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Report of Meeting

The Twenty-first Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities Brenna (Poland), February 2–5, 2022

The Twenty-first Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held in Hotel Kotarz in Brenna, Poland, from February 2 to February 5, 2022. The meeting was organized by the Institute of Mathematics of the University of Silesia.

11 participants came from the University of Debrecen (Hungary), 7 from the University of Silesia in Katowice (Poland), 2 from the Pedagogical University of Krakow (Poland), 1 from Budapest University of Technology and Economics (Hungary), 1 from the University of Rzeszów (Poland) and 1 with a dual affiliation University of Silesia (Poland) and Chernivtsi National University (Ukraine).

Professor Maciej Sablik opened the Seminar and welcomed the participants to Brenna.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iteration theory, equations on abstract algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity. Interesting discussions were generated by the talks.

There was also a Problem Session and a festive dinner.

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The closing address was given by Professor Mihály Bessenyei. His invitation to the Twenty-second Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in 2023 in Hungary was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in section 1, problems in section 2, and the list of participants in the final section.

1. Abstracts of talks

Mihály Bessenyei: *Convex separation in lack of convex combinations* (Joint work with Rezső L. Lovas and Dávid Cs. Kertész)

In the talk, we give a sufficient condition for pairs of functions to have a convex separator when the underlying structure is a reduced Birkhoff system. These kinds of systems include Cartan–Hadamard manifolds, as well. Surprisingly, in such manifolds our result reduces practically to the classical Euclidean case studied by Baron, Matkowski, and Nikodem.

Zoltán Boros: *Regularity conditions for additive functions involving particular curves*

Motivated by the algebraic conditions concerning alternative equations for additive functions (in particular, Theorem 6 in [2]) and a related regularity condition involving the unit circle ([1]), the following statement is established.

Theorem. *Let $\emptyset \neq I \subset \mathbb{R}$ be an open interval and $u, v\colon I \to \mathbb{R}$ be continuous functions such that u is non-constant on any non-void open subinterval of I. Moreover, let us assume that, for every positive integer k, there exist non-zero rationals a_k, b_k, c_k and d_k and a mapping $\gamma_k\colon I \to I$ such that*

$$\begin{pmatrix} u(\gamma_k(t)) \\ v(\gamma_k(t)) \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

and $\lim\limits_{k\to\infty} \gamma_k(t) = t$ locally uniformly in $t \in I$. If $f\colon \mathbb{R} \to \mathbb{R}$ is additive and the mapping

$$I \ni t \mapsto \Phi(t) := f(u(t))f(v(t))$$

is locally bounded, then f is continuous, i.e., $f(x) = cx$ for every $x \in \mathbb{R}$ with some fixed $c \in \mathbb{R}$.

Particular cases involve some elliptic or hyperbolic arcs.

References

[1] Z. Boros, W. Fechner, and P. Kutas, *A regularity condition for quadratic functions involving the unit circle*, Publ. Math. Debrecen **89** (2016), no. 3, 297–306.
[2] Z. Kominek, L. Reich, and J. Schwaiger, *On additive functions fulfilling some additional condition*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **207** (1998), 35–42.

Pál Burai: *What is the role of bisymmetry?* (Joint work with Gergely Kiss and Patrícia Szokol)

I formulate some results and open problems connected to bisymmetry.

Jacek Chmieliński: *On ε-Birkhoff–James orthogonality*

In a normed linear space X over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we consider an approximate Birkhoff–James orthogonality as defined in [1]:

$$(1) \qquad x \perp_{\mathrm{B}}^{\varepsilon} y \quad \Longleftrightarrow \quad \forall \lambda \in \mathbb{K}: \ \|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon \|x\| \, \|\lambda y\|,$$

where $x, y \in X$ and $\varepsilon \in [0, 1)$.

Some characterizations of the above definition were given later in [2] for real spaces, and recently extended to the complex case in [3].

The main purpose of the talk is to give yet one more condition equivalent to (1).

References

[1] J. Chmieliński, *On an ε-Birkhoff orthogonality*, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005), no. 3, Art. 79, 7 pp.
[2] J. Chmieliński, T. Stypuła, and P. Wójcik, *Approximate orthogonality in normed spaces and its applications*, Linear Algebra Appl. **531** (2017), 305–317.
[3] P. Wójcik, *Approximate orthogonality in normed spaces and its applications II*, Linear Algebra Appl. **632** (2022), 258–267.

Jacek Chudziak: *On solutions of the Gołąb–Schinzel type functional equation*

Assume that $(S, \circ)$ is a commutative semigroup. We determine the solutions of the functional equation

$$(1) \qquad f(x + g(x)y) = f(x) \circ f(y) \quad \text{for } x, y \in \mathbb{R},$$

in the class of pairs of functions (f, g) such that $f\colon \mathbb{R} \to S$ is not periodic and $g\colon \mathbb{R} \to \mathbb{R}$. Applying our result we give an answer to the question raised in [1].

Reference

[1] E. Jabłońska, *On locally bounded above solutions of an equation of the Gołąb-Schinzel type*, Aequationes Math. **87** (2014), no. 1–2, 125–133.

Borbála Fazekas: *Solutions of the biharmonic equation using enclosure methods*

Let $\Omega \subset \mathbb{R}^2$ be a domain. Let us consider the equation

$$\begin{aligned} \Delta^2 u &= F(u), \qquad \text{on } \Omega,\\ u &= 0, \qquad \text{on } \partial\Omega,\\ \frac{\partial u}{\partial \nu} &= 0, \qquad \text{on } \partial\Omega. \end{aligned}$$

We prove existence of solutions in case $F(u) = u^2 + \lambda$ and $F(u) = \lambda e^u$ on the unit square, on the unit disc and on a dumbbell-like domain, respectively, with the help of the enclosure method of Plum. We use C^1 finite elements for our calculations.

Reference

[1] M. Plum, *Existence and multiplicity proofs for semilinear elliptic boundary value problems by computer assistance*, Jahresber. Deutsch. Math.-Verein. **110** (2008), no. 1, 19–54.

Roman Ger: *Wright-convex functions that are measurable on segments*

Given a positive integer n and a nonempty open and convex subdomain D of a real Banach space $(X, \|\cdot\|)$ endowed with a cone C of positive elements with $C \cap (-C) = \{0\}$, we shall say that a map $F\colon D \longrightarrow \mathbb{R}$ is n-Wright-convex whenever for all elements $x \in D$ and $h_1, :, h_{n+1} \in C$ the inequality

$$\Delta_{h_1} \cdots \Delta_{h_{n+1}} F(x) \geq 0$$

holds true provided that $x + \varepsilon_1 h_1 + \cdots + \varepsilon_{n+1} h_{n+1}$ falls into D for every choice of scalars $\varepsilon_1, \ldots, \varepsilon_{n+1} \in \{0, 1\}$.

We show, among others, that a map $F\colon D \longrightarrow \mathbb{R}$ is n-Wright-convex if and only for every choice of scalars $\varepsilon_{1,j}, \ldots, \varepsilon_{n+1,j}$ from $\{0, 1\}$, $j \in \{1, \ldots, 2^{n+1}\}$, and for all nonnegative real numbers $\lambda_1, \ldots, \lambda_{n+1}$ summing up to 1 one has

$$y - x \in C \Longrightarrow \sum_{j=1}^{2^{n+1}} (-1)^{n+1-(\varepsilon_{1,j}+\cdots+\varepsilon_{n+1,j})} F\left((1-\mu_j)x + \mu_j y\right) \geq 0,$$

where

$$\mu_j := \sum_{i=1}^{n+1} \varepsilon_{i,j} \lambda_i, \quad j \in \{1, \ldots, 2^{n+1}\}.$$

Moreover, the following Mazur's type problem is discussed. Assume that a map $F\colon D \longrightarrow \mathbb{R}$ is n-Wright-convex and such that for all vectors x, y from D the function

$$[0,1] \ni t \longmapsto F((1-t)x + ty) \in \mathbb{R}$$

is Lebesgue measurable. Does it force F to be continuous?

Eszter Gselmann: *Polynomial equations for additive functions* (Joint work with Gergely Kiss)

The aim of this talk is to investigate classes of functional equations satisfied by additive functions. As a result, new characterizations for homomorphisms and derivations, resp., can be given. More exactly, the following type of functional equations are considered simultaneously

$$\sum_{i=1}^{n} f_i(x^{p_i})g_i(x^{q_i}) = 0 \quad (x \in \mathbb{F}),$$

$$\sum_{i=1}^{n} f_i(x)^{p_i}g_i(x)^{q_i} = 0 \quad (x \in \mathbb{F})$$

and

$$\sum_{i=1}^{n} f_i(x^{p_i})g_i(x)^{q_i} = 0 \quad (x \in \mathbb{F}),$$

where n is a positive integer, $\mathbb{F} \subset \mathbb{C}$ is a field and $f_i, g_i\colon \mathbb{F} \to \mathbb{C}$ are additive functions for all $i = 1, \dots, n$.

As our results show, although these equations seem very similar, their sets of general solutions differ significantly.

Mehak Iqbal: *On the existence of monotone sequences in generalized ordered sets* (Joint work with (work prepared under the supervision of Á. Száz))

We shall prove the following two basic theorems:

Theorem. *Suppose that $X(S)$ is a total, partially ordered set and $(x_n)_{n=1}^{\infty}$ is a sequence in X, then at least one of the following assertions holds:*

(1) $(x_n)_{n=1}^{\infty}$ *has a decreasing subsequence;*

(2) $(x_n)_{n=1}^{\infty}$ *has a strictly increasing subsequence.*

Theorem. *Suppose that* $X(S)$ *is a nonvoid preordered set,*

(1) φ *is a function of* X *to* $\mathbb{R}$;

(2) $\rho(x) = \sup\big((\varphi \circ S)(x)\big)$ *for all* $x \in X$;

(3) $a \in X$ *such that* $\rho(a) < +\infty$;

then there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ *in* $X(S)$, *with* $x_1 = a$, *such that:*

(4) $(\rho(x_n))_{n=1}^{\infty}$ *is decreasing;*

(5) $\lim\limits_{n\to\infty} \varphi(x_n) = \lim\limits_{n\to\infty} \rho(x_n) = \inf\limits_{n\in\mathbb{N}} \rho(x_n)$;

(6) $\varphi(x_k) \le \rho(x_n) \le \rho(a)$ *for all* $n \in \mathbb{N}$ *and* $n \le k \in \mathbb{N}$.

The first theorem can be used to prove metric completeness. While, the second one can be used to prove maximality principles.

References

[1] Z. Boros, M. Iqbal, and Á. Száz, *A relational improvement of a true particular case of Fierro's maximality theorem*, manuscript.

[2] M. Iqbal and Á. Száz, *An instructive treatment of the Brézis–Browder ordering and maximality principles*, manuscript.

[3] Á. Száz, *Altman type generalizations of ordering and maximality principles of Brézis, Browder and Brøndsted*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 4, 595–620.

Radosław Łukasik: *Vector valued invariant means, complementability and almost constrained subspaces*

Definition. Let $(S,+)$ be a left [right] amenable semigroup and X be a Banach space. A linear map $M\colon \ell_\infty(S,X) \to X$ is called a *left* [*right*] X*-valued invariant mean* if

$$\begin{aligned}
&\|M\| \le 1,\\
&M(c\mathbb{1}_S) = c, \quad c \in X,\\
&M({}_af) = M(f), \quad a \in S,\ f \in \ell_\infty(S,X),\\
&[M(f_a) = M(f), \quad a \in S,\ f \in \ell_\infty(S,X)],
\end{aligned}$$

where

$$\begin{aligned}
&{}_af(x) = f(a+x), \quad a,x \in S,\ f \in \ell_\infty(S,X),\\
&[f_a(x) = f(x+a), \quad a,x \in S,\ f \in \ell_\infty(S,X)].
\end{aligned}$$

If M is a left and right invariant mean, then M is called an *X-valued invariant mean.*

If in the above definition the norm of map M is equal to at most $\lambda \geq 1$, then M is called an *X-valued invariant λ-mean.*

Definition. A subspace X of a Banach space Y is said to be an almost constrained (AC) subspace of Y if any family of closed balls centered at points of X that intersects in Y also intersects in X.

Recall that for $\lambda \geq 1$ a subspace X of a Banach space Y is called λ-complemented if there is a projection of norm λ from Y on X.

In this talk, we will study a connection between some generalization of AC-subspaces, vector valued invariant λ-means and λ-complementability.

Oleksandr Maslyuchenko: *On extremal sections of continuous and separately continuous functions* (Joint work with Anastasija Kushnir)

We continue the research of interconnections between pairs of Hahn and separately continuous functions which was started by V. K. Maslyuchenko. A pair (g, h) of functions $f, g\colon X \to \overline{\mathbb{R}}$ on a topological space X is called *a pair of Hahn* if $g \leq h$, g is an upper semicontinuous function and h is a lower semicontinuous function. For a function $f\colon X \times Y \to \overline{\mathbb{R}}$ we consider *the vertical x-section* $f^x : Y \to \overline{\mathbb{R}}$ and *the horizontal y-section* $f_y : X \to \overline{\mathbb{R}}$ which is defined by $f^x(y) = f_y(x) = f(x, y)$ for any $x \in X$ and $y \in Y$. Recall that a function f is *separately continuous* if the sections f^x and f_y are continuous for any $x \in X$ and $y \in Y$. We define *the minimal and the maximal sections* $\wedge_f, \vee_f\colon X \to \overline{\mathbb{R}}$ of a function f by the formulae

$$\wedge_f(x) = \inf_{y \in Y} f(x, y) \text{ and } \vee_f(x) = \sup_{y \in Y} f(x, y) \text{ for any } x \in X.$$

It is not hard to show (see [2]) that for any separately continuous functions f the pair $(\wedge_f, \vee_f)$ is a pair of Hahn. We say that *a pair of Hahn (g, h) on X is generated by a function $f\colon X \times Y \to \overline{\mathbb{R}}$* if $\wedge_f = g$ and $\vee_f = h$. Moreover, in [2] it was proved that an arbitrary pair of Hahn (g, h) on $[a; b]$ is generated by a separately continuous function $f\colon [a; b] \times [c; d] \to \overline{\mathbb{R}}$ for any reals $a < b$ and $c < d$.

In [1], we generalize the previous result. In particular, we prove that for any perfectly normal topological space X and for any topological space Y having non-scattered compactification any pair of Hahn on X is generated by a separately continuous function on $X \times Y$. We also obtain that for any perfectly normal space X and non-pseudocompact space Y every pair of Hahn on X is generated by a continuous function on $X \times Y$.

In the talk, we will also discuss some new authors' results and open questions in this direction.

References

[1] A.S. Kushnir and O.V. Maslyuchenko, *Pairs of Hahn and separately continuous functions with the given extremal sections*, Bukovinian Math. Journal **9** (2021), no. 1, 210–229.
[2] G.A. Voloshin, V.K. Maslyuchenko, and V.S. Mel'nik, *Hahn's pairs and zero inverse problem*, Mat. Stud. **48** (2017), no. 1, 74–81.

Rayene Menzer: *An alternative equation for generalized monomials* (Joint work with Zoltán Boros)

In this presentation, we consider a generalized monomial or polynomial $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies the additional equation $f(x)f(y) = 0$ for the pairs $(x, y) \in D$, where $D \subset \mathbb{R}^2$ is given by some algebraic condition. In the particular cases when f is a generalized polynomial and there exist non-constant regular polynomials p and q that fulfill

$$D = \{\, (p(t), q(t)) \,|\, t \in \mathbb{R} \,\}$$

or f is a generalized monomial and there exists a positive rational m fulfilling

$$D = \{\, (x, y) \in \mathbb{R}^2 \,|\, x^2 - my^2 = 1 \,\},$$

we prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Our research is motivated by such results for additive functions ([2]) (for various particular choices of the set D), an analogous result for generalized polynomials when D is the unit circle ([1]) as well as by a counterexample (a non-zero solution) when f is additive and D is the hyperbola given by the equation $xy = 1$ ([3]).

References

[1] Z. Boros and W. Fechner, *An alternative equation for polynomial functions*, Aequationes Math. **89** (2015), no. 1, 17–22.
[2] Z. Kominek, L. Reich, and J. Schwaiger, *On additive functions fulfilling some additional condition*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **207** (1998), 35–42.
[3] P. Kutas, *Algebraic conditions for additive functions over the reals and over finite fields*, Aequationes Math. **92** (2018), no. 3, 563–575.

Gábor Marcell Molnár: *On convex and concave sequences and their applications* (Joint work with Zsolt Páles)

We extend the well-known definitions of convex, concave, affine squences and introduce the notions of q-convex, q-concave, and q-affine sequences with respect to a positive number q. A sequence $p = (p_n, \dots, p_m) \in \mathcal{S}(n,m)$ is called *q-convex* if, for all $i \in \{n+1, \dots, m-1\}$,

$$2qp_i \le p_{i-1} + p_{i+1}.$$

If, for all $i \in \{n+1, \dots, m-1\}$, the reversed inequality holds in the above inequality, then the sequence is termed *q-concave*. If a sequence is simultaneously q-convex and q-concave, then it is said to be *q-affine*.

In the talk, we will be talking about the basic properties of q-convex, q-concave, and q-affine sequences. The main result of the topic shows that q-concave sequences are the pointwise minima of q-affines sequences. We can apply the results to a nonlinear selfmap of the n-dimensional space and prove that it has a unique fixed point using the Banach Fixed Point theorem.

References

[1] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934, (first edition), 1952 (second edition).
[2] X.Z. Krasniqi, *On α-convex sequences of higher order*, J. Numer. Anal. Approx. Theory **45** (2016), no. 2, 177–182.

Andrzej Olbryś: *Remarks on generalized convexity*

We continue our research started in [2]. We examine functions $f\colon D \to \mathbb{R}$ which satisfy the functional inequality of the form

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \omega(x,y,t), \quad x, y \in D,\ t \in [0,1],$$

where $\omega\colon D \times D \times [0,1] \to \mathbb{R}$ is a given map and D stands for a convex subset of a real linear space.

References

[1] P. Cannarsa and C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations and Optimal Control*, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, Boston, 2004.
[2] A. Olbryś, *A support theorem for generalized convexity and its applications*, J. Math. Anal. Appl. **458** (2018), no. 2, 1044–1058.

Paweł Pasteczka: *Decision making via generalized Bajraktarević means* (Joint work with Zsolt Páles)

We define decision-making functions which arise from studying the multidimensional generalization of the weighted Bajraktarević means. It allows a nonlinear approach to optimization problems.

These functions admit several interesting (from the point of view of decision-making) properties, for example, delegativity (which states that each subgroup of decision-makers can aggregate their decisions and efforts), casuativity (each decision affects the final outcome except two trivial cases) and convexity-type properties.

Beyond establishing the most important properties of such means, we solve their equality problem, we introduce a notion of synergy and characterize the null-synergy decision-making functions of this type.

Reference

[1] Z. Páles, P. Pasteczka, *Decision making via generalized Bajraktarević means*. Available at arXiv:2007.04870.

Evelin Pénzes: *Support theorems for generalized monotone functions* (Joint work with Mihály Bessenyei)

We present a complete solution of the support problem for functions that are generalized monotone in the sense of Beckenbach. The key tool of the proof is Tornheim's uniform convergence theorem. As applications, we improve some known support results and give an abstract version of the Hermite–Hadamard inequality.

Maciej Sablik: *A note on a result by J. Sándor* (Joint work with Timothy Nadhomi)

In the present note, we are concerned with a result proved in [1]. We observe that one of the inequalities proposed by J. Sándor does not hold.

We are also presenting some results on adapting integral inequalities to the so called Sugeno integral.

Reference

[1] J. Sándor, *On upper Hermite-Hadamard inequalities for geometric-convex and log-convex functions*, Notes Number Theory Discrete Math. **20** (2014), no. 5, 25–30.

Justyna Sikorska: *A generalized multi-linear functional equation* (Joint work with Anna Bahyrycz)

General linear functional equations have been studied for years. During the talk, we shall discuss their counterpart for multivariable functions.

Let X, Y be linear spaces over a field $\mathbb{K}$ and $f\colon X^n \to Y$. For some fixed $a_{ji} \in \mathbb{K} \setminus \{0\}$, $C_{i_1\dots i_n} \in \mathbb{K}$ ($j \in \{1,\dots,n\}$, $i, i_1, \dots, i_n \in \{1,2\}$) we consider the following equation

$$(*) \quad f(a_{11}x_{11} + a_{12}x_{12}, \dots, a_{n1}x_{n1} + a_{n2}x_{n2}) = \sum_{1\le i_1,\dots,i_n \le 2} C_{i_1\dots i_n} f(x_{1i_1}, \dots, x_{ni_n}),$$

for all $x_{ji_j} \in X$, $j \in \{1,\dots,n\}$, $i_j \in \{1,2\}$.

We determine the general solution and present some stability results for $(*)$.

László Székelyhidi: *Invariant subspaces over the Heisenberg group*

In this talk, we study finite dimensional translation invariant linear spaces of continuous functions over the Heisenberg group. The description depends on some matrix functional equations.

Patricia Szokol: *Majorization and convexity generated by circulant, doubly stochastic matrices* (Joint work with Pál Burai)

In this talk, the set of all circulant, doubly stochastic matrices is examined. We present some properties and structural results of such matrices. Moreover, we introduce a convexity notion (cyclically convex functions) for multivariable real-valued functions generated by circulant, doubly stochastic matrices. Finally, we prove a characterization theorem of the so-obtained cyclically convex functions.

Tomasz Szostok: *A generalization of a theorem of Brass and Schmeisser*

Let n be an odd positive integer. It was proved by Brass and Schmeisser in [1] that for every quadrature

$$\mathcal{Q} = \alpha_1 f(x_1) + \dots + \alpha_m f(x_m),$$

(with positive weights) of order at least $n+1$ and for every $n-$convex function f, the value of Q on f lies between the values of Gauss and Lobatto quadratures of order $n+1$ calculated for the same function. We generalize this result in two directions, replacing Q by an integral with respect to a given measure and allowing the number n to any positive integer (for even n Radau quadratures replace Gauss and Lobatto ones).

Reference

[1] H. Brass and G. Schmeisser, *Error estimates for interpolatory quadrature formulae*, Numer. Math. **37** (1981), no. 3, 371–386.

Péter Tóth: *Characterization of particular quasisums in terms of functional equations*

In our presentation, we introduce the concept of translation invariant functions: considering an arbitrary set $\emptyset \neq S \subset \mathbb{R}^n$, the function $F\colon S \longrightarrow \mathbb{R}$ is translation invariant if $F(x) = F(y)$ implies $F(x+t) = F(y+t)$ for any vectors $x, y, t \in \mathbb{R}^n$ such that $x, y, x+t, y+t \in S$.

In our main results, we shall consider an open, connected set $\emptyset \neq D \subset \mathbb{R}^n$. We show that if $F\colon D \longrightarrow \mathbb{R}$ is a translation invariant, continuous function, then there exists a vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and a strictly monotone, continuous function f such that

$$F(x_1, \dots, x_n) = f(a_1 x_1 + \dots + a_n x_n)$$

holds for all $(x_1, \dots, x_n) \in D$.

Using this result we also obtain that continuous solutions $F\colon D \longrightarrow \mathbb{R}$ of the system

$$F(x_1, \dots, x_j + t_j, \dots, x_n) = \Psi_j(F(x_1, \dots, x_j, \dots, x_n), t_j) \quad (j = 1, \dots, n)$$

of functional equations can be represented as the composition of a strictly monotone, continuous function and a linear functional as well.

Finally, we will apply this theorem in order to describe the continuous solutions of a generalized version of this system of functional equations, namely, when the transformation in the variables of F is not necessarily the addition but any continuous group operation. Using these results we give the characterization of particular quasisums — such as some well-known utility functions — in terms of functional equations.

Thomas Zürcher: *On invariant measures for iterated function systems comprised of homeomorphisms* (Joint work with Janusz Morawiec)

Given a probabilistic iterated function system $((f_1, \dots, f_N), (p_1, \dots, p_N))$ of homeomorphisms $f_k\colon [0,1] \to [0,1]$ and probabilities p_k, we are interested in the existence and the uniqueness of invariant probability measures. These are probability measures μ such that

$$\mu(B) = \sum_{n=1}^{N} p_n \mu(f_n^{-1}(B))$$

for all Borel sets $B \subset [0,1]$.

Reference

[1] J. Morawiec and T. Zürcher, *A new take on random interval homeomorphisms*, Fund. Math. **257** (2022), no. 1, 1–17.

2. Problems and remarks

Open Problem. Characterize quasi-arithmetic expressions. More precisely: Is that true that a two-place function is a quasi-arithmetic expression if and only if it is reflexive, symmetric, bisymmetric and cancellative?

Open Problem. Homogeneity problem and translativity problem in the class of quasi-arithmetic expressions.

Hint: find all solutions (φ, b) of the following functional equation!

$$a_t(\varphi(x)) + b(t) = a_x(\varphi(t)) + b(x), \quad t, x \in \mathbb{R},$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an invertible function with Jensen-convex image, $a_t, a_x\colon \mathbb{R} \to \mathbb{R}$ are injective, additive functions for every $t, x \in \mathbb{R}$, and $b\colon \mathbb{R} \to \mathbb{R}$ is a function.

Pál Burai

3. List of Participants

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