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TRIALITY GROUPS ASSOCIATED WITH TRIPLE SYSTEMS AND THEIR HOMOTOPE ALGEBRAS

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Abstract. We introduce the notion of an (α, β, γ) triple system, which generalizes the familiar generalized Jordan triple system related to a construction of simple Lie algebras. We then discuss its realization by considering some bilinear algebras and vice versa. Next, as a new concept, we study triality relations (a triality group and a triality derivation) associated with these triple systems; the relations are a generalization of the automorphisms and derivations of the triple systems. Also, we provide examples of several involutive triple systems with a tripotent element.

1. Prelude

First, we start from a triple system equipped with a triple product (or a ternary product). The triple system V is a vector space over a field F whose characteristic is not two with a trilinear map $V \otimes V \otimes V \to V$. We denote the trilinear product by the juxtaposition $x\,y\,z \in V$ for x, y, $z \in V$. A well-studied example is an (ϵ, δ) Freudenthal–Kantor triple system (hereafter referred to as an (ϵ, δ) FKTS) with ϵ and δ being either $+1$ or -1 ([4, 9]). See also the references in [5, 11, 14, 19] for many earlier studies on the subject, in particular, for nonassociative algebras, refer to [18, 21]. The (ϵ, δ) FKTS is

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introduced by the relations

$$(1) \qquad \text{(i)} \qquad uv(xyz) = (uvx)yz + \epsilon x(vuy)z + xy(uvz),$$

$$\text{(ii)} \qquad L(x,y), K(x,y) \in \operatorname{End} V \quad \text{defined by}$$

$$(2) \qquad L(x,y)z := xyz \text{ and } K(x,y)z := xzy - \delta yzx,$$

which satisfy

$$(3) \qquad K(K(x,y)u,v) - L(v,u)K(x,y) + \varepsilon L(u,v)K(x,y) = 0$$

for u, v, x, y, $z \in V$.

The special case of $\epsilon = -1$ for Eq. (1) without assuming Eq. (3) defines a generalized Jordan triple system (GJTS) ([14]). Also, an (ϵ, δ) FKTS is called *balanced* ([4]) if there exists a nonzero bilinear form $(.|.)\colon V \otimes V \to F$ such that

$$K(x,y) = (x|y)1_V,$$

for any x, $y \in V$ with 1_V being the identity map in $\operatorname{End} V$.

Note that if the (ε, δ) FKTS is balanced, then in view of Eq. (3) we have $K(x,y) = L(x,y) - \varepsilon L(y,x) = (x|y)1_V$ for any $x, y \in V$.

For these triple systems, it is wellknown that there is a construction of several Lie (super)algebras associated with the triple systems and the construction is a representation without using systems of roots (for example, [4, 19]).

Second, one interesting problem for triple systems is their classification and realizations. If the underlying field F is of characteristic 0, then such a classification has been given in [4] for a finite-dimensional simple (ε, δ) FKTS defined by bilinear forms. In the case of F being an algebraically closed field of characteristic zero, Meyberg ([17]) classified another triple system that is essentially equivalent to a simple balanced $(1,1)$ FKTS by a simple transformation ([4]).

Next, the realization of a triple system in terms of some bilinear algebras becomes quite easy for some classes of triple systems when we also assume the existence of a privileged element $e \in V$, which behaves as an analogue of the identity element for bilinear algebras. In fact, essentially only one realization can often exist for such systems. To illustrate this, let us consider the case of the Jordan triple system J where we have

$$\text{(a)} \qquad \text{(i)} \qquad uv(xyz) = (uvx)yz - x(vuy)z + uv(xyz),$$

$$\text{(b)} \qquad \text{(ii)} \qquad xyz = zyx,$$

which is a special case of $\epsilon = -1$ and $\delta = +1$ with $K(x, y) = 0$ in Eqs. (1) and (2). Such a system, of course, possesses many different realizations. However, suppose that we impose an extra ansatz of Eq. (c),

(iii) i.e., there exists a privileged element $e \in J$ satisfying

$$exe = x \quad \text{for any } x \in J. \tag{c}$$

In this case, Loos ([16]) has proved the following.

PROPOSITION 1.1. *Let J be a Jordan triple system (for short, JTS) over a field F whose characteristic is not two and three, which satisfies the extra ansatz of Eq.* (c)*. Then, the homotope algebra $J^{(e)}$ $(= A)$ with the bilinear product defined by*

$$x \cdot y := xey \tag{d}$$

is a unital Jordan algebra with e being the unit element of A. Moreover, for any x, $y, z \in J$, we have

$$xyz := x \cdot (y \cdot z) - y \cdot (x \cdot z) + (x \cdot y) \cdot z. \tag{e}$$

Conversely, if A is a unital Jordan algebra with the unit element e, then the triple product xyz given by Eq. (e) *defines a Jordan triple system satisfying the extra condition Eq.* (c)*.*

The element e satisfying Eq. (c) call a *tripotent* of a triple system, because $eee = e$ (i.e., $e \cdot e = e$).

In [7] we have also proved the following proposition, which essentially generalizes the previous one.

PROPOSITION 1.2. *Let J be a generalized Jordan triple system (for short, GJTS) over a field F whose characteristic is not two and three possessing a privileged element $e \in J$ satisfying*

$$eex = xee = x, \tag{4}$$

for any $x \in J$. Then, the resulting homotope algebra $A (\equiv J^{(e)})$ defined in the vector space J with the multiplication given by $x \cdot y = xey$ and with the linear map $x \to \overline{x} = exe$ is a unital involutive noncommutative Jordan algebra (i.e., a flexible Jordan-admissible algebra, for a definition of the algebra, see [18]*) satisfying the following additional property: $D_{x,y} \in \operatorname{End} A$, defined by*

$$D_{x,y} := (x, y, \cdot) - (y, x, \cdot) = (\cdot, x, y) - (\cdot, y, x), \tag{5}$$

is a derivation of A, where $(x, y, \cdot)$ denotes the associator $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$. Moreover, the original triple product is expressed as

$$xyz = x \cdot (\overline{y} \cdot z) - \overline{y} \cdot (x \cdot z) + (\overline{y} \cdot x) \cdot z, \tag{6}$$

in terms of the bilinear product of A for any $x, y, z \in A$. Conversely, let $(A, \cdot, ^{-})$ be a unital involutive noncommutative Jordan algebra over a field F of characteristic not 2 satisfying the condition that $D_{x,y}$ defined by Eq. (5) *is a derivation of A. Then, the triple product xyz given by Eq.* (6) *defines a generalized Jordan triple system satisfying the extra relation Eq.* (4).

In Section 2, for a generalization of above results, we will first discuss a triple system V over a field F, which we call the (α, β, γ) triple system $(\alpha,\ \beta,\ \gamma \in F)$, defined by

$$uv(xyz) = \alpha(uvx)yz + \beta x(vuy)z + \gamma xy(uvz), \tag{7}$$

which possesses the privileged element $e \in V$ satisfying Eq. (4). For short, then it is said to be a *unital* (α, β, γ) *TS* (see [7]).

In this paper, triple products are generally denoted by xyz, (xyz), $\{xyz\}$, and $[xyz]$, as well as by $x \cdot y$, $x \circ y$, $[x, y]$, and $x * y$ for binary products.

To end this section, we remark that the (α, β, γ) TS is intimately related to a Lie algebra in the case of $\gamma = 1$ but to a Jordan algebra in the case of $\gamma = -1$ for the following reason. Introducing the multiplication operator $L(x, y)$ given by

$$L(x, y)z := xyz,$$

Eq. (7) is rewritten as

$$L(u, v)L(x, y) - \gamma L(x, y)L(u, v) = \alpha L(uvx, y) + \beta L(x, vuy)\,, \tag{8}$$

so that the set of $L(x, y) \in \operatorname{End} V$ defines a Lie algebra for $\gamma = 1$ and a Jordan algebra for $\gamma = -1$ since the set $L(V, V)$ of all left multiplications $L(x, y)$ of the triple system V is a subspace of $\operatorname{End} V$. That is, if $\gamma = 1$, by $[L(x, y), L(u, v)] := L(x, y)L(u, v) - L(u, v)L(x, y) = \alpha L(xyu, v) + \beta L(u, yxz)$, the subspace $L(V, V)$ has the structure of a Lie algebra and also if $\gamma = -1$, by $\{L(x, y), L(u, v)\} := L(x, y)L(u, v) + L(u, v)L(x, y) = \alpha L(uvx, y) + \beta L(x, vuy)$, the subspace $L(V, V)$ has the structure of a commutative Jordan algebra.

Some cases of $\gamma = -1$ satisfying $xyz = yxz$ and its super generalization have been discussed in ([6, 19]) for the construction of some Jordan superalgebras. Letting $x \leftrightarrow u$ and $y \leftrightarrow v$ in Eq. (8), it also gives

$$L(x,y)L(u,v) - \gamma L(u,v)L(x,y) = \alpha L(xyu,v) + \beta L(u,yxv).$$

Eliminating $L(x,y)L(u,v)$ from both relations, we obtain

$$\begin{aligned}(1-\gamma^2)&L(u,v)L(x,y)\\ &= \alpha L(uvx,y) + \beta L(x,vuy) + \alpha\gamma L(xyu,v) + \beta\gamma L(u,yxv).\end{aligned}$$

Our triple systems in this paper excluding Section 2 will deal with the case of $\gamma = \pm 1$, unless otherwise specified.

2. (α, β, γ) triple systems

In this section, we will study some properties of a unital (α,β,γ) TS so that the triple product satisfies

$$\begin{aligned}&\text{(i)} \qquad uv(xyz) = \alpha(uvx)yz + \beta x(vuy)z + \gamma xy(uvz),\\ \text{(9)}\qquad &\text{(ii)} \qquad eex = xee = x.\end{aligned}$$

We introduce a bilinear product $x \cdot y$ and a linear map $x \to \overline{x}$ by

$$x \cdot y := xey,$$

$$\overline{x} := exe.$$

We then obtain the following theorems (for the details, refer to [7]).

THEOREM 2.1. *Let V be a nonzero (i.e., $V \neq 0$) unital (α,β,γ) TS with $\beta \neq 0$. We then have*

(i) $\alpha + \beta + \gamma = 1$.

(ii) *The homotope algebra A ($\equiv V^{(e)}$) is unital with the unit element e, i.e.,*

$$e \cdot x = x \cdot e = x\,.$$

(iii) $\overline{\overline{x}} = x$ with $\overline{e} = e$.

(iv) *The original triple product in V can be expressed as*

$$xyz = \frac{1}{\beta}\{\overline{y}\cdot(x\cdot z) - \alpha(\overline{y}\cdot x)\cdot z - \gamma x\cdot(\overline{y}\cdot z)\}$$

in terms of bilinear product in A.

(v) *If the constants α, β, γ satisfy the condition*

$$(1-\alpha)(1-\gamma) = 0\,,$$

then $x \to \overline{x}$ is an involution of A, i.e., we have

$$\overline{x\cdot y} = \overline{y}\cdot\overline{x}\,.$$

(vi) *If the constants α, β, γ satisfy*

$$\alpha^2 + \beta^2 + \gamma^2 = 1\,,$$

then $x \to \overline{x}$ is an automorphism of A, i.e.,

$$\overline{x\cdot y} = \overline{x}\cdot\overline{y}\,.$$

(vii) *Suppose that $\gamma \neq 1$. Then,*

$$(1-\alpha)(\beta+2\gamma)(\overline{y}\cdot x - x\cdot\overline{y}) = 0$$

holds, and also, the associator defined by

$$(x,y,z) := (x\cdot y)\cdot z - x\cdot(y\cdot z)$$

satisfies

$$(1-\alpha)(x,y,z) = (1-\beta)(y,x,z)\,.$$

THEOREM 2.2. *Let V be a unital (α,β,γ) triple system such that $\alpha \neq 1$, $\gamma^2 \neq 1$, and $\beta + 2\gamma \neq 0$. Then, the associated homotope algebra A is a unital, involutive, commutative, and associative algebra. Moreover, the triple product is given by*

$$xyz = (x\cdot\overline{y})\cdot z = x\cdot(\overline{y}\cdot z)\,. \tag{10}$$

Conversely, if A is an involutive, commutative, and associative algebra, then the triple product given by Eq. (10) *defines an (α,β,γ) triple system with $\alpha+\beta+\gamma = 1$ for any α, β, γ $\in$ F.*

Here, in the following theorem, for the notation $[x,y] := x\cdot y - y\cdot x$, $\forall x,y \in A$, the relation $[A,[A,A]] = 0$ implies that $[x,[y,z]] = 0$, $\forall x,y,z \in A$.

THEOREM 2.3. *Let V be a unital $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ TS over a field F whose characteristic is not two and three. Then, the associated homotope algebra A is a unital associative algebra satisfying $[A,[A,A]] = 0$ with an automorphism $x \to \overline{x}$ of order 2. Moreover, the triple product is determined to be*

$$xyz = \frac{1}{2}(x\cdot\overline{y} + \overline{y}\cdot x)\cdot z\,. \tag{11}$$

Conversely, let A be an associative algebra satisfying $[A,[A,A]] = 0$ with an order-two automorphism $x \to \overline{x}$ over a field F of characteristic $\neq 2, \neq 3$. Then, the triple product given by Eq. (11) *determines a $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ triple system. Furthermore, if A is unital, then the triple system is also unital (i.e., satisfying Eq.* (9)*).*

3. $(1, 1, -1)$ triple systems

Here, we first give the following theorem.

THEOREM 3.1. *Let V be the unital $(1,1,-1)$ triple system over a field F of characteristic not 2. Then, the homotope algebra A (see Theorem* 2.1*) is a unital, involutive, and alternative algebra. Moreover, the original triple product is expressed as*

$$xyz = (x\cdot\overline{y})\cdot z \tag{12}$$

in terms of bilinear product of A. Conversely, if A is a unital, involutive, and alternative algebra, then the triple product given by Eq. (12) *defines a unital $(1,1,-1)$ triple system.*

PROPOSITION 3.2. *Let A be an involutive alternative algebra over a field F with the bilinear product $x\cdot y$ and with the involutive map $x \to \overline{x}$. Then, the triple product given by Eq.* (12) *defines a $(1,1,-1)$ TS. Moreover, if A is unital with the unit element e, then the triple system is also unital, i.e., it satisfies Eq.* (9).

Corollary. *Let V be a $(1,1,-1)$ triple system over a field F of characteristic not 2 with a symmetric, bilinear, and nondegenerate form $(.|.)$ satisfying*

$$xxy = yxx = (x|x)y. \tag{13}$$

Then, for any element $e \in V$ satisfying $(e|e) = 1$, V becomes a unital $(1,1,-1)$ triple system. Moreover, the associated homotope algebra A is a Hurwitz algebra (i.e., a unital composition algebra) satisfying

$$(x \cdot y|x \cdot y) = (x|x)(y|y).$$

Conversely, if A is a Hurwitz algebra with the unit element e, then the triple product xyz given by Eq. (12) defines a unital $(1,1,-1)$ triple system satisfying Eq. (13) .

4. $(-1,1,1)$ triple systems

We give the following theorem in this section.

Theorem 4.1. *Let V be a unital $(-1,1,1)$ TS over a field F of characteristic not 2. Then, its homotope algebra A is a unital, involutive, and associative algebra. Moreover, it satisfies an additional constraint of*

$$[A,[A,A]] = 0. \tag{14}$$

The original triple product is now expressed as

$$xyz = (2\overline{y} \cdot x - x \cdot \overline{y}) \cdot z \tag{15}$$

in terms of bilinear product of A. Conversely, if A is a unital, involutive, and associative algebra satisfying Eq. (14), then the triple product xyz given by Eq. (15) defines a unitary $(-1,1,1)$ TS.

Proposition 4.2. *Let A be an involutive associative algebra. Then, the triple product xyz given by Eq. (15) satisfies*

$$uv(xyz) + (uvx)yz - x(vuy)z - xy(uvz) = 2[[x,\overline{y}],w] \cdot z,$$

with $w := 2\overline{v} \cdot u - u \cdot \overline{v}$. In particular, if we have $[[A,A],A] = 0$, then xyz defines a $(-1,1,1)$ triple system.

5. $(1,-1,1)$ triple systems and structurable algebras

The case of $\alpha=\gamma=1$ and $\beta=-1$ is perhaps the most interesting case since the equations $(1-\alpha)(x,y,z)=(1-\beta)(y,x,z)$ and $(1-\alpha)(\beta+2\gamma)(\overline{y}\cdot x-x\cdot\overline{y})=0$ in Section 2 need not be considered so as not to give new constraints. However, this case has already been treated in Theorem 2.1 since a $(1,-1,1)$ TS is equivalent to a generalized Jordan triple system. However, if we modify the unital condition in the system by some additional ansatz for a structurable algebra ([1, 7, 8]), we have the following result:

THEOREM 5.1. *Let V be a $(-1,1)$ FKTS over a field F whose characteristic is not two and three. Suppose that V has a privileged element $e\in V$ satisfying modified unital conditions of*

$$\text{(16)} \qquad \text{(i)} \quad eex=x\,,$$

$$\text{(17)} \qquad \text{(ii)} \quad exe+2xee=3x$$

for any $x\in V$. We then introduce a linear map $x\to\overline{x}$ and a bilinear product $x\cdot y$ in V by

$$\text{(i)} \qquad \overline{x}:=2x-xee\,,$$

$$\text{(ii)} \qquad x\cdot y:=yex-\overline{x}\,\overline{y}e+\overline{x}ey\,.$$

The resulting bilinear algebra $(A,\ x\cdot y)$ is then a unital involutive algebra with the unit element e and with the involution given by $x\to\overline{x}$, i.e., $\overline{x\cdot y}=\overline{y}\cdot\overline{x}$. Moreover, the original triple product is expressed as

$$xyz=(z\cdot\overline{y})\cdot x-(z\cdot\overline{x})\cdot y+(x\cdot\overline{y})\cdot z \tag{18}$$

in terms of the bilinear product, implying that the algebra A is structurable. Conversely, if A is a structurable algebra with the unit element e and involution map $x\to\overline{x}$, then the triple product given by Eq. (18) *defines a $(-1,1)$ FKTS satisfying Eqs.* (16) *and* (17).

Here, to make the paper as self-contained as possible, we note that the contents from Section 2 to Section 5 are based on a summary of results appearing in [4], which are mainly needed in Section 6 and 7.

For an inner structure of alternative or octonion algebras, and for a characterization of structurable algebras, see [1, 7, 8, 18, 21]. On the other hand, for the structure of a Peirce decomposition of an (ϵ,δ) FKTS and also for the characterization of the triple system as well as the structure of the algebras, refer to, for example, [3, 5, 8, 14]. In a final comment of this section, note that the balanced $(1, 1)$-FKTS induced from an exceptional Jordan algebra is closely related to a 56 dimensional meta symplectic geometry due to H. Freudenthal ([4, 9, 11] and the earlier references therein).

6. Main theorems

This section deals with a triple system's variation for a triality group and a triality derivation of nonassociative algebras with involution ([11]). That is, as a new idea, we consider a triality relation of triple systems with involution.

To consider several triple systems, from now on, instead of (xyz), we will often use the notations $\{xyz\}$, $[xyz]$ with respect to the triple product (ternary product) of triple systems.

Let $(V, \{-, -, -\})$ be a triple system with an element e and an endomorphism $x \to \bar{x}$, equipped with $\bar{x} = M(e, e)x = \{exe\}$ satisfying

$$\{eex\} = \{xee\} = x \text{ and } \{e\{xey\}e\} = \{\{eye\}e\{exe\}\}, \ \ \{e\{exe\}e\} = x \tag{19}$$

for any $x, y \in V$ and also equipped with an endomorphism D_j satisfying

$$[D_j, R(x, y)] = R(\overline{D}_{j+1}x, y) + R(x, \overline{D}_{j+1}y). \tag{20}$$

Hence, the element e is said to be a *unit* element of V, because it satisfies $\{eee\} = e$. Here, indices j are defined by modulo 3, i.e., we denote $j = j \pm 3$ $(j = 0, 1, 2)$ and $R(x, y)z = \{zxy\}$. Also, for $D_j \in \operatorname{End} V$, $\overline{D}_j$ is denoted by

$$\overline{D}_j(x) = \overline{D_j(\overline{x})}.$$

Moreover, suppose that the exponential map is well-defined, i.e., we mean that

$$\xi_j(t) = \exp(t\ D_j) = \mathrm{Id} + t\ D_j + \frac{t^2}{2}(D_j)^2 + \cdots, \tag{21a}$$

where t is an indeterminate variable, and $t \in F$, is well-defined to satisfy Stone's theorem;

$$\frac{d}{dt}\xi_j(t) = D_j\xi_j(t) = \xi_j(t)D_j. \tag{21b}$$

Of course, we have $\xi_j(-t) = (\xi_j(t))^{-1}$, since $\xi_j(t)\xi_j(-t) = Id$.

Furthermore, for these $R(x,y)$, D_j, and $\xi_j(t)$, assuming

$$\text{(21c)} \qquad \left\{ \begin{array}{l} \frac{d}{dt}R(\xi_j(t)x,y) = R(D_j\xi_j(t)x,y), \\ \frac{d}{dt}R(x,\xi_j(t)y) = R(x,D_j\xi_j(t)y) \end{array} \right\},$$

then we have the following theorems.

THEOREM 6.1. *Under the above assumptions, that is, if D_j satisfies Eq.* (20) *and ξ_j satisfies Eqs.* (21)*, then $\xi_j(t)$ satisfies*

$$\text{(22)} \qquad \xi_j(t)R(x,y)\xi_j(t)^{-1} = R(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y) \quad (\textit{global triality}),$$

where $\overline{\xi_j(t)}$ is induced from $\exp(t\ \overline{D}_j)$.

Conversely, the validity of Eq. (22) *with Eqs.* (21) *implies that of the local triality relation Eq.* (20)*, i.e., we have the correspondence of local triality for D_j $\leftrightarrow$ global triality for ξ_j. This means that*

$$\xi_j = \exp\ D_j = \sum_{k=0}^{\infty} \frac{(D_j)^k}{k!}.$$

PROOF. Set $G(t) = \xi_j(-t)R(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y)\xi_j(t)$. In view of Eqs. (21), we have

$$\begin{aligned} \frac{d}{dt}(G(t)) &= \xi_j(-t)(-D_jR(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y)\xi_j(t) \\ &\quad + \xi_j(-t)(R(\overline{D}_{j+1}\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y)\xi_j(t) \\ &\quad + \xi_j(-t)(R(\overline{\xi_{j+1}(t)}x, \overline{D}_{j+1}\overline{\xi_{j+1}(t)}y)\xi_j(t) \\ &\quad + \xi_j(-t)(R(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y)D_j\xi_j(t) \\ &= \xi_j(-t)\{-D_jR(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y) + R(\overline{D}_{j+1}\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y \\ &\quad + R(\overline{\xi_{j+1}(t)}x, \overline{D}_{j+1}\overline{\xi_{j+1}(t)}y) + R(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y)D_j\}\xi_j(t). \end{aligned}$$

By changing $\overline{\xi_{j+1}(t)}x \to x$ and $\overline{\xi_{j+1}(t)}y \to y$, we obtain

$$\xi(-t)\{-D_jR(x,y) + R(\overline{D}_{j+1}x,y) + R(x,\overline{D}_{j+1}y) + R(x,y)D_j\}\xi_j(t) = 0$$

through the assumption of Eq. (20).

From the fact that $F(t)$ is independent for the value of t, we have

$$G(t) = G(0) = R(x, y).$$

By using $\xi_j(t)\xi_j(-t) = \mathrm{Id}$, we obtain

$$\xi_j(t)R(x, y)\xi_j(-t) = R(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y).$$

Conversely, if $\xi_j(t)R(x, y)\xi_j(-t) = R(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y)$ holds, then we have

$$G(t) = \xi_j(-t)R(\overline{\xi_{j+1}(t)}x, \overline{\xi_{j+1}(t)}y)\xi_j(t) = R(x, y).$$

Therefore, we get

$$\frac{d}{dt}G(t) = 0.$$

From this and the fact that

$$\xi_j(t)(\frac{d}{dt}G(t))\xi_j(-t) = -[D_j, R(x, y)] + R(\overline{D}_{j+1}x, y) + R(x, \overline{D}_{j+1}y),$$

we obtain Eq. (20), i.e.,

$$[D_j, R(x, y)] = R(\overline{D}_{j+1}x, y) + R(x, \overline{D}_{j+1}y).$$

This completes the proof. □

Note that if a binary product $*$ in V is defined by $x * y = \{xey\}$, then Eq. (19) implies the validity of $\overline{x * y} = \overline{y} * \overline{x}$, $\overline{\overline{x}} = x$ (i.e., involution). Also, from now on, we want to use the product $*$ instead of the product $\cdot$ for convenience and to use $*$ with respect to the standard product of algebras in future work.

THEOREM 6.2. *Under the assumption in Theorem* 6.1, *we have the validity*

$$\text{(23)} \quad \frac{d}{dt}[(\exp\ tD_j)R(x, y)(\exp\ tD_j)^{-1}]_{t=0} = [D_j, R(x, y)]\ (= D_jR(x, y) - R(x, y)D_j).$$

PROOF. By straightforward calculations, we can verify Eq. (23) and we omit the proof. Note that $(\exp\ tD_j)^{-1} = \exp(-tD_j)$. □

Remark. The well-definedness of the exponential map is verified if V is a finite dimensional triple system over the real or complex number field F, since

$$\exp tD_j = \sum_{n=0}^{\infty} \frac{1}{n!}(tD_j)^n$$

is convergent in some suitably chosen topology.

Note that ξ_j (resp. D_j) may be regarded as a generalization of the automorphism (resp. derivation) in triple systems and ξ_j is a global relation (resp. D_j is a local relation). This ξ_j (resp. D_j) is said to be an element of a *triality group* (resp. a *triality derivation*) of V.

Remark. Let A^* with the product $*$ be an associative or alternative algebra equipped with involution. Here, we denote the product by $*$ instead of $\cdot$ in Section 1. We note that if the triple products are defined by $\{xyz\} = (x*\overline{y})*z$ as in Theorem 2.2 and Theorem 3.1, then these triple systems are (α, β, γ) triple systems, that is, for (α, β, γ) triple systems induced from algebras with involution, we have the following correspondence between triple systems and the homotope algebras:

(α, β, γ) triple system (with an assumption) $\longleftrightarrow$ associative algebra,

$(1, 1, -1)$ triple system $\longleftrightarrow$ alternative algebra.

Furthermore, we note the following.

Remark. Let A^* be a commutative Jordan algebra with the product $x*y$. Then, it is well known that $(A^*, (xyz))$ is a Jordan triple system with respect to the triple product

$$(xyz) = (x*y)*z + (z*y)*x - (z*x)*y.$$

That is, it is a $(1, -1, 1)$ triple system with $(xyz) = (zyx)$ (i.e., $K(x, y) \equiv 0$).

Remark. As we have obtained Theorem 6.1 and Theorem 6.2 for triple systems in this paper, we may apply the Theorems 6.1 and 6.2 in this section to (α, β, γ) triple systems and the homotope algebras with involution. For the structurable algebra described in Section 5, we will discuss the details of a triality relation in the future paper, but we will briefly consider such a relation in Section 7. On the other hand, we have studied the triality relations on nonassociative algebras with involution (for example, [8, 10, 11]).

Hence, from these results we may obtain the concept of triality group for a triple system as follow.

Let $(V, \{xyz\})$ be a triple system. If there exists a privileged element $e \in V$ satisfying

$$\{eee\} = e,\ \{e\{exe\}e\} = x$$

and

$$\{e\{xey\}e\} = \{\{eye\}e\{exe\}\}$$

for any $x, y \in V$, then the triple system V is said to be *involutive.* Now, we denote $\overline{x}$ by $\overline{x} = M(x, y)e = \{exe\}$.

Theorem 6.3. *Let V be an involutive triple system. If we introduce a bilinear product $x * y$ and a transformation $g_j \in \operatorname{End} V$ $(j = 0, 1, 2)$ satisfying*

$$x * y = \{xey\} \text{ and } g_j R(e, y) = R(e, \overline{g}_{j+1} y)\overline{g}_{j+2},$$

then we have

$$g_j(x * y) = (\overline{g}_{j+2}x) * (\overline{g}_{j+1}y).$$

This implies that

$$g_j(x \cdot y) = (g_{j+1}x) \cdot (g_{j+2}y)$$

with respect to the product $x \cdot y = \overline{x * y}$, that is, g_j is a triality group of the algebra $(A, \cdot)$, which we denote by $A = V^{(e)}$ (this is said to be a homotope algebra induced from the triple system V).

Proof. From the relation

$$g_j R(e, y)x = R(e, \overline{g}_{j+1} y)\overline{g}_{j+2}x,$$

it follows that

$$g_j\{xey\} = \{\overline{g}_{j+2}xe\overline{g}_{j+1}y\},$$

and so

$$g_j(x * y) = \overline{g}_{j+2}(x) * \overline{g}_{j+1}(y).$$

Thus, using $x \cdot y = \overline{x * y}$, we obtain

$$g_j(x \cdot y) = (g_{j+1}(x)) \cdot (g_{j+2}(y)). \qquad \square$$

Corollary. *Under the assumption in Proposition* 1.2 *(i.e., $J=V$ is a GJTS with the condition Eq.* (4)*), if a transformation g_j of J satisfies the relation*

$$g_j R(e,y) = R(e, \overline{g}_{j+1} y)\overline{g}_{j+2}, \tag{24}$$

then we have

$$g_j(x \cdot y) = (g_{j+1}x) \cdot (g_{j+2}y).$$

This Corollary implies that g_j is a triality group of the homotope algebra $A = J^{(e)}$ $(= V^{(e)})$ with respect to the product $x \cdot y = \overline{x * y}$. Indeed, this GJTS J in Proposition 1.2 satisfies the property of involutive triple system, since $\overline{x} = \{exe\}$ and

$$\overline{x * y} = \overline{y} * \overline{x} = \{e\{xey\}e\} = x \cdot y.$$

Furthermore, note that the homotope algebra A is a noncommutative Jordan algebra with involution and so this g_j is a triality group of the algebra.

Example of Theorem 6.3 (the homotope algebra of a Jordan triple system). Note that if J is a JTS with an element e of the Eq. (c) in Section 1, then $(J^{(e)}, \cdot)$ is an involutive commutative Jordan algebra with respect to the bilinear product $x * y$ defined by $\{xey\}$ and so the above transformation g_j (satisfying Eq. (24)) is a triality group of the Jordan algebra (i.e., homotope algebra) $A = J^{(e)}$ induced from the JTS J. In this case, we have

$$x \cdot y = \overline{x * y} = x * y = y * x = y \cdot x \quad \text{(commutative property)},$$

because the involution is the identity map.

Case of a field of complex numbers C From the result of Section 3, by means of the product $x * y$ ($x * y$ is the standard product and $\overline{x}$ is the conjugation of $\mathbf{C}$), we note that $\{xyz\} = (x * \overline{y}) * z$ is a $(1, 1, -1)$ triple system, since the complex number is associative (a special case of alternative) with respect to the standard product $*$. That is, from $x \cdot y = \overline{x * y}$, we obtain $\{xey\} = x * y = \overline{x \cdot y}$, where e denotes the identity element of $\mathbf{C}$. Hence in the new product $x \cdot y$, for $j = 0, 1, 2$, by straightforward calculation (by $g_j(e) = g_{j+1}(e) \cdot g_{j+2}(e)$), we have the relation

$$g_j(x \cdot y) = (g_{j+1}x) \cdot (g_{j+2}y), \quad \text{such that} \quad g_j = e^{\sqrt{-1}\theta_j},\ \theta_0 + \theta_1 + \theta_2 = 0.$$

Hence we get $D_j = \sqrt{-1}\theta_j$. That is, these imply that g_j is the triality group and D_j is the triality derivation of $\mathbf{C}$ with respect to the new product $x \cdot y$. In particular, for endomorphisms $g = e^{\frac{2}{3}\pi\sqrt{-1}}$ and $D = \frac{2}{3}\pi\sqrt{-1}$ of $\mathbf{C}$, we can easily see that $g(x \cdot y) = g(x) \cdot g(y)$ (automorphism) and $D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$ (derivation) respectively, where the period is 2π.

Case of a matrix algebra For triality relations of matrix algebras, we note that matrix algebras $M(n, F)$ ($n \times n$ matrix sets) equipped with the standard product $x * y$ are associative and that Theorem C can be applied by means of the triple product

$$\{xyz\} = (x *^t y) * z,$$

where ${}^t y$ denotes the transpose matrix of $y \in M(n, F)$.

Since the involution is defined by the transpose matrix, the relation ${}^t(x * y) =^t y *^t x$ holds (i.e., $\overline{x * y} = \overline{y} * \overline{x}$). Note that $x * y = \{xId_n y\}$.

More precisely, for $a_j \in O(n, F) = \{a_j \in M(n, F)|\ a_j *^t a_j = Id_n\}$ ($j = 0, 1, 2$) (orthogonal matrix set), we define an endomorphism $g_j(a)$ by

$$g_j(a)x = a_{j+1} * x *^t a_{j+2},$$

then by straightforward calculation, we have

$$g_j(a)(x * y) = (\overline{g_{j+2}(a)}x) * (\overline{g_{j+1}(a)}y).$$

By the new product $x \cdot y = \overline{x * y}$, this relation means that

$$g_j(a)(x \cdot y) = (g_{j+1}(a)x) \cdot (g_{j+2}(a)y)$$

(triality group of the matrix algebra with respect to the product $\cdot$),

where $x \cdot y =^t (x * y)$ (new product induced from $x * y$). This $g_j(a)$ (denoted by g_j) satisfies the relation of the triple system;

$$g_j R(x, y) = R(\overline{g_{j+1}}x, \overline{g_{j+1}}y)g_j.$$

For the triality relations of matrix algebras with respect to the binary product $*$, refer to [10, 11], however this case will be discussed also in Section 7, to the reason of an interesting example of a linear Lie group related to the triple system and the corresponding Lie algebra.

Theorem 6.4. *Let* $(V, \{xyz\})$ *be involutive and* g_j *be a transformation of the triple system* V *satisfying*

$$g_j R(x, y) = R(\overline{g}_{j+1}x, \overline{g}_{j+1}y)g_j \quad (j = 0, 1, 2)$$

and

$$\overline{g}_{j+2} = R(e, g_{j+1}e)g_j \tag{25}$$

for any $x, y \in V$. *(It is said to be a triality group of a triple system.)*

If the bilinear product $x * y$ *defined by* $x * y = \{xey\}$ *satisfies the relation*

$$(x * \overline{y}) * z = \{xyz\},$$

then this g_j *is a triality group of the homotope algebra* $A = V^{(e)}$ *with respect to the product* $x \cdot y = \overline{x * y}$ *and with involution, that is,*

$$\overline{\overline{x}} = x, \ \overline{x \cdot y} = x * y = \overline{\overline{y} * \overline{x}} = \overline{y} \cdot \overline{x}.$$

Proof. From the assumption of

$$g_j\{xyz\} = \{g_j x \overline{g}_{j+1} y \overline{g}_{j+1} z\},$$

it follows that

$$\begin{aligned} g_j\{xez\} &= \{g_j x \overline{g}_{j+1} e \overline{g}_{j+1} z\} \\ &= (g_j x * \overline{\overline{g}_{j+1} e}) * \overline{g}_{j+1} z \\ &= \overline{g}_{j+2} x * \overline{g}_{j+1} z \end{aligned}$$

in view of the two relations

$$\{xyz\} = (x * \overline{y}) * z \ \text{ and } \ \overline{g}_{j+2} = R(e, g_{j+1}e)g_j.$$

Thus, we obtain

$$g_j(x * y) = \overline{g}_{j+2} x * \overline{g}_{j+1} y$$

and so

$$g_j(x \cdot y) = (g_{j+1}x) \cdot (g_{j+2}y)$$

since $x \cdot y = \overline{x * y}$ and $\overline{\overline{g}_j x} = g_j \overline{x}$. This completes the proof. □

For the Eq. (25), if there is an element e satisfying $e * x = x * e = x$, for any $x \in V$, then the condition Eq. (25) does not need.

Remark. These theorems may be applied to (α, β, γ) triple systems.

Example of Theorem 6.4. Theorem 3.1 can be applied to the triality group associated with Theorem 6.4.

From Section 1, we recall the notation of a balanced $(-1, -1)$ FKTS as follows. Such a FKTS is said to be balanced if it satisfies the relation $K(x, y) = (x|y)1_V$ with a symmetric bilinear form $(x|y) \in F$.

An algebra A over a field F is said to be *quadratic* if it is unital and for any $x \in A$, $1, x, x \cdot x$ are linearly dependent.

Applying the concept of these triple systems and algebras, by Theorem 4.4 in [2], we have the following:

Lemma 6.5. *Let $(V, (xyz))$ be a balanced $(-1, -1)$ FKTS over a field F and let $e \in V$ such that $(e|e) \neq 0$. Define a bilinear product on V by*

$$x \cdot y = \frac{1}{(e|e)}(exy)$$

for any $x, y \in V$, then $(V, \cdot)$ is a quadratic algebra. Moreover, the original triple product on V is related to the binary product by

$$(xyz) = (e|e)((\bar{x} \cdot y) \cdot z - \bar{x} \cdot (y \cdot z) + y \cdot (\bar{x} \cdot z))$$

for any $x, y \in V$, where $x \to \bar{x}$ denotes the standard involution of the quadratic algebra.

Applying Lemma 6.5 in results of these triple system and algebra in this section, we have the following.

Theorem 6.6. *Under the assumption in Lemma 6.5, then $(V, \{xyz\})$ is a generalized Jordan triple system (i.e., a $(1, -1, 1)$ triple system) with respect to the new product defined by $\{xyz\} = (yxz)$, and the homotope algebra $(V^{(e)}, x * y)$ is a noncommutative Jordan algebra with respect to the binary product $x * y = \{xey\} = \frac{1}{(e|e)}(exy) = x \cdot y$. Furthermore, $(V, \{xyz\})$ is an involutive triple system equipped with $\bar{x} = \{exe\}$.*

Proof. In the relations of a balanced $(-1, -1)$ FKTS;

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \quad \text{and}$$

$$(xzy) + (yzx) = (xyz) + (yxz) = (x|y)z,$$

changing the notation of triple product by $\{xyz\} = (yxz)$, we obtain the relation of GJTS;

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}.$$

Also, by straightforward calculations, for the triple product $\{xyz\}$, we obtain the involutive relations

$$\{e\{exe\}e\} = x \quad \text{and} \quad \{e\{xey\}e\} = \{\{eye\}e\{exe\}\}.$$

That is, for the homotope algebra $V^{(e)}$ induced from V, these relations imply that $\bar{\bar{x}} = x$ and $\overline{x * y} = \bar{y} * \bar{x}$. □

These results mean that the theory of triality relations in triple systems may be applied to the balanced $(-1,-1)$ FKTS and the homotope algebra (i.e., a quadratic algebra).

As a generalization of automorphisms and derivations for involutive triple systems V, these theorems in this section lead us to the following concept:

($\sharp$) *(triality group)* $\quad \mathrm{Trig}\, V := \{(g_0, g_1, g_2) \in \mathrm{Epi}\, V |$

$$g_j R(x,y) = R(\overline{g_{j+1}}x, \overline{g_{j+1}}y) g_j,\ \forall x, y \in V\},$$

($\sharp\sharp$) *(triality derivation)* $\quad \mathrm{Trid}\ V := \{(D_0, D_1, D_2) \in \mathrm{Epi}\, V |$

$$D_j R(x,y) = R(\overline{D_{j+1}}x, y) + R(x, \overline{D_{j+1}}y) + R(x,y)D_j,\ \forall x, y \in V\},$$

where $\mathrm{Epi}\, V$ denotes the set of epimorphisms of V.

Note that if $g_0 = g_1 = g_2$ (resp. $D_0 = D_1 = D_2$), then its concept is an automorphism (resp. a derivation) of the triple system.

Of course, for the simplest case of $V = F$ (a field), we have $\mathrm{Trig}\, F \simeq K_4$ (Klein's four group). Indeed, from $(Id, -Id, -Id)$, $(-Id, Id, -Id)$, $(-Id, -Id, Id)$, $(Id, Id, Id) \in \mathrm{Trig}\, F$, it follows that $(Id)(x \cdot y) = ((-Id)x) \cdot ((-Id)y)$, where the product is denoted by $\cdot$ and the conjugation by the identity map.

As in nonassociative algebras with involution ([11]), in view of straightforward calculation, if the triple system is involutive, then we obtain the following.

($\sharp\sharp\sharp$) *S_4 (symmetric group of order 4) is an invariant group of* $\mathrm{Trig}\, V$.

Indeed, we can show that $\mathrm{Trig}\, V$ is invariant under actions of the alternative group A_4 as follows. Let $\phi \in \mathrm{End}(\mathrm{Trig}(V))$ be given by

$$\phi : g_1 \to g_2 \to g_3 \to g_1,$$

which satisfies $\phi^3 = Id$ and leaves ($\sharp$) invariant. Thus, $\mathrm{Trig}\, V$ is invariant under actions of the cyclic group Z_3 generated by ϕ. We next introduce $\tau_\mu \in \mathrm{End}(\mathrm{Trig}(V))$ for $\mu = 1, 2, 3$ by

$$\tau_1 : g_1 \to g_1,\ g_2 \to -g_2,\ g_3 \to -g_3$$

$$\tau_2 : g_1 \to -g_1,\ g_2 \to g_2,\ g_3 \to -g_3$$

$$\tau_3 : g_1 \to -g_1,\ g_2 \to -g_2,\ g_3 \to g_3,$$

which leaves ($\sharp$) invariant. The group generated by $< \phi, \tau_\mu >_{gen}$ is the alternative group A_4 (alternative group of order 4).

Moreover, the endomorphism $\theta \in \mathrm{End}(\mathrm{Trig}(V))$ defined by

$$\theta : g_1 \to \overline{g}_2,\ g_2 \to \overline{g}_1,\ g_3 \to \overline{g}_3$$

also yields an invariant operation of $\mathrm{Trig}\, V$ and so leaves ($\sharp$) invariant.

Therefore, the group generated by $< \phi, \tau_\mu, \theta >_{gen}$ gives the symmetric group S_4. This shows that S_4 is an invariant group of $\mathrm{Trig}\, V$. That is, ($\sharp\sharp\sharp$) is verified.

7. Miscellaneous examples

In this section, to consider triality relations of algebras associated with triple systems, we give several examples of a structurable algebra and a non-commutative Jordan algebra.

First, following [9], we recall the notation of a triple system V:

$$S(x, y)z = (L(x, y) + \epsilon L(y, x))z, \quad A(x, y)z = (L(x, y) - \epsilon L(y, x))z,$$

$$K(x, y)z = (xzy) - \delta(yzx),$$

where $\epsilon = \pm 1$, $\delta = \pm 1$, and $x, y, z \in V$.

Here, for the triple system V satisfying Eq. (1), we remark that $S(x, y)$ is a derivation of V and $A(x, y)$ is an anti-derivation of V.

For a structurable algebra $(V, *)$ defined by $L(x, y)z = (xyz) = (x * \overline{y}) * z + (z * \overline{y}) * x - (z * \overline{x}) * y$ with respect to (w.r.t.) the triple product of Eq. (18) (where the binary product notation is changed by $x * y$), if we introduce the two new triple products

$$[xyz]_+ = \frac{1}{2}(S(x, y) + K(x, y))z, \quad [xyz]_- = \frac{1}{2}((A(x, y) + K(x, y))z),$$

then by straightforward calculation, we have several interesting structures as follows:

$$L(x,y)z = (xyz) = [xyz]_+ + [xyz]_-,$$

$$[xyz]_+ \text{ is a Lie triple system,}$$

$$[xyz]_- = (x * \bar{y}) * z$$

for any $x, y, z \in V$.

Note that the structurable algebra is a special case of $\varepsilon = -1$ and $\delta = +1$ on the (ε, δ) FKTS by Theorem 5.1 in Section 5.

Furthermore, we use the following notation:

$\mathbf{A} = \{A(x,y)\}_{span}$, $\mathbf{K} = \{K(x,y)\}_{span}$, $\mathbf{S} = \{S(x,y)\}_{span}$ for all $x, y \in V$, $[X,Y] = XY - YX$, $[XYZ] = [Z,[X,Y]]$, $\{XYZ\} = XYZ + ZYX$ for all $X, Y, Z \in \operatorname{End} V$.

Hence, we can verify the validity of

$$[XYZ] = \{YXZ\} - \{XYZ\}.$$

Under the above notation, for the structurable algebra, we have the following ([9]):

$$\mathbf{A} := \{A(x,y)\}_{span} \text{ is a Lie triple system}$$

$$\text{w.r.t. the triple product } [XYZ] \in \mathbf{A} \text{ for any } X, Y, Z \in \mathbf{A},$$

$$\mathbf{K} := \{K(x,y)\}_{span} \text{ is a Lie triple system}$$

$$\text{w.r.t. the triple product } [XYZ] \in \mathbf{K} \text{ for any } X, Y, Z \in \mathbf{K},$$

$$\text{and also } \mathbf{K} \text{ is a Jordan triple system}$$

$$\text{w.r.t. the triple product } \{XYZ\} \in \mathbf{K} \text{ for any } X, Y, Z \in \mathbf{K},$$

$$\mathbf{S} := \{S(x,y)\}_{span} \text{ is a Lie triple system}$$

$$\text{w.r.t. the triple product } [XYZ] \in \mathbf{S} \text{ for any } X, Y, Z \in \mathbf{S}.$$

Remark. The structurable algebra is a $(-1,1)$ FKTS w.r.t. the triple product defined by Eq. (18) in Section 5, and there is an unit element e satisfying $e * x = x * e = x$ for all $x \in V$, but it is not a $(1,-1,1)$ TS, because $(xee) \neq x$.

REMARK. The homotope algebra $(\mathbf{K}^{(\mathbf{Id})}, \{X,Y\})$ induced from the Jordan triple system $(\mathbf{K}, \{XYZ\})$ is a Jordan algebra w.r.t. the product $\{X,Y\} = XY + YX,\ X,Y \in \operatorname{End} V$ since $\{XIdY\} = XY + YX = \{X,Y\}$.

As these subsets $\mathbf{A}$, $\mathbf{K}$, $\mathbf{S}$ into $\operatorname{End} V$ have the structure of several triple systems, it seems that there is a triality relation associated with the triple systems, however the details will be discussed in a future paper. For the triple product $[xyz]_-$ only, we consider now about the triality relation.

First, for the triple system $[xyz]_-$ induced from the structurable algebra, we have

$$[xyz]_- = (x * \overline{y}) * z \quad \text{is an involutive triple system.}$$

Indeed, by means of the involution $\overline{x * y} = \overline{y} * \overline{x}$ in the structurable algebra, we have $[exe]_- = \overline{x}$ and $[e[xey]_-e]_- = [[eye]_-e[exe]_-]_-$. Thus, we can apply Theorem C to this involutive triple system $(V, [xyz]_-)$ associated with the structurable algebra $(V, x * y)$.

Second, let J be a generalized Jordan triple system. Then from Proposition 1.2 in Section 1, we can obtain $V = J^{(e)}$ with the product $x \cdot y$ $(= xey)$, called the homotope algebra associated with a generalized Jordan triple system J, and so the algebra V is a noncommutative Jordan algebra equipped with unit element $e \in V$, i.e., $e \cdot x = x \cdot e = x$ for any element $x \in V$.

If we introduce a new binary product defined by

$$x \circ y = \frac{1}{2}(x \cdot y + y \cdot x),$$

then $(V,\ x \circ y)$ is a commutative Jordan algebra with an identity involution $\bar{x} = x$ satisfying

$$x \circ y = \overline{x \circ y} = \bar{y} \circ \bar{x} = y \circ x, \quad \text{and} \quad \bar{\bar{x}} = x.$$

Furthermore, setting

$$\{xyz\} = (x \circ y) \circ z + x \circ (y \circ z) - y \circ (x \circ z) \quad \text{and}$$

$$[xyz]_- = \frac{1}{2}(\{xyz\} + \{yxz\}),$$

then with respect to the triple product, we have

$$[xyz]_- = (x \circ y) \circ z.$$

Hence, $(V,[xyz]_-)$ is a triple system with $[xyz]_- = [yxz]_-$, and putting $l(x,y)z = [xyz]_-$, then $l(x,y)$ is an anti-derivation of $(V,\ [xyz]_-)$ and also this product $[xyz]_-$ is an involutive triple system with $[xey]_- = x \circ y$.

On the other hand, the above triple system $(V,\{xyz\})$ is a $(1,-1,1)$ triple system with $\{xyz\} = \{zyx\}$ (that is, a JTS). Moreover it is involutive, because it is clear that the triple system V inherits the properties, $\bar{x} = \{exe\} = x$ and $\{e\{xey\}e\} = \overline{x \circ y} = x \circ y = y \circ x = \bar{y} \circ \bar{x} = \{\{eye\}e\{exe\}\}$.

Following [14], next we recall a $(-1,1)$ Freudenthal–Kantor triple system (or GJTS of second order) V defined by the triple product

$$\{xyz\} = x *^t y * z + z *^t y * x - y *^t x * z, \quad \text{for any } x,y,z \in Mat(n,k;F),$$

where ${}^t x$ denotes the transpose of x and the product of right hand is expressed in terms of the standard matrix product $*$.

Then, by straightforward calculation, this triple system $(V,[xyz]_-)$ has the structure of an involutive triple system with respect to the product

$$[xyz]_- = \frac{1}{2}(A(z,y)x + K(z,y)x) = x *^t y * z, \tag{26}$$

where $A(x,y)z = \{xyz\} + \{yxz\}$, and $K(x,y)z = \{xzy\} - \{yzx\}$ for the case of $\varepsilon = -1,\ \delta = 1$.

Thus, for the matrix algebra $M(n,n;F)$, $M(n,n;F)$ $(= V^{(Id_{n,n})})$ is an associative algebra (homotope algebra) induced from Eq. (26) with respect to the binary product

$$x \circ y = [xId_{n,n}y]_- = x({}^t Id_{n,n})y = x * y \quad \text{(product of matrix algebra)}$$

equipped with an involution $\overline{x * y} =^t (x * y) =^t y *^t x = \overline{y} * \overline{x}$.

This matrix algebra is the homotope algebra in a special case of $(-1,1)$ FKTS, because the involution is the transpose.

These examples mean that there are involutive triple systems in view of use above deformations from well-known triple systems (for example, the JTS, the GJTS, the (ε,δ) FKTS, and the structurable algebra).

Remark. Triple systems and algebras given above can be applied to the theory of triality relations described in Section 6.

Examples of triple systems that are not involutive

a) Let V be a vector space with a bilinear form $(\ |\ )$. Then, $(V,\{xyz\})$ is not involutive with respect to the triple product $\{xyz\} = (y|z)x$, but if $(x|y) = -\varepsilon(y|x)$, where $\varepsilon = \pm 1$ and $K(x,y)z = \{xzy\} - \delta\{yzx\}$ (the case of $\delta = \pm 1$), then, it is a (ε, δ) FKTS. Furthermore, if there is an endomorphism g_j satisfying $(g_j x | g_j y) = (x|y)$ $(j = 0, 1, 2)$, then we have

$$g_j\{xyz\} = \{g_j x g_{j+1} y g_{j+1} z\}, \quad \text{i.e.,} \quad g_j R(x,y) = R(g_{j+1}x, g_{j+1}y) g_j.$$

For example, if $g_j \in O(n, F)$, then the validity of $(g_j x | g_j y) = (x|y)$ holds.

b) Let V be a Lie algebra with a binary product $[x, y]$ and an involution $\overline{x} = -x$. Then, $(V, [xyz])$ is a Lie triple system with respect to the triple pruduct defined by

$$[xyz] = [z, [x, y]]. \tag{27}$$

Hence, if $(g_1, g_2, g_3) \in \mathrm{Trig}(V, [x, y])$, then we have

$$g_j R(x,y) = R(g_{j+1}x, g_{j+1}y) g_j, \quad \text{i.e.,} \quad g_j[xyz] = [g_j x g_{j+1} y g_{j+1} z],$$

for all $x, y, z \in V$ $(j = 0, 1, 2)$.

Indeed, from the fact that $g_j[x, y] = [g_{j+1}x, g_{j+2}y]$ and in view of $\overline{x} = -x$, $\overline{[x, y]} = [\overline{y}, \overline{x}]$, we obtain

$$\overline{g}_j(x) = g_j(x) \quad \text{and}$$

$$g_j([z, [x, y]]) = [g_{j+1}z, g_{j+2}[x, y]] = [g_{j+1}z, [g_j x, g_{j+1} y]].$$

This implies that the triple product defined by Eq. (27) has the validity of

$$g_j R(x,y) = R(g_{j+1}x, g_{j+1}y) g_j.$$

This concept is a triality group of the Lie triple system induced from the Lie algebra's triality group with an involution $\overline{[x, y]} = [\overline{y}, \overline{x}]$.

Note that a Lie triple system is a GJTS without the tripotent element since $[xxx] = 0$ for any element x.

Case of the matrix algebra (Revised) As a continuation of the triple system V induced from the matrix algebra (see Section 6), we may provide some additional comments associated with $g = (g_j, g_{j+1}, g_{j+2}) \in \mathrm{Trig}(V, \{xyz\})$ as follows.

For any element x of $V = \mathrm{Mat}(n, n; F)$, if we set $g_j(a)x = a_{j+1} * x * a_{j+2}^{-1}$, for any $a_j \in O(n, F) := \{a_j | \ a_j^t a_j = Id_{n,n}, \ j = 0, 1, 2\}$, then we obtain

$$g_j(x \cdot y) = (g_{j+1}x) \cdot (g_{j+2}y), \quad \text{for any } x, y \in V \quad \text{(global triality relation)},$$

where $x \cdot y =^t (x * y)$, ${}^t x$ is the transpose of x, and $x * y$ is the standard product of the matrix algebra. Hence, if we set $D_j(P)x = P_{j+1} * x - x * P_{j+2}$, for any $P_j \in \ Alt(n, F) := \{P_j | \ {}^t P_j = -P_j\}$ (alternative matrix), where $D_j(P)$ is denoted by D_j, then we have

$$D_j(x \cdot y) = (D_{j+1}x) \cdot y + x \cdot (D_{j+2}y) \quad \text{(local triality relation).}$$

Indeed, by straightforward calculation, we have $g_j(x \cdot y) = a_{j+1} * \overline{x * y} * a_{j+2}^{-1} = (g_{j+1}x) \cdot (g_{j+2}y)$. By means of the Cayley transformation (assuming that well-defined), the relation $(1 - a_j) * (1 + a_j)^{-1} = P_j$ implies that

$$a_j \in O(n, F) \iff_{iff} P_j \in Alt(n, F).$$

Hence, from $D_j(x \cdot y) = P_{j+1} * (x \cdot y) - (x \cdot y) * P_{j+2} = P_{j+1} * (\overline{x * y}) - (\overline{x * y}) * P_{j+2}$ and from the fact that $(D_{j+1}x) \cdot y + x \cdot (D_{j+2}y) = (P_{j+2} * x) \cdot y - (x * P_j) \cdot y + x \cdot (P_j * y) - x \cdot (y * P_{j+1}) = \overline{(P_{j+2} * x) * y} - \overline{x * (y * P_{j+1})} = P_{j+1} * \overline{y} * \overline{x} - \overline{y} * \overline{x} * P_{j+2}$, and $\overline{x} =^t x$, then we obtain the local triality relation;

$$D_j(x \cdot y) = (D_{j+1}x) \cdot y + x \cdot (D_{j+2}y),$$

as a generalization of the derivation with respect to the new product $x \cdot y$, because the transpose of the matrix algebra is the involution in the product of $x \cdot y$, by $x \cdot y = \overline{(x * y)} =^t (x * y) =^t y *^t x$.

8. Conclusion

- An application (toward a Chern-Simon gauge theory and a field theory) to physics of triple systems is discussed in [12, 13].
- For a Cayley algebra (octonion algebra) and triple systems, refer to the book [18] on mathematical physics.
- For the correspondence of Hermitian Jordan triple systems with symmetric domains, see [20]
- For the relationship between a Lie triple system and a symmetric space, it is useful for the book [15], in particular, for a complex structure and a Riemannian curvature tensor related to the (ε, δ)-FKTS, see [9].

As the final comment in this section, we here emphasize why one should study triple systems as follows:

We can construct Lie (super)algebras from triple systems and characterize a structure of the homotope algebras associated with their triple systems. The study of triple systems provides an important common ground for various *branches of mathematics, not only Lie theory and Jordan theory, but also differential geometry (symmetric and homogeneous spaces) and mathematical physics.*

Briefly summarizing about this paper, it seems that the triality relations were first appeared in a "principle of triality" of the Lie algebra of type D_4 (called to a local triality relation) (see, [21]), hence our results may be regarded to a variation (generalization) of this principle with respect to the (α, β, γ) triple systems and the involutive triple systems. That is, by means of terms in triple systems, it is shown that we can represent several properties on algebras.

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