — an answer which, however, consists in reducing the above problems to problems of pure logic […]" (B. Russell, *The Principles of Mathematics*, London, George Allen and Unwin LTD 1956 (1903), 4—5.

tempt to approach their positions by virtue of a common goal, which is to seek the basis of arithmetic.[2]

I would also like to mention a great renaissance of interest in Frege's logic and logical notation introduced by him, as evidenced by:

1. The third German edition of Grundgesetze der Arithmetik;[3]
2. The first full English translation and edition of the above two-volume book;[4] and
3. Noteworthy books on Frege's logic.[5]

What is more, in the field of information science and in research on artificial intelligence, there is a useful concept of contextual definition, strongly connected with Frege's context principle: „never to ask for the meaning [*Bedeutung*] of a word in isolation, but only in context of a proposition" (Frege, 1953, p. X).

Frege's correspondence with Russell shows how many interesting topics remain hitherto unexplored. The goal of this article is to give an account of some of them rather than present the rich literature which discusses the relationship between Frege's and Russell's philosophy (and logic as well).[6]

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[2] Frege investigated bases of arithmetic, Russell bases of whole mathematics. They did not discuss the difference in their approaches. Polish researchers who worked on this subject were Leon Chwistek and Stanisław Leśniewski.

[3] G. Frege: *Grundgesetze der Arithmetik. Begriffsschrift abgeleitet*. Bd. 1 und 2, in moderne Formelnotation transkribiert und mit einem ausführlichen Sachregister versehen von T. Müller, B. Schröder und R. Stuhlmann-Laeisz. Mentis, Paderborn 2009 (Bd. 1 — 1893; Bd. 2 — 1903; 1963).

[4] G. Frege: *Basic Laws of Arithmetic*. Trans. P.A. Ebert, M. Rossberg. Oxford, Oxford University Press 2013.

[5] See: P. Blanchette: *Frege's Conception of Logic*. Oxford, Oxford University Press 2012; R.G. Heck, Jr: *Frege's Theorem*. Clarendon Press, Oxford 2011; G. Landini: *Frege's Notations. What They Are and How They Mean*. New York, Palgrave Macmillan 2012.

[6] See, for example, M. Beaney, Frege and Russell. In: *The Cambridge Companion to Bertrand Russell*. Ed. by N. Griffin. Cambridge University Press, pp. 128—170; A.P. Kubiak, P. Lipski: *Getting Straight on how Russell Underestimated Frege*. „Roczniki Filozoficzne" Tom 62 (2014), nr 4, 121—134.

The letters and their themes

16.06.1902 Russell to Frege;
22.06.1902 Frege to Russell;
24.06.1902 Russell to Frege;
29.06.1902 Frege to Russell;
10.07.1902 Russell to Frege;
24.07.1902 Russell to Frege;
28.07.1902 Frege to Russell;
03.08.1902 Frege to Russell;
08.08.1902 Russell to Frege;
23.09.1902 Frege to Russell;
29.09.1902 Russell to Frege;
20.10.1902 Frege to Russell;
12.12.1902 Russell to Frege;
28.12.1902 Frege to Russell;
20.02.1903 Russell to Frege;
21.05.1903 Frege to Russell;
24.05.1903 Russell to Frege;
13.11.1904 Frege to Russell;
12.12.1904 Russell to Frege;
09.06.1912 Frege to Russell.

Here are the main topics of the Frege—Russell correspondence:

1. Antinomy: possibility of constructing antinomy; sources of antinomy and ways to avoid it; linguistic syntactic antinomy.
2. Class: abstract object or a whole consisting of many objects; can classes be eliminated from arithmetic?
3. Binary functor from an argument and a course of value of a first-level function.
4. Function as a variable: The most general form of function.
5. The improvement on Law V from *Grundgesetze der Arithmetik*.
6. Can we speak of the Frege—Russell theory of type?
7. Ontology: logical objects, improper objects, a division of the universe into the function and objects; truth and falsehood.
8. Semantics: proposition, proper name, sense and reference, compositional principle and extensional principle; (in)direct speech; truth/falsehood as a reference of a proposition.
9. Others: abstract definition, logical basic fact.

Frege's basic philosophical assumptions

I confine here only to these philosophical assumptions to which I will later refer in the presentation of Frege's correspondence.

Frege assumed adequate and disjunctive (according to him) division of the universe into functions and objects.[7] An object (which corresponds to a logical notion of function's argument) is understood as a saturated element which does not require a complementation. A function (which corresponds to a function's letter) is understood as an unsaturated element which requires a complementation. Frege treated a function's course of value [*Wertverlauf*] (or a class) as an object. Identity was treated as a relation between names referring to objects, not to functions. And the relation of identity can occur between senses or references. Frege took for granted that one cannot speak of equality of two functions because they are always unsaturated, incomplete. However, the sign of equality can be put between two courses of value of a particular function because they are objects, thus saturated and complete. Propositions, proper names and predicates have their own sense [*Sinn*] and reference [*Bedeutung*]. Objects correspond to proper names in language, functions to predicates.

Possibility of constructing an antinomy

It seems to be widely known in which circumstances an antinomy was found in Frege's system. Sometimes we read that Russell noticed it in Frege's *Grundlagen der Arithmetik*. However, an attentive reading of the first two letters discussed here shows that this is a misconception.

In his first letter to Frege of June 16th 1902, Russell announced that on the basis of the chapter *Function* from *Begriffsschrift*,[8] where

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[7] This distinction was precisely introduced by Frege in G. Frege: *Über Begriff und Gegenstand*. „Vierteljahrsschrift für wissenschaftliche Philosophie" 1892, Bd. 16.

[8] Russell referred here to Frege's first book: G. Frege: *Begriffsschrift und andere Aufsätze*. Hrsg. I. Angelelli. Georg Olms Verlag, Hildesheim, Zürich, New York 1998 (1879).

the difference between function and variable was presented for the first time, an antinomy could be formulated:

> I have encountered a difficulty only at one point. You assert […] that a function could also constitute the indefinite element. This is what I used to believe, but this view now seems to me dubious because of the following contradiction. Let w be the predicate of being a predicate which cannot be predicated of itself. Can w be predicated of itself? From either answer follows its contradictory.[9]

Here Russell referred to the following words from Frege's *Begriffsschrift*:

> If, in an expression (whose content needs not be assertible), a simply or a complex symbol occurs in one or more places and we imagine it as replaceable by another [symbol] (but the same one each time) in all or some of these places, then we call the part of expression that shows itself invariant [under such replacement] a function and the replaceable part its argument.[10]

> It can also happen that, conversely, the argument is determinate, but the function is indeterminate.[11]

Next, the noticed difficulty Frege referred to his logical system presented in the first volume of the *Grundgesetze der Arithmetik*, whose second volume, after ten years, was about to appear. Precisely, he referred to Law V, which serves as truths too fundamental to be proven.[12] Frege's original notation is shown below:[13]

$$\vdash (\acute{\varepsilon}f(\varepsilon) = (\acute{a}g(\alpha)) = (\underset{\alpha}{\smile} f(\alpha) = g(\alpha))$$

Reading the above formula: equality of the courses of value of two functions is equal to the general equality of those functions for each argument. In other words: if the courses of value of functions are equal, then these functions are equal for each argument. The adapta-

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[9] Russell to Frege, 16.06.1902, p. 130 (German edition S. 211).

[10] G. Frege: *Conceptual Notation. A Formula language of Pure Thought Modelled upon the Formula Language of Arithmetic*. In: Frege: *Conceptual Notation and Related Articles*. Trans., Ed. T.W. Bynum, p. 127.

[11] Ibid., p. 128.

[12] Patricia Blanchette wrote: „I am not sure why he doesn't use the word "axiom" in *Grundgesetze*… for the basic laws of logic." Private correspondence.

[13] G. Frege: *Grundgesetze der Arithmetik, Begriffsschlift abgeleitet*. Bd. 1. Verlag von Hermann Pohle, Jena 1893, § 20, S. 36.

tion of such a rule was necessary because functions are incomplete and one cannot write the following formula: $f = g$, where f, g are names of functions. However, as it was mentioned above, courses of value are treated by Frege as objects, and one can establish the relation of equality between them. Later, he used the equality of value of functions to define numbers. He wrote: "[…] all objects of arithmetic are introduced as ranges of values."[14]

To sum up this section, I would like to emphasise that Frege found an antinomy in *Grundgesetze der Arithmetik* by himself, after Russell's brilliant remark regarding *Begriffsschrift*. And I wish to add that all topics raised in their correspondence are more or less significantly connected with the above Law V, leading to an antinomy.

Other cases of antinomy

In the second letter to Frege, we find the information that Russell read (at least a part of) the *Grundgesetze der Arithmetik*. Russell referred to particular pages and formulated another antinomy, which I propose to call a linguistic syntactic one. Russell was inspired by the following sentence:

> […] a function name can never occupy the place of a proper name, because it carries with it empty places that answer to the unsaturatedness of the functions.[15]

On this basis, Russell noticed the following difficulty:

> From what you say […] that a function name can never take the place of a proper name […] there arises a philosophical difficulty […] For „ξ can never take the place of a proper name" is a false proposition if ξ is a proper name, but otherwise it is not a proposition at all.[16]

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[14] XXXVI/8 Frege to Russell 3.08.1902, [in:] G. Frege: *Philosophical and Mathematical Correspondence*, p. 142.

[15] G. Frege: *The Basic Laws of Arithmetic. Exposition of the System*. Trans., ed. intr. M. Furth. University of California Press, Berkeley and Los Angelos 1967, § 21, p. 73.

[16] XXXVI/3 Russell to Frege 24.06.1902…, p. 134.

In his answer, Frege mentioned a distinction between a name (a linguistic sign) and its (non-linguistic) reference. He wrote:

> 4_Frege-Russell 29.06.1902: Concerning your doubts regarding my proposition that a function name can never take the place of a proper name, we must distinguish sharply between a name or sign and its meaning. When I use a proper name in a proposition, I am not talking about this proper name but about the object it designates. But it can happen that I want to talk about the name itself; I then enclose it within quotation marks.[17]

This explanation did not satisfy Russell, who presented another case of this difficulty:

> Concerning function names, there still seems to me be a difficulty. If we leave aside names altogether and speak merely of what they mean, then we must admit that there is no proposition in which a function takes the place of a subject. But the proposition „A function never takes the place of a subject" is self-contradictory; and it seems to me that this contradiction does not rest on a confusion of a name with what it means.[18]

Frege answered that this was an apparent difficulty, which had already been described by him in the article *Concept and Object*.[19]

Giving another example, Russell asked: if m is a class of all sentences, then does a sentence

$$\text{"} p\varepsilon m \cdot \supset_p \cdot p\text{"}$$

belong to m or not? (Russell—Frege, 29.09.1902). And a similar example was formulated by Frege with regard to a class of all thoughts.

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[17] XXXVI/4 Frege an Russell 29.06.1902…, p. 217.
[18] XXXVI/5 Russell an Frege, 10.07.1902…, p. 220.
[19] XXXVI/7 Frege an Russell 28.07.1902…, p. 224. Frege referred here to his paper, *Über Begriff und Gegenstand*, op.cit.

How to understand a class?

What is a class? was a widely discussed subject in the correspondence. Frege considered a class as a special case of value-course,[20] as a logical object, not a physical one.[21] He distinguished class from system (whole), which is constituted by relation and

> [...] it is not yet determined what we are to envisage as its part. As parts of a regiment I can regard the battalions, the companies or the individual soldiers, and as parts of a sand pile, the grains of sand or the silicon and oxygen atoms.[22]

As an example of a system (a whole) was given an army, which is established by special relations between soldiers and which can be destroyed even if they are still alive.[23] A system (a whole) of physical objects is a physical object as well. A class of physical objects is not a physical object but a logical one. However, items belonging to one class can simultaneously create a system, a whole. Frege explained it as follows:

> The class of atoms that form the chair on which I am sitting is not the chair itself. A whole whose parts are material is itself material.[24]

For Russell, a class is a sum of objects which create a system, a whole. A class containing many objects is not one object but many. A class which contains only one object is identical with this object. Below are some relevant quotations:

> I believe that classes cannot always be admitted as proper names. A class consisting of more than one object is in the first place not one object but many. Now an ordinary class does form one whole; thus soldiers for example form an army. But this does not seem to me to be a necessity of thought, though it is essential if we want to use a class as a proper name. I believe I can therefore say without contradiction that certain classes [...] are

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[20] XXXVI/8 Frege an Russell 3.08.1902..., p. 225.
[21] XXXVI/7 Frege an Russell 28.07.1902..., p. 140.
[22] Ibid.
[23] Ibid.
[24] Ibid.

mere manifolds and do not form wholes at all. (Russell-Frege, 10.07.1902)

For me the extension of a concept or a class is only a special case of a range of values […] what you would call a class is properly speaking a system, whole, or aggregate, and cannot replace what I call a class. (Frege-Russell, 3.08.1902, p. 73)

Frege's last letter was written to Richard Hönigswald three months before Frege's death. It is important to mention that there Frege treated as synonyms: "set of Fs," "extension of F," "class of Fs" and "system of Fs."[25]

What is function? Function as a variable

Another noteworthy topic relates to how to understand a function. Both Frege and Russell concordantly assumed that the basic form of a function should be most general. This is the reason why Russell asked: „whether the φ in $\varphi(x)$ can be regarded as anything at all?"[26] Russell and Frege had doubts about such a solution, but they used different arguments. Frege invoked the adopted "rigid" distinction between a function and its argument. Thus, a function cannot be understood as "something" because the word "something" represents a proper name.[27] Later he added: „Instead of using the imprecise expression »ξ is a function«, we can say: »() 3 + 4« is a function name."[28] In this way, Frege moved from the general form of all functions to an example of a function. But in the next letter, Russell presented (and later rejected) another solution, which seemed a better option because it was written in language of *Grundgesetze der Arithmetik*, using an application operator:[29]

$$x \cap u$$

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[25] Frege to Hönigswald, 26.04.1925, p. 55 (German edition 86).
[26] XXXVI/3 Russell to Frege 24.06.1902, p. 134.
[27] XXXVI/4 Frege an Russell 29.06.1902, p. 136.
[28] Op XXXVI/4 Frege an Russell 29.06.1902, p. 136.
[29] XXXVI/6 Russell an Frege 24.07.1902…, p. 221.

In his answer, Frege wrote:

> You are correct in writing that not all functions can be designated by the form
>
> $$\xi \cap u$$
>
> But the proof that there is no one-one relation between all objects and all functions strikes me as dubious.[30]

Then the most general form of function is the following: something belongs to a class (falls under an extension of a concept).

The next investigated topic makes reference to the distinction between function and its argument as well. A question posed by Russell was substantially connected with the difficulty which was the basis of antinomy and read as follows: Can a function be treated as its own argument? He wrote:

> [...] the contradiction arises only if the argument itself is a function of the function, i.e. if function and argument cannot vary independently. In the function $\varphi\{\acute{\varepsilon}\varphi(\varepsilon)\}$, φ is the only variable, and the argument $\acute{\varepsilon}\varphi(\acute{\varepsilon})$ is itself [...] a function of φ. It seems that functions of the form $\varphi\{F(\varphi)\}$, where F is constant and φ variable are certainly permitted for every value of φ, though dangerous where the extension is in question.[31]

In the next letter, Frege asked whether in connection with this difficulty Russell proposed to prohibit the use of formulas in which an argument was a course of value of the function. Frege maintained that in such a situation Russell would contradict himself because earlier he had assumed that there was a sign for a class. This means that a class has its own proper name and as a consequence has to be treated as an object.[32]

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[30] XXXVI/8 Frege an Russell, 3.08.1902..., p. 143. Here Frege changed the variable letter from x (as used by Russell) to ξ. This means Frege rewrote this formula in accordance with the language of his logic.

[31] XXXVI/3 Russell an Frege 26.06.1902..., p. 215.

[32] XXXVI/4 Frege an Russell 29.06.1902..., p. 217.

Some attempts to avoid antinomy

Below some important attempts to eliminate the antinomy are collected, made by Frege and Russell in their correspondence.

1. An introduction of a new type of objects: **improper objects** (Frege). It was one of Frege's first ideas how to handle the difficulty leading to an antinomy, noticed by Russell. According to Frege, value-ranges (classes) are special objects which fulfil two conditions:
   a) It would be impossible "[…] that certain predicates could be either ascribed to them or denied them";[33]
   b) Names of these objects "[…] cannot appear in all argument places of the first kind."[34]

   This project, more philosophical than logical in nature, was described by Frege in most detail in the fifth letter to Russell, and was as follows:

   > A class would not then be an object in the full sense of the word, but — so to speak — an improper object for which the law of excluded middle did not hold because there would be predicates that could be neither truly affirmed nor truly denied of it. Numbers would then be improper objects. We should also distinguish different argument places of the first kind, namely those that could take the names of both proper and improper objects, those that could take names of proper ones, and those that could only take names of improper ones.[35]

2. **Arithmetic without classes (Russell)**. In one of the last letters, Russell informed Frege about his new discovery that, as he wrote, "[…] we can do arithmetic without classes. And this seems to me a way to avoid the contradiction."[36] Frege answered this letter half a year later; the break in their correspondence was very long. He considered Russell's attempt as unsuccessful,[37] and a month later Russell replied that now he maintained the same opinion.[38] They

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[33] Ibidem, p. 135. See as well G. Frege: *Nachwort…*, p. 550.
[34] XXXVI/10 Frege to Russell 23.09.1902…, p. 145.
[35] Ibid.
[36] XXXVI/17 Russell an Frege 24.05.1903…, p. 159.
[37] XXXVI/18 Frege an Russell 13.11.1904… p. 243.
[38] XXXVI/19 Russell an Frege 12.12.1904…, p. 248.

together agreed that the elimination of class from arithmetic led to isolation of function letters.[39]

3. **Restriction of function extension** (Frege), which leads to the introduction of an improvement in Law V from *Grundgesetze der Arithmetik* (Frege).[40] See the following expression from Frege's article "What is a Function?": "[...] the delimitation of its range appears irrelevant to the question what a function essentially is."[41]

4. **Distinction of logical types** (Frege, Russell). In 1903 Russell published a short paper on the theory of types, as an appendix to *The Principles of Mathematics*.[42] The correspondence described here indicates a substantial influence of Frege on the idea of logical types as a way to avoid antinomy. See Frege's expression below:

> I do not quite understand your meaning here. A relation between relations is of a different logical type from the one between objects. For the former is a second-level function, the latter a first-level one. But it may be asked whether relations between other objects, for domains of relations are objects if they are admissible at all [...] But if domains of relations are conceived as proper objects, then a relation between domains of relations will be of the same logical type as one between objects in general.[43]

In another letter, Frege considered an example similar to the one given by Russell but with a class of thought and asked whether a thought that expressed it belonged to the class or not. What is very interesting, Frege used there three kinds of fonts to express it and the antinomy does not appear.[44] It seems to be an excellent example of using the theory of type in a context similar to the one given by Russell. As a result, we do not have any antinomy here. Hence my question, maybe we should speak about Frege—Russell type theory? This problem has already been investigated, for example by Hans Sluga, but the name Frege—Russell type theory has not been used until now.[45]

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[39] XXXVI/18 Frege an Russell 13.11.1904..., p. 243. Por. XXXVI/19 Russell an Frege 12.12.1904..., p. 248.

[40] G. Frege: *Nachwort*..., p. 549—563.

[41] G. Frege: *Collected Papers on Mathematic, Logic, and Philosophy*. Ed. B. McGuinness, 1984, p. 289.

[42] B. Russell: *The Principles of Mathematics*..., 523—528. Frege thanked for this book in a letter of 21.05.1903.

[43] XXXVI/10 Frege to Russell 23.09.1902..., s. 146. (German edition S. 229)

[44] See Frege to Russell 28.12.1902, p. 153—154. German edition, S. 236—237.

[45] H. Sluga: *Frege und die Typetheorie*. In: *Logik und Logikkalkül*. Hrsg. M. Käsbaur. Münich 1962, S. 195—209.

In Frege's last letter, there is another hint how to avoid antinomy. He pointed out that one should bear in mind the fundamental distinction between objects and concepts (broadly: functions) and not replace a concept by its extension (an extension was treated as an object) because this leads to an antinomy.[46]

Frege's reasons for distinction between sense and reference

Russell did not understand why Frege differentiated between sense and reference of language expressions (proposition, proper name and predicate). However, it is worth pointing out that J.S. Mill's distinction between extension and intention of language expressions was obvious for Russell, and he referred to it in the correspondence. Below there are arguments used by Frege to convince his unconvinced opponent.[47]

1. **Semiotic** (different names, different senses):

"It frequently happens that different signs designate the same object but are not necessarily interchangeable because they determine the same object in different ways. It could be said that they lead to it from different directions" (14_Frege to Russell, 28.12.1902, p. 152. (German edition S. 234).

2. **From theory of knowledge**:

"The words 'morning star' and 'evening star' designate the same planet, Venus; but to recognize this, a special act of recognition is required [*Erkenntnisthat*]; it cannot simply be inferred from the principle of identity. (14_Frege to Russell, 28.12.1902, p. 152. (German edition S. 234).

"[…] the equations „$3^2 = 3^2$" and „$2^3 + 1 = 3^2$" do not have the same cognitive value even though their truth-value is the same. The difference is one of sense: the thoughts expressed are different" (14_Frege to Russell, 28.12.1902, p. 152. (German edition S. 235).

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[46] Frege to Hönigswald, 24.04—4.05 1903, p. 55 (German edition S. 86).

[47] Presented here arguments only partially cover these presented by Frege in his others papers, published and unpublished.

3. **From identity**:

There are some names which have different senses (like 'morning star' and 'evening star') but refer to the same object. It means on the level of reference 'morning star' = 'evening star', it is like a = a. But on the level of sense ~('morning star' = 'evening star'), so it is like "a = b".

> 'The moon is identical with heavenly body closest to the earth.' We can say that 3 + 4 is identical with 8 − 1; i.e., that the meaning of '3 + 4' coincides with the meaning of '8 − 1.' But this meaning, namely the number 7, is not a component part of the sense of '3 + 4.' The identity is not an identity of sense, nor of part of the sense, but of meaning.[48]

Conclusion

When we read Frege's scientific correspondence, we can state that he was treated with respect by both philosophers and mathematicians.

This discussion on the sense-reference distinction can be seen as a next step — after the chapter *Denoting in Principles of Mathematics*[49] — in developing the theory of descriptions by Russell. One may ask whether Russell would have written his famous article on denoting in 1905 if he had not corresponded with Frege on this subject. What is more, one may wonder whether Russell would have convinced the editor to publish this paper, if he had not been aware that it had been discussed for a long time with a genius-Frege. The influence of this correspondence on the theory of descriptions needs further investigation.

At the end I would like to stress that neither Frege nor Russell changed substantially their opinions on the discussed subjects. And the stakes were constantly extremely high — to relieve (arithmetic) mathematics of an accusation of contradiction.

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[48] Frege to Russell, 13.11.1904, p. 162.
[49] B. Russell: *The Principles of Mathematics*. London 1956, pp. 53—65.

Bibliography

Frege Gottlob: *Nachgelassene Schriften und wissenschaftlicher Briefwechsel*. Band 2: Wissenschaftlicher Briefwechsel. Hrsg., bearb., eingel. und mit Anmerkungen versehen von G. Gabriel. Hamburg Meiner, 1976 (2013).

Frege Gottlob: *Philosophical and Mathematical Correspondence*. Ed. G. Gabriela and others. Oxford 1980.

Frege Gottlob: *The Basic Laws of Arithmetic. Exposition of the System*. Trans., ed., intr. M. Furth. University of California Press, Berkeley and Los Angelos 1967.

Gottlob Freges Briefwechsel mit D. Hilbert, E. Husserl, B. Russell, sowie ausgewählte Einzelbriefe Freges. Einl., Anm., Reg. G. Gabriel, F. Kambartel, Ch. Thiel. Hamburg 1980.

Russell Bertrand: *The Principles of Mathematics*. London (1903) 1956.