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# RESULTS IN STRONGLY MINIHEDRAL CONE AND SCALAR WEIGHTED CONE METRIC SPACES AND APPLICATIONS

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**Abstract.** The convergence of sequences and non-unique fixed points are established in $\mathcal{M}$-orbitally complete cone metric spaces over the strongly minihedral cone, and scalar weighted cone assuming the cone to be strongly minihedral. Appropriate examples and applications validate the established theory. Further, we provide one more answer to the question of the existence of the contractive condition in Cone metric spaces so that the fixed point is at the point of discontinuity of a map. Also, we provide a negative answer to a natural question of whether the contractive conditions in the obtained results can be replaced by its metric versions.

## 1. Introduction

$\mathcal{K}$-metric and $\mathcal{K}$-normed spaces were familiarized ([1], [5], [13]) using an ordered Banach space as the range of a metric, in place of the set of real numbers. Bogdan Rzepecki ([11]), familiarized a generalized metric $d_{\mathcal{E}}\colon \mathcal{U} \times \mathcal{U} \to \mathcal{S}$, where, $\mathcal{S}$ is a normal cone in a Banach space $\mathcal{E}$ with a partial order $\preceq$. Later on, Shy-Der Lin ([8]) studied $d\colon \mathcal{U}\times\mathcal{U} \to \mathcal{K}$ by substituting real numbers in the metric function with cone $\mathcal{K}$. Huang and Zhang ([3]) re-initiated it as

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cone metric spaces and defined convergence and Cauchy sequences in terms of the interior points of the cone under consideration.

In this manuscript, the convergence of sequences and non-unique fixed points are established in an $\mathcal{M}$-orbitally complete strongly minihedral cone metric space and scalar weighted cone metric space. The established results and illustrative examples show that the cone metric space is a real generalization of a metric space. The solution of equations that are significant in engineering and sciences, integral equations, and a boundary value problem illustrates its usefulness. Our results generalize and extend many earlier obtained results. In the sequel, we provide novel answers in cone metric spaces to the open question posed by Rhoades ([10]) regarding the existence of a contractive map having the discontinuity at a fixed point. Also, we provide a negative answer to a natural question of whether the contractive conditions can be replaced by its metric versions. It is worth mentioning here that Khamsi ([7]) claimed that the majority of the cone fixed point results are identical to the classical results and consequently, extensions of known fixed point theorems to cone metric spaces are superfluous.

## 2. Preliminaries

Let $\mathcal{E} := (\mathcal{E}, \|\cdot\|)$ be a real Banach space and $\mathcal{P} := \mathcal{P}_{\mathcal{E}}$, a closed non-empty subset of $\mathcal{E}$. $\mathcal{P}$ is a cone if $au + bv \in \mathcal{P}, u, v \in \mathcal{P}$, $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$, $\mathcal{P} \neq \{0\}$ and $a, b$ are non-negative real numbers. A partial ordering with respect to $\mathcal{P}$ is defined by $u \preceq v$ if and only if $v - u \in \mathcal{P}$. $u \prec v$ shows that $u \preceq v$ and $u \neq v$, and $u \prec\prec v$ means $v - u \in int\mathcal{P}$, the interior of $\mathcal{P}$. We assume that $int\mathcal{P} \neq \phi$. The cone $\mathcal{P}$ is normal if there is a least positive number $\mathcal{K} \geq 1$ (normal constant) for which $\theta \preceq u \preceq v$ implies that $\|u\| \leq \mathcal{K}\|v\|$, $u, v \in \mathcal{E}$, and $\theta$ is the zero element of cone $\mathcal{P}$ in a normed linear space $\mathcal{E}$. The cone $\mathcal{P}$ is regular if for any sequence $\{u_n\}_{n\geq 1}$ such that $u_1 \preceq u_2 \cdots \preceq v$, $v \in \mathcal{E}$, there exists $u \in \mathcal{E}$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$.

Lemma 2.1 ([9]).

(i) *Every regular cone is normal.*

(ii) *For each $k > 1$, there is a normal cone with a normal constant $\mathcal{K} > k$.*

(iii) *The cone $\mathcal{P}$ is regular if every decreasing sequence which is bounded from below is convergent.*

Definition 2.2 ([3]). Let $\mathcal{U}$ be a non-empty set. Suppose that the map $d \colon \mathcal{U} \times \mathcal{U} \to \mathcal{E}$ satisfies:

(i) $\theta \preceq d(u, v)$ and $d(u, v) = \theta$ if and only if $u = v$;

(ii) $d(u, v) = d(v, u)$;
(iii) $d(u, v) \preceq d(u, z) + d(z, v)$.
Then, the pair $(\mathcal{U}, d)$ is called a *cone metric space.*

EXAMPLE 2.3 ([6]). Let $\mathcal{E} = \mathbb{R}^3$, $\mathcal{P} = \{(u, v, z) \in \mathcal{E} : u, v, z \geq 0\}$, and $\mathcal{U} = \mathbb{R}$. Define $d\colon \mathcal{U} \times \mathcal{U} \to \mathcal{E}$ by $d(u, \hat{u}) = (\mathbf{a}\,|u - \hat{u}|\,, \mathbf{b}\,|u - \hat{u}|\,, \mathbf{c}\,|u - \hat{u}|)$, where, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are positive constants. Then, $(\mathcal{U}, d)$ is a cone metric space. However, it is not a usual metric space. Note that the cone $\mathcal{P}$ is normal with $\mathcal{K} = 1$. Clearly, the cone is strongly minihedral.

DEFINITION 2.4 ([6]). Let $u \in \mathcal{U}$ and $\{u_n\}_{n\geq 1}$ be a sequence in a cone metric space $(\mathcal{U}, d)$. Then,

(i) $\{u_n\}_{n\geq 1}$ *converges to* $u$ whenever for every $c \in \mathcal{E}$ with $\theta \prec\prec c$, there is a natural number $N$ satisfying $d(u_n, u) \prec\prec c$, $n \geq N$. We write $\lim_{n\to\infty} u_n = u$.
(ii) $\{u_n\}_{n\geq 1}$ is a *Cauchy sequence* whenever for every $c \in \mathcal{E}$ with $\theta \prec\prec c$ there is a natural number $N$ such that $d(u_n, u_m) \prec\prec c$, $n, m \geq N$.
(iii) $(\mathcal{U}, d)$ is a *complete* cone metric space if every Cauchy sequence is convergent.

DEFINITION 2.5 ([2]). $\mathcal{P}$ is called a *minihedral cone* if $\sup\{u, v\}$ exists, $u, v \in \mathcal{E}$, and a *strongly minihedral cone* if every subset of $\mathcal{E}$ which is bounded from above has a supremum.

LEMMA 2.6 ([6]).

(i) *Every strongly minihedral normal (not necessarily closed) cone is regular.*
(ii) *Every strongly minihedral (closed) cone is normal.*

EXAMPLE 2.7 ([6]). Let $\mathcal{E} = \mathcal{C}[0, 1]$ with the supremum norm and $\mathcal{P} = \{f \in \mathcal{E} : f \succeq \theta\}$. Then, $\mathcal{P}$ is a cone with a normal constant $\mathcal{K} = 1$ which is not regular, since the sequence $\{x^n\}$ is monotonically decreasing, but not uniformly convergent to $\theta$. This cone is not strongly minihedral.

DEFINITION 2.8 ([6]). A map $\mathcal{M}$ on a cone metric space $(\mathcal{U}, d)$ is *orbitally continuous* if $\lim_{j\to\infty} \mathcal{M}^{n_j} u = z$ implies that $\lim_{j\to\infty} \mathcal{M}(\mathcal{M}^{n_j} u) = \mathcal{M}z$. A cone metric space $(\mathcal{U}, d)$ is $\mathcal{M}$-*orbitally complete* if every Cauchy sequence of the form $\{\mathcal{M}^{n_j} u\}_{n=1}^{\infty}, u \in \mathcal{U}$ converges in $(\mathcal{U}, d)$.

REMARK 2.9. Orbital continuity of $\mathcal{M}$ implies orbital continuity of $\mathcal{M}^m$, $m \in \mathbb{N}$.

DEFINITION 2.10 ([6]). The *scalar weight* of the cone metric $d$ is given as $d_s(u, v) = \|d(u, v)\|$.

Noticeably, the scalar weight of the cone metric $d_s$ acts like a metric on $\mathcal{U}$ for normal cone $\mathcal{P}$ ($\mathcal{K}=1$).

## 3. Main results

First, we demonstrate that the iterated sequence converges to a fixed point of an orbitally continuous self-map in an orbitally complete cone metric space $(\mathcal{U}, d)$.

THEOREM 3.1. *Let $\mathcal{M}$ be an orbitally continuous self-map on an $\mathcal{M}$-orbitally complete cone metric space $(\mathcal{U}, d)$ over a strongly minihedral normal cone $\mathcal{P}$ such that there exists a real number $\eta$ satisfying:*

$$\begin{aligned} \theta &\preceq \min\{d(u,\mathcal{M}u), d(v,\mathcal{M}v), d(\mathcal{M}u,\mathcal{M}v)\} \\ &\quad + \eta \min\{d(u,\mathcal{M}v), d(v,\mathcal{M}u)\} \\ &\preceq \alpha \max\{d(u,v), \min\{d(u,\mathcal{M}u), d(v,\mathcal{M}v)\}\} + \beta d(u,v), \end{aligned} \tag{3.1}$$

*$u, v \in \mathcal{U}$, where $\theta$ is the zero element of the cone $\mathcal{P}$ in a normed linear space $\mathcal{E}$, $\alpha+\beta<1$, $\alpha$ and $\beta$ are non-negative real numbers. Then, the iterated sequence $\{\mathcal{M}^n u\}$, $u \in \mathcal{U}$ converges to a fixed point of $\mathcal{M}$.*

PROOF. Fix $u_0 \in \mathcal{U}$ and set $u_1 = \mathcal{M}u_0$. Repeatedly, $u_{n+1} = \mathcal{M}u_n = \mathcal{M}^{n+1}u_0$, $n \geq 1$. Evidently, the sequence $\{u_n\}$ is Cauchy, when $u_{n+1} = u_n$, for some $n \in \mathbb{N}$.

Let $u_{n+1} \neq u_n, n \in \mathbb{N}$. Taking $u = u_{n-1}$ and $v = u_n$ in inequality (3.1) we get

$$\begin{aligned} \theta &\preceq \min\{d(u_{n-1},\mathcal{M}u_{n-1}), d(u_n,\mathcal{M}u_n), d(\mathcal{M}u_{n-1},\mathcal{M}u_n)\} \\ &\quad + \eta \min\{d(u_{n-1},\mathcal{M}u_n), d(u_n,\mathcal{M}u_{n-1})\} \\ &\preceq \alpha \max\{d(u_{n-1},u_n), \min\{d(u_{n-1},\mathcal{M}u_{n-1}), d(u_n,\mathcal{M}u_n)\}\} + \beta d(u_{n-1},u_n), \end{aligned}$$

that is,

$$\begin{aligned} \theta &\preceq \min\{d(u_{n-1},u_n), d(u_n,u_{n+1}), d(u_n,u_{n+1})\} \\ &\quad + \eta \min\{d(u_{n-1},u_{n+1}), d(u_n,u_n)\} \\ &\preceq \alpha \max\{d(u_{n-1},u_n), \min\{d(u_{n-1},u_n), d(u_n,u_{n+1})\}\} + \beta d(u_{n-1},u_n). \end{aligned}$$

Two cases arise: either $d(u_{n-1},u_n)\prec d(u_n,u_{n+1})$ or $d(u_n,u_{n+1})\prec d(u_{n-1},u_n)$.

Case (i) If $d(u_{n-1},u_n)\prec d(u_n,u_{n+1})$, then

$$\begin{aligned}\theta \preceq d(u_{n-1},u_n)+\eta\cdot\theta &\preceq \alpha d(u_{n-1},u_n)+\beta d(u_{n-1},u_n)\\ &=(\alpha+\beta)d(u_{n-1},u_n)\\ &\prec d(u_{n-1},u_n), \quad \text{a contradiction.}\end{aligned}$$

Case (ii) If $d(u_n,u_{n+1})\prec d(u_{n-1},u_n)$, then

$$\begin{aligned}\theta \preceq d(u_n,u_{n+1})+\eta\cdot\theta &\preceq \alpha d(u_{n-1},u_n)+\beta d(u_{n-1},u_n)\\ &=(\alpha+\beta)d(u_{n-1},u_n)\\ &\prec d(u_{n-1},u_n).\end{aligned}$$

Repeatedly, $d(u_n,u_{n+1})\prec d(u_{n-1},u_n)\prec d(u_{n-2},u_{n-1})\prec\cdots\prec d(u_0,u_1)$. So, $\{d(u_n,u_{n+1})\}_{n\geq 0}$ is a decreasing sequence bounded below by $\theta$ and consequently, converges to some real number $\theta\preceq t$. Since

$$\begin{aligned}t\preceq d(u_n,u_{n+1}) &\preceq (\alpha+\beta)d(u_{n-1},u_n)\\ &\preceq(\alpha+\beta)^2 d(u_{n-2},u_{n-1})\\ &\preceq \vdots\\ &\preceq(\alpha+\beta)^n d(u_0,u_1)\to\theta, \quad \text{as } n\to\infty,\end{aligned}$$

then $\lim_{n\to\infty} d(u_n,u_{n+1})=\theta$. Now, for $n>m$

$$\begin{aligned}d(u_m,u_n) &\prec d(u_m,u_{m+1})+d(u_{m+1},u_{m+2})+\cdots+d(u_{n-1},u_n)\\ &\preceq\big((\alpha+\beta)^m+(\alpha+\beta)^{m+1}+\cdots+(\alpha+\beta)^n\big)d(u_0,u_1)\\ &\preceq\frac{(\alpha+\beta)^m(1-(\alpha+\beta)^{n-m})}{1-\alpha-\beta}d(u_0,u_1)\to\theta, \quad \text{as } n,m\to\infty.\end{aligned}$$

Hence, $\{u_n\}$ is a Cauchy sequence in $(\mathcal{U},d)$. Since, $(\mathcal{U},d)$ is $\mathcal{M}$-orbitally complete, there exists $z\in\mathcal{U}$ satisfying

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\mathcal{M}^n u_0=z.$$

Since $\mathcal{M}$ is orbitally continuous, $\mathcal{M}z=\lim_{n\to\infty}\mathcal{M}(\mathcal{M}^n u_0)=z$, that is, $z$ is a fixed point of $\mathcal{M}$. $\square$

Example 3.2. Let $\mathcal{U} = \{0\} \cup \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\}$, $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{(u,v) : u,v \geq 0\}$ and $d\colon \mathcal{U}\times\mathcal{U} \to \mathcal{E}$ be defined by $d(u,v) = (\mathbf{a}\,|u-v|\,, \mathbf{b}\,|u-v|)$, where $\mathbf{a}, \mathbf{b} \geq 0$. We define a self map $\mathcal{M}$ on $\mathcal{U}$ by $\mathcal{M}0 = 0$, $\mathcal{M}(\frac{1}{3^n}) = \frac{1}{3^{n+1}}, n \geq 1$. Here, $O(\frac{1}{3}) = \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\}$. Clearly, $(\mathcal{U}, d)$ is $\mathcal{M}$-orbitally complete cone metric space over a strongly minihedral normal cone $\mathcal{P}$ and $\mathcal{M}$ is orbitally continuous at 0. $\mathcal{M}$ satisfies inequality (3.1) for $u, v \in \mathcal{U}$, $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$, and $\eta \in (-\infty, 0]$. Thus, all the assumptions of Theorem 3.1 are verified and $\mathcal{M}$ has a fixed point at $u = 0$, which is a point of discontinuity of a map $\mathcal{M}$. Also, there exists an iterated sequence $\{\mathcal{M}^n u\} = \left\{\frac{1}{3^n}\right\}$ converging to a fixed point 0 of $\mathcal{M}$. Noticeably, $(\mathcal{U}, d)$ is not a usual metric space.

Next, we establish a fixed point of an orbitally continuous self-map.

Theorem 3.3. *If in Theorem 3.1, the sequence $\{\mathcal{M}^n u_0\}$ has a cluster point $z \in \mathcal{U}$, for some $u_0 \in \mathcal{U}$, then $z$ is a fixed point of $\mathcal{M}$.*

Proof. Let $\mathcal{M}^m u_0 = \mathcal{M}^{m-1} u_0$, for some $m \in \mathbb{N}$. So, $\mathcal{M}^n u_0 = \mathcal{M}^m u_0 = z$, $n \geq m$. Evidently, $z$ is a required point. Let $\mathcal{M}^m u_0 \neq \mathcal{M}^{m-1} u_0$, for all $m \in \mathbb{N}$. Since $\{\mathcal{M}^n u_0\}$ has a cluster point $z \in \mathcal{U}$, $\lim_{i\to\infty} \mathcal{M}^{n_i} u_0 = z$. On taking $u = \mathcal{M}^{n-1} u_0$ and $v = \mathcal{M}^n u_0$ in inequality (3.1), we get

$$\begin{aligned}
(3.2)\quad \theta &\preceq \min\left\{d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0), d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0), d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)\right\} \\
&\quad + \eta \min\left\{d(\mathcal{M}^{n-1}u_0, \mathcal{M}^{n+1} u_0), d(\mathcal{M}^n u_0, \mathcal{M}^n u_0)\right\} \\
&\preceq \alpha \max\{d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0), \min\left\{d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0), d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)\right\}\} \\
&\quad + \beta d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0).
\end{aligned}$$

There are two cases: $d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0) \preceq d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)$ or $d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0) \preceq d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0)$.

Case (i) If $d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0) \preceq d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)$, inequality (3.2) gives

$$\begin{aligned}
\theta &\preceq d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0) + \eta \cdot \theta \\
&\prec \alpha d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0) + \beta d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0) \\
&= (\alpha + \beta) d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0) \\
&\prec d(\mathcal{M}^{n-1}u_0, \mathcal{M}^n u_0), \quad \text{a contradiction.}
\end{aligned}$$

Case (ii) If $d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0) \preceq d(\mathcal{M}^{n-1} u_0, \mathcal{M}^n u_0)$, inequality (3.2) gives

$$\begin{aligned}\theta \preceq d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0) + \eta \cdot \theta \prec \alpha d(\mathcal{M}^{n-1} u_0, \mathcal{M}^n u_0) + \beta d(\mathcal{M}^{n-1} u_0, \mathcal{M}^n u_0)\\ = (\alpha+\beta) d(\mathcal{M}^{n-1} u_0, \mathcal{M}^n u_0)\\ \prec d(\mathcal{M}^{n-1} u_0, \mathcal{M}^n u_0).\end{aligned}$$

So, the sequence $\left\{d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)\right\}_{n\geq 0}$ is decreasing. Since the cone $\mathcal{P}$ is strongly minihedral, it follows from Lemma 2.1(iii) and 2.6(i) that the sequence $\left\{d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)\right\}_{n\geq 0}$ is convergent. Also, by $\mathcal{M}$-orbital continuity $\lim_{i\to\infty} d(\mathcal{M}^{n_i} u_0, \mathcal{M}^{n_i+1} u_0) = d(z, \mathcal{M}z)$. But $\left\{d(\mathcal{M}^{n_i} u_0, \mathcal{M}^{n_i+1} u_0)\right\}_{n\geq 0} \subseteq \left\{d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)\right\}_{n\geq 0}$. So, $\lim_{n\to\infty} d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0) = d(z, \mathcal{M}z)$. Note that $\lim_{i\to\infty} \mathcal{M}^{n_i+1} u_0 = \mathcal{M}z$, $\lim_{i\to\infty} \mathcal{M}^{n_i+2} u_0 = \mathcal{M}^2 z$ and $\left\{d(\mathcal{M}^{n_i+1} u_0, \mathcal{M}^{n_i+2})\right\}_{n\geq 0} \subseteq \left\{d(\mathcal{M}^n u_0, \mathcal{M}^{n+1} u_0)\right\}_{n\geq 0}$, therefore $d(\mathcal{M}z, \mathcal{M}^2 z) = d(z, \mathcal{M}z)$.

Suppose that $\mathcal{M}z \neq z$, that is, $\theta \prec d(z, \mathcal{M}z)$. Taking $u = z$ and $v = \mathcal{M}z$ in inequality (3.1), we get

$$\begin{aligned}\theta \preceq \min\{d(z, \mathcal{M}z), d(\mathcal{M}z, \mathcal{M}^2 z), d(\mathcal{M}z, \mathcal{M}^2 z)\}\\ + \eta \min\{d(z, \mathcal{M}^2 z), d(\mathcal{M}z, \mathcal{M}z)\}\\ \preceq \alpha \max\{d(z, \mathcal{M}z), \min\{d(z, \mathcal{M}z), d(\mathcal{M}z, \mathcal{M}^2 z)\}\} + \beta d(z, \mathcal{M}z),\end{aligned}$$

that is,

$$\begin{aligned}\theta \preceq d(z, \mathcal{M}z) + \eta \cdot \theta \preceq \alpha d(z,\mathcal{M}z) + \beta d(z, \mathcal{M}z)\\ = (\alpha+\beta) d(z, \mathcal{M}z)\\ \prec d(z, \mathcal{M}z), \quad \text{a contradiction.}\end{aligned}$$

Hence, $\mathcal{M}z = z$. □

Next, we demonstrate that the iterated sequence converges to a fixed point on scalar weighted cone metric spaces with no restriction on a normal constant $\mathcal{K}$. It is interesting to see that assumption of strongly minihedral is not required.

THEOREM 3.4. *Let $\mathcal{M}$ be an orbitally continuous self-map on an $\mathcal{M}$-orbitally complete scalar weighted cone metric space $(\mathcal{U}, d_s)$ over a normal* *cone $\mathcal{P}$ with a normal constant $\mathcal{K}$ such that there exists a real number $\eta$ satisfying:*

$$0 \leq \min\{d_s(u,\mathcal{M}u), d_s(\mathcal{M}u,\mathcal{M}v), d_s(v,\mathcal{M}v)\} + \eta\min\{d_s(u,\mathcal{M}v), d_s(v,\mathcal{M}u)\} \leq \alpha\max\{d_s(u,v), \min\{d_s(u,\mathcal{M}u), d_s(v,\mathcal{M}v)\}\} + \beta d_s(u,v), \tag{3.3}$$

*$u, v \in \mathcal{U}$, $\alpha+\beta<1$, $\alpha$ and $\beta$ are non-negative real numbers. Then the iterated sequence $\{\mathcal{M}^n u\}$ converges to a fixed point of $\mathcal{M}$, for each $u \in \mathcal{U}$.*

PROOF. Let $u_0 \in \mathcal{U}$. Set $u_1 = \mathcal{M}u_0$. Repeatedly, $u_{n+1} = \mathcal{M}u_n = \mathcal{M}^{n+1}u_0$, $n \geq 1$. Evidently, the sequence $\{u_n\}$ is a Cauchy sequence when $u_{n+1} = u_n$, for some $n \in \mathbb{N}$. So, let $u_{n+1} \neq u_n$, for all $n \in \mathbb{N}$. On taking $u = u_{n-1}$ and $v = u_n$, in inequality (3.3), we get

$$\begin{aligned} 0 &\leq \min\{d_s(u_{n-1},\mathcal{M}u_{n-1}), d_s(\mathcal{M}u_{n-1},\mathcal{M}u_n), d_s(u_n,\mathcal{M}u_n)\} \\ &\quad + \eta\min\{d_s(u_{n-1},\mathcal{M}u_n), d_s(u_n,\mathcal{M}u_{n-1})\} \\ &\leq \alpha\max\{d_s(u_{n-1},u_n), \min\{d_s(u_{n-1},\mathcal{M}u_{n-1}), d_s(u_n,\mathcal{M}u_n)\}\} \\ &\quad + \beta d_s(u_{n-1},u_n), \end{aligned}$$

that is,

$$\begin{aligned} 0 &\leq \min\{d_s(u_{n-1},u_n), d_s(u_n,u_{n+1}), d_s(u_n,u_{n+1})\} \\ &\quad + \eta\min\{d_s(u_{n-1},u_{n+1}), d_s(u_n,u_n)\} \\ &\leq \alpha\max\{d_s(u_{n-1},u_n), \min\{d_s(u_{n-1},u_n), d_s(u_n,u_{n+1})\}\} + \beta d_s(u_{n-1},u_n). \end{aligned}$$

There are two cases:

$$d_s(u_n,u_{n+1}) \leq d_s(u_{n-1},u_n) \quad \text{or} \quad d_s(u_{n-1},u_n) \leq d_s(u_n,u_{n+1}).$$

Case (i) If $d_s(u_{n-1},u_n) \leq d_s(u_n,u_{n+1})$, we have

$$\begin{aligned} 0 &\leq d_s(u_{n-1},u_n) + \eta\cdot 0 \\ &\leq \alpha\max\{d_s(u_{n-1},u_n), d_s(u_{n-1},u_n)\} + \beta d_s(u_{n-1},u_n) \\ &= (\alpha+\beta)d_s(u_{n-1},u_n) \\ &< d_s(u_{n-1},u_n), \quad \text{a contradiction.} \end{aligned}$$

Case (ii) If $d_s(u_n, u_{n+1}) \leq d_s(u_{n-1}, u_n)$, we have

$$\begin{aligned}0 &\leq d_s(u_n, u_{n+1}) + \eta \cdot 0\\ &\leq \alpha \max\{d_s(u_{n-1}, u_n), d_s(u_n, u_{n+1})\} + \beta d_s(u_{n-1}, u_n)\\ &= (\alpha + \beta) d_s(u_{n-1}, u_n)\\ &< d_s(u_{n-1}, u_n).\end{aligned}$$

This implies that $\{d_s(u_n, u_{n+1})\}_{n\geq 0}$ is a decreasing sequence of positive real numbers and consequently converges to $t \geq 0$. We assert that $t = 0$. Because if $t > 0$, on making $n \to \infty$, $d_s(u_n, u_{n+1}) \leq (\alpha + \beta) d_s(u_{n-1}, u_n)$, yields $t \leq (\alpha + \beta)t < t$, a contradiction. Hence, $t = 0$. By using similar arguments, we may prove that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{U}, d_s)$. Since $(\mathcal{U}, d_s)$ is $\mathcal{M}$-orbitally complete, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \mathcal{M}^n u_0 = z$, for some $z \in \mathcal{U}$. By the orbital continuity of $\mathcal{M}$, $\mathcal{M}z = \lim_{n\to\infty} \mathcal{M}^{n+1} u_0 = z$, that is, $z$ is a fixed point of $\mathcal{M}$ and for $u \in \mathcal{U}$, the iterated sequence $\{\mathcal{M}^n u\}$ converges to a fixed point of $\mathcal{M}$. □

Example 3.5. Let $\mathcal{U} = \{1\} \cup \left\{1 + \frac{1}{3^{n-1}} : n \in \mathbb{N}\right\} \cup \left\{2 - \frac{1}{4^{n-1}} : n \in \mathbb{N}\right\} \cup \{2\}$, $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{(u, v) : u, v \geq 0\}$ and $d \colon \mathcal{U} \times \mathcal{U} \to \mathcal{E}$ be defined as $d(u, v) = (\mathbf{a}\,|u - v|\,, \mathbf{b}\,|u - v|)$, where, $\mathbf{a}, \mathbf{b} \geq 0$. We define a self map $\mathcal{M}$ on $\mathcal{U}$ as: $\mathcal{M}1 = 1$, $\mathcal{M}(1 + \frac{1}{3^n}) = 1 + \frac{1}{3^{n+1}}, n \geq 1$, $\mathcal{M}(2 - \frac{1}{4^n}) = 2 - \frac{1}{4^{n+1}}, n \geq 1$, $\mathcal{M}2 = 2$. Here, $O(1 + \frac{1}{3}) = \left\{1 + \frac{1}{3^n} : n \in \mathbb{N}\right\}$ and $O(2 - \frac{1}{4}) = \left\{2 - \frac{1}{4^n} : n \in \mathbb{N}\right\}$. Clearly, $(\mathcal{U}, d)$ is $\mathcal{M}$-orbitally complete scalar weighted cone metric space and $\mathcal{M}$ is orbitally continuous at 1 and 2. $\mathcal{M}$ satisfies inequality (3.3) for $u, v \in \mathcal{U}$ and $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, \eta \in (-\infty, 0]$. Thus, all the assumptions of Theorem 3.4 are verified and $\mathcal{M}$ has two fixed points $u = 1$ and $u = 2$, which are also points of discontinuity of $\mathcal{M}$. Also, there exist iterated sequences: $\{\mathcal{M}^n u\} = \left\{1 + \frac{1}{3^n}\right\}$ converging to a fixed point 1, and $\{\mathcal{M}^n u\} = \left\{2 - \frac{1}{4^n}\right\}$ converging to a fixed point 2 of $\mathcal{M}$. Noticeably, $(\mathcal{U}, d)$ is not a usual metric space.

Now, we establish a fixed point of an orbitally continuous self-map on a scalar weighted cone metric space.

Theorem 3.6. *If in Theorem* 3.4*, the sequence* $\{\mathcal{M}^n u_0\}$ *has a cluster point* $z \in \mathcal{U}$*, for some* $u_0 \in \mathcal{U}$*, then* $z$ *is a fixed point of* $\mathcal{M}$*.*

Proof. The proof of Theorem 3.6 is similar to Theorem 3.3 except the fact that we may conclude the decreasing sequence to be convergent without the supposition of strongly minihedrality of the cone $\mathcal{P}$ since we are using the scalar weight of a cone metric in Theorem 3.6 . □

Theorem 3.7. *Theorem* 3.4 *remains true even if inequality* (3.3) *is replaced by*

$$0 \leq \min\left\{(d_s(u,\mathcal{M}u))^2, d_s(u,v)d_s(\mathcal{M}u,\mathcal{M}v), (d_s(v,\mathcal{M}v))^2\right\} + \eta \min\{d_s(u,\mathcal{M}v)d_s(\mathcal{M}u,v), d_s(v,\mathcal{M}v)d_s(u,\mathcal{M}u)\} \leq \alpha \max\{(d_s(u,v))^2, \min\{d_s(u,\mathcal{M}u)d_s(v,\mathcal{M}v), d_s(v,\mathcal{M}u)d_s(u,\mathcal{M}v)\}\} + \beta(d_s(u,v))^2. \tag{3.4}$$

Proof. On taking $v = \mathcal{M}u$ in inequality (3.4), we get

$$0 \leq \min\left\{(d_s(u,\mathcal{M}u))^2, d_s(u,\mathcal{M}u)d_s(\mathcal{M}u,\mathcal{M}^2u), (d_s(\mathcal{M}u,\mathcal{M}^2u))^2\right\} + \eta \min\{d_s(u,\mathcal{M}^2u)d_s(\mathcal{M}u,\mathcal{M}u), d_s(\mathcal{M}u,\mathcal{M}^2u)d_s(u,\mathcal{M}u)\} \leq \alpha \max\{(d_s(u,\mathcal{M}u))^2, \min\{d_s(u,\mathcal{M}u)d_s(\mathcal{M}u,\mathcal{M}^2u), d_s(\mathcal{M}u,\mathcal{M}u)d_s(u,\mathcal{M}^2u)\}\} + \beta(d_s(u,\mathcal{M}u))^2,$$

that is,

$$0 \leq \min\left\{(d_s(u,\mathcal{M}u))^2, d_s(u,\mathcal{M}u)d_s(\mathcal{M}u,\mathcal{M}^2u), (d_s(\mathcal{M}u,\mathcal{M}^2u))^2\right\} \leq \alpha d_s(u,\mathcal{M}u))^2 + \beta(d_s(u,\mathcal{M}u))^2.$$

There are two cases:

$$d_s(u,\mathcal{M}u) \leq d_s(\mathcal{M}u,\mathcal{M}^2u) \quad \text{or} \quad d_s(\mathcal{M}u,\mathcal{M}^2u) \leq d_s(u,\mathcal{M}u).$$

Case (i) If $d_s(u,\mathcal{M}u) \leq d_s(\mathcal{M}u,\mathcal{M}^2u)$, then

$$\begin{aligned} 0 \leq (d_s(u,\mathcal{M}u))^2 &\leq \alpha(d_s(u,\mathcal{M}u))^2 + \beta(d_s(u,\mathcal{M}u))^2 \\ &= (\alpha+\beta)(d_s(u,\mathcal{M}u))^2 \\ &< (d_s(u,\mathcal{M}u))^2, \text{ a contradiction.} \end{aligned}$$

Case (ii) If $d_s(\mathcal{M}u,\mathcal{M}^2u) \leq d_s(u,\mathcal{M}u)$, then

$$\begin{aligned} 0 \leq (d_s(\mathcal{M}u,\mathcal{M}^2u))^2 &\leq \alpha(d_s(u,\mathcal{M}u))^2 + \beta(d_s(u,\mathcal{M}u))^2 \\ &= (\alpha+\beta)(d_s(u,\mathcal{M}u))^2 \\ &< (d_s(u,\mathcal{M}u))^2, \end{aligned}$$

that is, $d_s(\mathcal{M}u, \mathcal{M}^2u) \le \sqrt{(\alpha+\beta)}(d_s(u, \mathcal{M}u)) \le d_s(u, \mathcal{M}u)$. Repeatedly, we get $d_s(\mathcal{M}^2u, \mathcal{M}^3u) \le d_s(\mathcal{M}u, \mathcal{M}^2u) \le d_s(u, \mathcal{M}u)$. Continuing like this,

$$d_s(\mathcal{M}^n u, \mathcal{M}^{n+1}u) \le d_s(\mathcal{M}^{n-1}u, \mathcal{M}^n u) \le \cdots \le d_s(u, \mathcal{M}u). \tag{3.5}$$

Following Theorem 3.4, let $u_0 \in \mathcal{U}$ and $u_1 = \mathcal{M}u_0$. Repeatedly, $u_{n+1} = \mathcal{M}u_n = \mathcal{M}^{n+1}u_0$, $n \ge 1$. Evidently, the sequence $\{u_n\}$ is Cauchy when $u_{n+1} = u_n$, for some $n \in \mathbb{N}$. Let $u_{n+1} \ne u_n$, for all $n \in \mathbb{N}$. Now, using inequality (3.5) with $u = u_0$, we get $d_s(u_n, u_{n+1}) \le d_s(u_{n-1}, u_n) \le \ldots d_s(u_0, \mathcal{M}u_0)$. By routine calculation, we see that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{U}, d_s)$. Since, $(\mathcal{U}, d_s)$ is $\mathcal{M}$ orbitally complete, there exists $z \in \mathcal{U}$ satisfying $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \mathcal{M}^n u_0 = z$, that is, $z$ is a fixed point of $\mathcal{M}$. Consequently, the iterated sequence $\{\mathcal{M}^n u\}$ converges to a fixed point of $\mathcal{M}$, for each $u \in \mathcal{U}$. □

Our next result is more interesting as two cone metrics are being used to establish a fixed point.

THEOREM 3.8. *Let $\mathcal{U}$ be a non-empty set equipped with two cone metrics $d, \rho$ and $\mathcal{M}$ be a self map of $\mathcal{U}$. Let*

(i) *$\mathcal{U}$ be an orbitally complete space with respect to $d_s$,*
(ii) *$d_s(u, v) \le \rho_s(u, v)$, $u, v \in \mathcal{U}$,*
(iii) *$\mathcal{M}$ be orbitally continuous with respect to $d_s$,*
(iv) *$\mathcal{M}$ be such that there exists a real number $\eta$ satisfying:*

$$\begin{aligned} 0 &\le \min\left\{(\rho_s(u, \mathcal{M}u))^2, \rho_s(u, v)\rho_s(\mathcal{M}u, \mathcal{M}v), (\rho_s(v, \mathcal{M}v))^2\right\} \\ &\quad + \eta \min\{\rho_s(u, \mathcal{M}v)\rho_s(\mathcal{M}u, v), \rho_s(v, \mathcal{M}v)\rho_s(u, \mathcal{M}u)\} \\ &\le \alpha \max\left\{(\rho_s(u, v))^2, \min\left\{\rho_s(u, \mathcal{M}u)\rho_s(v, \mathcal{M}v),\right.\right. \\ &\quad \left.\left.\rho_s(v, \mathcal{M}u)\rho_s(u, \mathcal{M}v)\right\}\right\} + \beta(\rho_s(u, v))^2, \quad u, v \in \mathcal{U}, \end{aligned} \tag{3.6}$$

*where, $\alpha + \beta < 1$, $\alpha$ and $\beta$ are non-negative real numbers. Then $\mathcal{M}$ has a fixed point in $\mathcal{U}$.*

PROOF. Following Theorem 3.4, let $u_0 \in \mathcal{U}$ and $u_1 = \mathcal{M}u_0$. Repeatedly, $u_{n+1} = \mathcal{M}u_n$, $n \ge 1$. Evidently, the sequence $\{u_n\}$ is a Cauchy sequence when $u_{n+1} = u_n$, for some $n \in \mathbb{N}$. Let $u_{n+1} \ne u_n$, for all $n \in \mathbb{N}$. Taking $u = u_{n-1}$ and $v = u_n$ in inequality (3.6), and following the steps similar to Theorem 3.7, we get $0 \le \rho_s(u_n, u_{n+1}) < \rho_s(u_{n-1}, u_n)$. Repeatedly, we observe that $\rho_s(u_n, u_{n+1}) < \rho_s(u_{n-1}, u_n) < \cdots < \rho_s(u_0, \mathcal{M}u_0)$.

By routine calculation, we see that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{U}, \rho_s)$. In view of (ii), $\{u_n\}$ is also a Cauchy sequence in $(\mathcal{U}, d_s)$. Since, $(\mathcal{U}, d_s)$ is $\mathcal{M}$-orbitally complete, there exists $z \in \mathcal{U}$ satisfying $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \mathcal{M}^n u_0 = z$.

By the orbital continuity of $\mathcal{M}$, $\mathcal{M}z = \lim_{n\to\infty} \mathcal{M}\mathcal{M}^n u_0 = \lim_{n\to\infty} \mathcal{M}^{n+1} u_0 = z$, that is, $z$ is a fixed point of $\mathcal{M}$. □

REMARK 3.9.

(i) Examples 3.2 and 3.5 provide a negative answer to a natural question of whether the contractive conditions in the statements of Theorems 3.1 and 3.4 can be replaced by its metric versions. M.A. Khamsi ([7]) claimed that the majority of the cone fixed point theorems are duplications of the classical results and that any extensions of existing fixed point theorems to cone metric spaces are superfluous. Further, Banach spaces and the related cone subsets under consideration are not required. But, cone metric spaces are different from usual metric spaces due to the fact that here the distance function is not a positive real number, but elements of a cone are in some normed spaces ([12]) or topological vector spaces ([4]).

(ii) One may notice that we have not assumed the map to be continuous (not even at a fixed point) in any of our results (see Examples 3.2 and 3.5). Consequently, we have provided more answers to the question (Rhoades [10]) of the existence of contractive condition admitting fixed point at the point of discontinuity of the map in an $\mathcal{M}$-orbitally complete cone metric spaces over the strongly minihedral cone and a scalar weighted cone assuming the cone to be strongly minihedral.

## 4. Applications

Now, we utilize our result to the following elementary equations:

$$u = \sinh^{-1} \mu u, \quad v = \mu \tan^{-1} v, \quad 0 < \mu < 1, \tag{4.1}$$

and to show that it has a solution in $\mathbb{R}^2$.

PROOF. Let $\mathcal{U} = \left\{0, \frac{1}{2^n}, \frac{1}{3^n}, \dots, 1\right\} \times \left\{0, \frac{1}{2^n}, \frac{1}{3^n}, \dots, 1\right\}$, $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{(u, v) : u, v \geq 0\}$ and for any $u = (u_1, u_2)$, $v = (v_1, v_2)$, $d \colon \mathcal{U} \times \mathcal{U} \to \mathcal{E}$ be defined by $d(u, v) = (|u_1 - v_1|, |u_2 - v_2|)$. Then $(\mathcal{U}, d)$ is a complete cone metric space over a Banach algebra. Define a map $\mathcal{M} \colon \mathcal{U} \to \mathcal{U}$ by $\mathcal{M}u = \mathcal{M}(u_1, u_2) = (\sinh^{-1} \mu u_1, \mu \tan^{-1} u_2)$. Now, for $u, v \in \mathcal{U}$, we have

$$\begin{aligned} d(u, \mathcal{M}u) &= \left(\left|u_1 - \sinh^{-1} \mu u_1\right|, \left|u_2 - \mu \tan^{-1} u_2\right|\right) \\ &\preceq (u_1, u_2), \end{aligned}$$

$$\begin{aligned}d(u,\mathcal{M}v) &= \left(\left|u_1 - \sinh^{-1}\mu v_1\right|, \left|u_2 - \mu\tan^{-1} v_2\right|\right)\\ &\preceq \left(\left|u_1 - v_1\right|, \left|u_2 - v_2\right|\right),\\ d(\mathcal{M}u,\mathcal{M}v) &= \left(\left|\sinh^{-1}\mu u_1 - \sinh^{-1}\mu v_1\right|, \left|\mu\tan^{-1} u_2 - \mu\tan^{-1} v_2\right|\right).\end{aligned}$$

By the Mean Value Theorem, there exist: $\xi$ between $u_1$ and $v_1$, and $\gamma$ between $u_2$ and $v_2$ so that

$$\begin{aligned}d(\mathcal{M}u,\mathcal{M}v) &\preceq \Big(\frac{\mu}{\sqrt{(1+\xi^2)}}\left|u_1 - v_1\right|, \frac{\mu}{1+\gamma^2}\left|u_2 - v_2\right|\Big)\\ &\preceq \left(\mu\left|u_1 - v_1\right|, \mu\left|u_2 - v_2\right|\right).\end{aligned}$$

So,

$$\begin{aligned}\min\{d(\mathcal{M}u,\mathcal{M}v), d(u,\mathcal{M}u), d(v,\mathcal{M}v)\} &= d(\mathcal{M}u,\mathcal{M}v)\\ &\preceq \left(\mu\left|u_1 - v_1\right|, \mu\left|u_2 - v_2\right|\right),\\ \min\{d(u,\mathcal{M}u), d(v,\mathcal{M}u)\} &= d(u,\mathcal{M}u) \preceq \left(\left|u_1 - v_1\right|, \left|u_1 - v_1\right|\right),\end{aligned}$$

and $\max\{d(u,v), \min\{d(u,\mathcal{M}u), d(v,\mathcal{M}u)\}\} = d(u,v)$. Now we conclude, for $\eta = \frac{-1}{7}$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$, all the assumptions of Theorem 3.1 are satisfied. Hence, the iterated sequence $\{\mathcal{M}^n u\}$ converges to a fixed point of $\mathcal{M}$, for each $u \in \mathcal{U}$. $\square$

Next, we apply our main result to a non-linear integral equation. Let $I = [0,1]$ and $\mathcal{U} = C[I,\mathbb{R}]$ denotes the set of continuous functions on $[0,1]$. Define $d\colon \mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ by $d(u,v) = \sup_{t\in[0,1]}\|u(t)-v(t)\|$. Clearly, $(\mathcal{U}, d)$ is a complete cone metric space.

THEOREM 4.1. *Consider the following homogeneous integral equation*

$$u(t) = \int_0^t K(s,t)\gamma(s,u(s))ds. \tag{4.2}$$

*Suppose that the following hold:*

(i) $K\colon [0,1]\times[0,1] \to \mathbb{R}$ *is integrable with respect to* $s$ *on* $I$,
(ii) $\gamma\colon \mathcal{U}\times[0,1] \to \mathbb{R}$ *is an orbitally continuous function,*
(iii) $\gamma(t,u(t)) \preceq \sup_{t\in[0,1]} u(t)$,
(iv) $\int_0^1 K(s,t)dt \preceq 1$.

*Then, integral equation* (4.2) *has a solution in* $\mathcal{U}$.

Proof. Consider a map $\mathcal{M}\colon \mathcal{U} \to \mathcal{U}$ given by

$$\mathcal{M}u(t) = \int_0^t K(s,t)\gamma(s,u(s))ds. \tag{4.3}$$

Then, $u$ is a solution of (4.2) if and only if $u$ is a fixed point of $\mathcal{M}$. Now, for $u, v \in \mathcal{U}$, we have

$$\begin{aligned}
d(u,\mathcal{M}u) &= \sup_{t\in[0,1]} \left\| u(t) - \int_0^t K(s,t)\gamma(s,u(s))ds \right\| \\
&\preceq \sup_{t\in[0,1]} \left\| u(t) - \int_0^t \gamma(s,u(s))ds \right\| \\
&\preceq \sup_{t\in[0,1]} \|u(t) - u(t)\| = 0, \\
d(v,\mathcal{M}u) &= \sup_{t\in[0,1]} \left\| v(t) - \int_0^t K(s,t)\gamma(s,u(s))ds \right\| \\
&\preceq \sup_{t\in[0,1]} \left\| v(t) - \int_0^t \gamma(s,u(s))ds \right\| \\
&\preceq \sup_{t\in[0,1]} \|v(t) - u(t)\| = d(u,v), \\
d(v,\mathcal{M}v) &= 0, \\
d(\mathcal{M}u,\mathcal{M}v) &= \sup_{t\in[0,1]} \left\| \int_0^t K(s,t)\gamma(s,u(s))ds - \int_0^t K(s,t)\gamma(s,v(s))ds \right\| \\
&\preceq \sup_{t\in[0,1]} \left\| \int_0^t K(s,t)(\gamma(s,u(s)) - \gamma(s,v(s)))ds \right\| \\
&\preceq \sup_{t\in[0,1]} \|u(t) - v(t)\| = d(u,v).
\end{aligned}$$

So, $\min\{d(\mathcal{M}u,\mathcal{M}v), d(u,\mathcal{M}u), d(v,\mathcal{M}v)\} = 0$, $\min\{d(u,\mathcal{M}u), d(v,\mathcal{M}u)\} = \sup_{t\in[0,1]} \|u(t) - v(t)\|$, and

$$\max\{d(u,v), \min\{d(u,\mathcal{M}u), d(v,\mathcal{M}u)\}\} = \sup_{t\in[0,1]} \|u(t) - v(t)\|.$$

Now, we conclude for $\eta = \frac{1}{6}, \alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$, all the assumptions of Theorem 3.1 are satisfied. Hence, the iterated sequence $\{\mathcal{M}^n u\}$ converges to a fixed point of $\mathcal{M}$, for each $u \in \mathcal{U}$. $\square$

Further, as an application of main result, we show that a non-linear two point boundary value problem has a solution. Let $\mathcal{U} = \mathcal{C}[0,1]$ be a set of real continuous functions on $[0,1]$ and $\mathcal{E} = \mathbb{R}^2$, then cone metric $d\colon \mathcal{U} \times \mathcal{U} \to \mathcal{E}$ is defined by $d(u,v) = (\sup_{t\in[0,1]} \|u(t) - v(t)\|, \sup_{t\in[0,1]} \|u(t) - v(t)\|)$. Clearly, $(\mathcal{C}[0,1], d)$ is a complete cone metric space over a Banach algebra.

THEOREM 4.2. *Consider a boundary value problem*

$$\frac{d^2u}{dt^2} = \zeta(t, u(t)), \quad t \in [0,1], \quad u(0) = 0, \quad u(1) = 0, \tag{4.4}$$

*where* $\zeta\colon [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ *is a continuous function. If there exists* $\varpi \in [1, \infty)$ *such that*

$$\sup_{t\in[0,1]} \|\zeta(t, u(t)) - \zeta(t, v(t))\| \preceq \varpi \sup_{t\in[0,1]} \|u(t) - v(t)\|,$$

*then boundary value problem* (4.4) *has a solution in* $\mathcal{U}$.

PROOF. The boundary value problem (4.4) is equivalent to

$$u(t) = \int_0^1 G(t,\xi)\zeta(\xi, u(\xi))d\xi, t \in [0,1], \tag{4.5}$$

where, $G$ is the Green function given by

$$G(t,\xi) = \begin{cases} (1-t)\xi, & 0 \leqslant \xi \leqslant t \leqslant 1, \\ (1-\xi)t, & 0 \leqslant t \leqslant \xi \leqslant 1. \end{cases}$$

Now, $u \in \mathcal{U}$ is a solution of equation (4.5) if and only if it is the solution of a boundary value problem (4.4). Define a map $\mathcal{M}\colon \mathcal{U} \to \mathcal{U}$ by

$$\mathcal{M}u(t) = \int_0^1 G(t,\xi)\zeta(\xi, u(\xi))d\xi.$$

If we assume that $\sup_{t\in[0,1]} |u(t)| \geq \sup_{t\in[0,1]} |v(t)|$, then

$$\begin{aligned} d(\mathcal{M}u, \mathcal{M}v) &= (\sup_{t\in[0,1]} \|\mathcal{M}u(t) - \mathcal{M}v(t)\|, \sup_{t\in[0,1]} \|\mathcal{M}u(t) - \mathcal{M}v(t)\|) \\ &= \Big( \sup_{t\in[0,1]} \Big\| \int_0^1 G(t,\xi)\zeta(\xi, u(\xi))d\xi - \int_0^1 G(t,\xi)\zeta(\xi, v(\xi))d\xi \Big\|, \\ &\qquad \sup_{t\in[0,1]} \Big\| \int_0^1 G(t,\xi)\zeta(\xi, u(\xi))d\xi - \int_0^1 G(t,\xi)\zeta(\xi, v(\xi))d\xi \Big\| \Big) \end{aligned}$$

$$
\begin{aligned}
&= \Big( \sup_{t\in[0,1]} \Big\| \int_0^1 G(t,\xi)(\zeta(\xi,u(\xi)) - \zeta(\xi,v(\xi)))d\xi \Big\|, \\
&\quad \sup_{t\in[0,1]} \Big\| \int_0^1 G(t,\xi)(\zeta(\xi,u(\xi)) - \zeta(\xi,v(\xi)))d\xi \Big\| \Big) \\
&\preceq \Big( \sup_{t\in[0,1]} \int_0^1 |G(t,\xi)(\zeta(\xi,u(\xi)) - \zeta(\xi,v(\xi)))|\, d\xi, \\
&\quad \sup_{t\in[0,1]} \int_0^1 \|G(t,\xi)(\zeta(\xi,u(\xi)) - \zeta(\xi,v(\xi)))\| d\xi \Big) \\
&\preceq \Big( \sup_{t\in[0,1]} \|\zeta(t,u(t)) - \zeta(t,v(t))\| \int_0^1 G(t,\xi)d\xi, \\
&\quad \sup_{t\in[0,1]} \|\zeta(t,u(t)) - \zeta(t,v(t))\| \int_0^1 G(t,\xi)d\xi \Big) \\
&\preceq \Big( \varpi \sup_{t\in[0,1]} \|u(t) - v(t)\| \int_0^1 G(t,\xi)d\xi, \varpi \sup_{t\in[0,1]} \|u(t) - v(t)\| \int_0^1 G(t,\xi)d\xi \Big) \\
&\preceq \varpi \frac{1}{8} \Big( \sup_{t\in[0,1]} \|u(t) - v(t)\|, \sup_{t\in[0,1]} \|u(t) - v(t)\| \Big),
\end{aligned}
$$

that is, $d(\mathcal{M}u, \mathcal{M}v) \preceq \frac{1}{8}\varpi d(u,v)$ (since $\int_0^1 G(t,\xi)d\xi = \int_0^t (1-t)\xi d\xi + \int_t^1 (1-\xi)t d\xi = \frac{1}{2}t(1-t) \leq \frac{1}{8}$),

$$
\begin{aligned}
d(u,\mathcal{M}u) &= \Big( \sup \Big\| u - \int_0^1 G(t,\xi)\zeta(\xi,u(\xi))d\xi \Big\|, \sup \Big\| u - \int_0^1 G(t,\xi)\zeta(\xi,u(\xi))d\xi \Big\| \Big) \\
&\preceq \Big( u + \frac{1}{8}\varpi u, u + \frac{1}{8}\varpi u \Big), \\
d(v,\mathcal{M}v) &\preceq \Big( v + \frac{1}{8}\varpi v, v + \frac{1}{8}\varpi v \Big), \\
d(u,\mathcal{M}v) &= \Big( \sup \Big\| u - \int_0^1 G(t,\xi)\zeta(\xi,v(\xi))d\xi \Big\|, \sup \Big\| u - \int_0^1 G(t,\xi)\zeta(\xi,v(\xi))d\xi \Big\| \Big) \\
&\preceq \Big( u + \frac{1}{8}\varpi v, u + \frac{1}{8}\varpi v \Big).
\end{aligned}
$$

Now, $\min\{d(u,\mathcal{M}u), d(v,\mathcal{M}v), d(\mathcal{M}u,\mathcal{M}v)\} = d(\mathcal{M}u,\mathcal{M}v)$, $\min\{d(u,\mathcal{M}v), d(v,\mathcal{M}u)\} = d(v,\mathcal{M}u)$, and $\max\{d(u,v), \min\{d(u,\mathcal{M}u), d(v,\mathcal{M}u)\}\} = d(u,\mathcal{M}u)$. We conclude, for $\eta = \frac{1}{7}, \alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$, all the assumptions of Theorem 3.1 are satisfied. Hence, the iterated sequence $\{\mathcal{M}^n u\}$ converges to a fixed point of $\mathcal{M}$, for each $u \in \mathcal{U}$, which is a solution to the problem (4.4). $\square$

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