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GENERALIZED FRACTIONAL INEQUALITIES OF THE HERMITE–HADAMARD TYPE FOR CONVEX STOCHASTIC PROCESSES

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Abstract. A generalization of the Hermite–Hadamard (HH) inequality for a positive convex stochastic process, by means of a newly proposed fractional integral operator, is hereby established. Results involving the Riemann–Liouville, Hadamard, Erdélyi–Kober, Katugampola, Weyl and Liouville fractional integrals are deduced as particular cases of our main result. In addition, we also apply some known HH results to obtain some estimates for the expectations of integrals of convex and p-convex stochastic processes. As a side note, we also pointed out a mistake in the main result of the paper [*Hermite–Hadamard type inequalities, convex stochastic processes and Katugampola fractional integral*, Revista Integración, temas de matemáticas **36** (2018), no. 2, 133–149]. We anticipate that the idea employed herein will inspire further research in this direction.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\lambda \in [0, 1]$, the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

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For this class of functions, the following theorem is known:

THEOREM 1.1 (Hermite–Hadamard Inequality). *Let $f\colon I \to \mathbb{R}$ be a convex function, and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Inequalities akin to the above double inequality have been established for different classes of functions. In this article, we shall discuss (1.1) within the frame work of the convex stochastic processes. Now, let $(\Omega, \mathcal{F}, P)$ be a probability space. In 1980, Nikodem [5] introduced the notion of convex stochastic processes and proposed the following definition: a stochastic process $X\colon I \times \Omega \to \mathbb{R}$ is said to be convex if

$$X(\lambda a + (1-\lambda)b, \cdot) \leq \lambda X(a, \cdot) + (1-\lambda)X(b, \cdot)$$

holds almost everywhere for all $a, b \in I$ and $\lambda \in [0,1]$. If we put $\lambda = \frac{1}{2}$ in the above inequality, then the process X is Jensen-convex or $\frac{1}{2}$-convex. A stochastic process X is termed concave if $-X$ is convex. For a stochastic process, we have the following theorem:

THEOREM 1.2 ([3]). *Let $X\colon I \times \Omega \to \mathbb{R}$ be a convex and mean square continuous process in the interval $I \times \Omega$. Then*

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{b-a}\int_a^b X(t, \cdot)dt \leq \frac{X(a, \cdot)+X(b, \cdot)}{2} \tag{1.2}$$

holds almost everywhere.

Recently, this concept was extended in the following definitions:

DEFINITION 1.3 ([6]). A stochastic process $X\colon I \subset (0,\infty) \times \Omega \to \mathbb{R}$ is called *p-convex* if the inequality

$$X([\lambda a^p + (1-\lambda)b^p]^{\frac{1}{p}}, \cdot) \leq \lambda X(a, \cdot) + (1-\lambda)X(b, \cdot)$$

holds almost everywhere for all $a, b \in I \subset (0,\infty)$, $p \in \mathbb{R}$ and $\lambda \in [0,1]$.

Definition 1.4 ([7]). A stochastic process $X\colon I\subset(0,\infty)\times\Omega\to\mathbb{R}$ is said to be *exponentially p-convex* if the inequality

$$X([\lambda a^p+(1-\lambda)b^p]^{\frac{1}{p}},\cdot)\le\lambda\frac{X(a,\cdot)}{e^{\alpha a}}+(1-\lambda)\frac{X(b,\cdot)}{e^{\alpha b}}$$

holds almost everywhere for all $a,b\in I\subset(0,\infty)$, $p\in\mathbb{R}\setminus\{0\}$ and $\lambda\in[0,1]$.

For these classes of functions, the following theorems have been established:

Theorem 1.5 ([6]). *Let $X\colon I\subset(0,\infty)\times\Omega\to\mathbb{R}$ be a p-convex stochastic process and mean-square integrable on $[a,b]$ where $a,b\in I$ and $a<b$. Then*

$$X\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\cdot\right)\le\frac{p}{b^p-a^p}\int_a^b\frac{X(t,\cdot)}{t^{1-p}}dt\le\frac{X(a,\cdot)+X(b,\cdot)}{2}. \tag{1.3}$$

Theorem 1.6 ([7]). *Let $X\colon I\subset(0,\infty)\times\Omega\to\mathbb{R}$ be an exponentially p-convex stochastic process. Let $a,b\in I$ with $a<b$. If X is mean-square integrable on $[a,b]$, then for $p\in\mathbb{R}\setminus\{0\}$ and $\alpha\in\mathbb{R}$, we have almost everywhere*

$$\begin{aligned}X\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\cdot\right)&\le\frac{p}{b^p-a^p}\int_a^b\frac{X(s,\cdot)}{s^{1-p}e^{\alpha s}}ds\\&\le A_1(\alpha)\frac{X(a,\cdot)}{e^{\alpha a}}+A_2(\alpha)\frac{X(b,\cdot)}{e^{\alpha b}},\end{aligned} \tag{1.4}$$

where

$$A_1(\alpha)=\int_0^1\frac{\lambda d\lambda}{e^{\alpha\left(\lambda a^p+(1-\lambda)b^p\right)}},\quad A_2(\alpha)=\int_0^1\frac{(1-\lambda)d\lambda}{e^{\alpha\left(\lambda a^p+(1-\lambda)b^p\right)}}.$$

The case of $\alpha=0$ gives (1.3).

Aside extensions by means of convexity, analogues of inequality (1.1) (with the Riemann–Liouville, Hadamard, Erdélyi–Kober, Katugampola, Weyl and Liouville fractional integrals) are bound in the literature. Recently, Katugampola [2] unified the aformentioned six integral operators as follows:

Suppose $X_c^p(a,b)$, $c\in\mathbb{R}$ denotes the set of complex valued Lebesgue measurable functions f on $[a,b]$ with the norm

$$\|f\|_{X_c^p}=\left(\int_a^b|t^cf(t)|^p\frac{dt}{t}\right)^{\frac{1}{p}}<\infty,\quad 1\le p<\infty,$$

and

$$\|f\|_{X_c^\infty} = \sup_{t\in(a,b)} \operatorname{ess} |t^c f(t)|.$$

Let $f \in X_c^p(a,b)$, $\alpha > 0$ and $\beta,\rho,\eta,\kappa \in \mathbb{R}$. Then the left (and respectively the right) fractional integrals of f are given by

$$\left({}^\rho I^{\alpha,\beta}_{a^+,\eta,\kappa} f\right)(t) = \frac{\rho^{1-\beta} t^\kappa}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho(\eta+1)-1} f(s)ds, \ \ 0 \le a < t < b \le \infty$$

and

$$\left({}^\rho I^{\alpha,\beta}_{b^-,\eta,\kappa} f\right)(t) = \frac{\rho^{1-\beta} t^{\rho\eta}}{\Gamma(\alpha)} \int_t^b (s^\rho - t^\rho)^{\alpha-1} s^{\kappa+\rho-1} f(s)ds, \ \ 0 \le a < t < b \le \infty.$$

For $\eta = 0$, $\beta = \alpha$ and $\kappa = 0$, one obtains, from the above defined operators, the so-called Katugampola integrals. For this, Hernández and Gômez recently proved the following theorem:

Theorem 1.7 ([1]). *Let $\alpha > 0$ and $\rho > 0$. Let $X\colon [a^\rho, b^\rho] \times \Omega \to \mathbb{R}$ be a positive stochastic process with $0 \le a < b$ and $X(t,\cdot) \in X_c^p(a^\rho, b^\rho)$. If $X(t,\cdot)$ is convex, then the following inequality holds almost everywhere*

$$\begin{aligned} X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) &\le \frac{\Gamma(\alpha+1)}{2\rho^{-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho J^\alpha_{b^\rho -} X(a^\rho,\cdot) + {}^\rho J^\alpha_{a^\rho +} X(b^\rho,\cdot)\right) \\ &\le \frac{X(a^\rho,\cdot) + X(b^\rho,\cdot)}{2\rho\alpha}, \end{aligned} \tag{1.5}$$

where ${}^\rho J^\alpha_{b^\rho -} X(a^\rho,\cdot) = {}^\rho I^{\alpha,\alpha}_{b^\rho -,0,0} X(a^\rho,\cdot)$ and ${}^\rho J^\alpha_{a^\rho +} X(b^\rho,\cdot) = {}^\rho I^{\alpha,\alpha}_{a^\rho +,0,0} X(b^\rho,\cdot)$.

However, we observed that there is a mistake in the proof of Theorem 1.7 and hence, inequality (1.5) should read:

$$\begin{aligned} X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) &\le \frac{\Gamma(\alpha+1)}{2\rho^{-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho J^\alpha_{b^\rho -} X(a^\rho,\cdot) + {}^\rho J^\alpha_{a^\rho +} X(b^\rho,\cdot)\right) \\ &\le \frac{X(a^\rho,\cdot) + X(b^\rho,\cdot)}{2}. \end{aligned} \tag{1.6}$$

Remark 1.8. We were informed that recently, the authors of [1] submitted a follow-up paper with a corrected version of Theorem 1.7.

The goal of this article is two-fold. Namely,

1. Give a broader generalization of inequality (1.6) by means of the generalized fractional integral operators. From this, inequalities involving the Riemann–Liouville, Hadamard, Erdélyi–Kober, Katugampola, Weyl and Liouville fractional integrals are deduced as particular cases.
2. As an application, we provide new Hermite–Hadamard type estimates for expectations of integrals of convex and p-convex stochastic processes.

This article is arranged as follows: Section 2 contains two subsections – the first subsection houses the generalization of the Hermite–Hadamard inequality in the fractional sense. Thereafter, some new estimates involving the expectation of integrals of positive–convex stochastic processes are given in the next subsection.

2. Main Results

In this section, we start by presenting a generalization of Theorem 1.7.

2.1. Generalized Fractional HH Inequality

THEOREM 2.1. *Let $\alpha > 0, \rho > 0, \beta > 0$ and $\eta > 0$. Suppose that $X\colon [a^{\rho(\eta+1)}, b^{\rho(\eta+1)}] \times \Omega \to \mathbb{R}$ is a positive stochastic process with $0 \le a < b$ and $X(t,\cdot) \in X_c^p(a^{\rho(\eta+1)}, b^{\rho(\eta+1)})$. If $X(t,\cdot)$ is convex, then the following inequality holds almost everywhere*

$$X\left(\frac{a^{\rho(\eta+1)} + b^{\rho(\eta+1)}}{2}, \cdot\right) \le \frac{(\eta+1)\Gamma(\alpha+1)}{2\rho^{-\beta}(b^{\rho(\eta+1)} - a^{\rho(\eta+1)})^{\alpha}}$$

$$\times \left[\frac{1}{a^{k\rho(\eta+1)}}\, {}^{\rho}I^{\alpha,\beta}_{b^{\rho(\eta+1)-},\eta,\kappa} X(a^{\rho(\eta+1)},\cdot) + \frac{1}{b^{\rho^2\eta(\eta+1)}}\, {}^{\rho}I^{\alpha,\beta}_{a^{\rho(\eta+1)+},\eta,\rho\eta} X(b^{\rho(\eta+1)},\cdot)\right]$$

$$\le \frac{X(a^{\rho(\eta+1)},\cdot) + X(b^{\rho(\eta+1)},\cdot)}{2}.$$

PROOF. Let $\tilde{\rho} := \rho(\eta+1)$, $t \in [a,b]$ and $u, v \in [a,b]$. We now define $u^{\tilde{\rho}}$ and $v^{\tilde{\rho}}$ as follows:

$$u^{\tilde{\rho}} = t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}, \quad v^{\tilde{\rho}} = (1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}.$$

Then, $u^{\tilde{\rho}} + v^{\tilde{\rho}} = a^{\tilde{\rho}} + b^{\tilde{\rho}}$. Since X is a convex stochastic process, we have:

$$X\left(\frac{u^{\tilde{\rho}} + v^{\tilde{\rho}}}{2}\right) \leq \frac{X(u^{\tilde{\rho}}) + X(v^{\tilde{\rho}})}{2},$$

and obtain

$$2X\left(\frac{a^{\tilde{\rho}} + b^{\tilde{\rho}}}{2}\right) \leq X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}) + X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}).$$

Now, multiplying both sides of the above inequality by $t^{\alpha\tilde{\rho}-1}$, $\alpha, \tilde{\rho} > 0$ and integrating over t in the interval $[0, 1]$ we obtain:

$$\begin{aligned}\text{(2.1)} \qquad \frac{2}{\alpha\rho(\eta+1)} X\left(\frac{a^{\tilde{\rho}} + b^{\tilde{\rho}}}{2}\right) &\leq \int_0^1 t^{\alpha\tilde{\rho}-1} X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}) dt \\ &\quad + \int_0^1 t^{\alpha\tilde{\rho}-1} X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}) dt.\end{aligned}$$

From the definition of $u^{\tilde{\rho}}$ above, we have $t^{\tilde{\rho}} = \frac{b^{\tilde{\rho}} - u^{\tilde{\rho}}}{b^{\tilde{\rho}} - a^{\tilde{\rho}}}$ and hence $\frac{u^{\tilde{\rho}-1}}{b^{\tilde{\rho}} - u^{\tilde{\rho}}} du = -\frac{1}{t} dt$.

Computing the right hand side of the inequality and using the definition of the generalized integral, one gets:

$$\begin{aligned}\int_0^1 t^{\alpha\tilde{\rho}-1} X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}) dt &= \int_0^1 t^{\alpha\tilde{\rho}} X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}) t^{-1} dt \\ &= -\int_b^a \left(\frac{b^{\tilde{\rho}} - u^{\tilde{\rho}}}{b^{\tilde{\rho}} - a^{\tilde{\rho}}}\right)^{\alpha} X(u^{\tilde{\rho}}) \frac{u^{\tilde{\rho}-1}}{b^{\tilde{\rho}} - u^{\tilde{\rho}}} du \\ &= \frac{1}{\left(b^{\tilde{\rho}} - a^{\tilde{\rho}}\right)^{\alpha}} \int_a^b \frac{u^{\tilde{\rho}-1}}{\left(b^{\tilde{\rho}} - u^{\tilde{\rho}}\right)^{1-\alpha}} X(u^{\tilde{\rho}}) du \\ &= \frac{\Gamma(\alpha)}{\rho^{1-\beta} (b^{\tilde{\rho}})^{\rho\eta}} \frac{1}{\left(b^{\tilde{\rho}} - a^{\tilde{\rho}}\right)^{\alpha}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}}+,\eta,\rho\eta} X(b^{\tilde{\rho}}, \cdot).\end{aligned}$$

Following similar steps, we also obtain:

$$\int_0^1 t^{\alpha\tilde{\rho}-1} X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}) dt = \frac{\Gamma(\alpha)}{\rho^{1-\beta} (a^{\tilde{\rho}})^{k}} \frac{1}{\left(b^{\tilde{\rho}} - a^{\tilde{\rho}}\right)^{\alpha}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}}-,\eta,\kappa} X(a^{\tilde{\rho}}, \cdot).$$

Substituting the integrals into (2.1), we get:

$$\frac{2}{\alpha\tilde{\rho}} X\left(\frac{a^{\tilde{\rho}} + b^{\tilde{\rho}}}{2}\right) \leq \frac{\Gamma(\alpha)}{\rho^{1-\beta}} \frac{1}{\left(b^{\tilde{\rho}} - a^{\tilde{\rho}}\right)^{\alpha}} \times \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}}+,\eta,\rho\eta} X(b^{\tilde{\rho}}, \cdot) + \frac{1}{(a^{\tilde{\rho}})^{k}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}}-,\eta,\kappa} X(a^{\tilde{\rho}}, \cdot)\right). \tag{2.2}$$

To obtain the other part of the inequality, we use the convex property of the process X as follows:

$$X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}) \leq t^{\tilde{\rho}} X(a^{\tilde{\rho}}) + (1 - t^{\tilde{\rho}}) X(b^{\tilde{\rho}}),$$

$$X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}) \leq (1 - t^{\tilde{\rho}}) X(a^{\tilde{\rho}}) + t^{\tilde{\rho}} X(b^{\tilde{\rho}}).$$

Adding the two inequalities, we obtain:

$$X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}) + X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}) \leq X(a^{\tilde{\rho}}) + X(b^{\tilde{\rho}}).$$

Multiplying through by $t^{\alpha\tilde{\rho}-1}$, $\alpha, \tilde{\rho} > 0$ and integrating the resulting inequality over t, in the interval $[0, 1]$, we obtain

$$\frac{\Gamma(\alpha)}{\rho^{1-\beta}} \frac{1}{\left(b^{\tilde{\rho}} - a^{\tilde{\rho}}\right)^{\alpha}} \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}}+,\eta,\rho\eta} X(b^{\tilde{\rho}}, \cdot) + \frac{1}{(a^{\tilde{\rho}})^{k}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}}-,\eta,\kappa} X(a^{\tilde{\rho}}, \cdot)\right) \leq \frac{X(a^{\tilde{\rho}}) + X(b^{\tilde{\rho}})}{\alpha\tilde{\rho}}. \tag{2.3}$$

Thus, combining (2.2) and (2.3), we get:

$$\frac{2}{\alpha\tilde{\rho}} X\left(\frac{a^{\tilde{\rho}} + b^{\tilde{\rho}}}{2}\right) \leq \frac{\Gamma(\alpha)}{\rho^{1-\beta}} \frac{1}{\left(b^{\tilde{\rho}} - a^{\tilde{\rho}}\right)^{\alpha}} \times \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}}+,\eta,\rho\eta} X(b^{\tilde{\rho}}, \cdot) + \frac{1}{(a^{\tilde{\rho}})^{k}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}}-,\eta,\kappa} X(a^{\tilde{\rho}}, \cdot)\right) \leq \frac{X(a^{\tilde{\rho}}) + X(b^{\tilde{\rho}})}{\alpha\tilde{\rho}}. \tag{2.4}$$

It is enough to multiply now all sides of (2.4) by $\frac{\alpha\tilde{\rho}}{2}$ and the intended result follows. $\square$

We deduce the generalized Hermite–Hadamard inequality for the generalized Katugampola fractional integrals of any convex function f:

Corollary 2.2. *Let* $f \in X_c^p(a^{\rho(\eta+1)}, b^{\rho(\eta+1)})$. *If* $f\colon [a^{\rho(\eta+1)}, b^{\rho(\eta+1)}] \to \mathbb{R}$ *is a convex function with* $0 \leq a < b, \alpha > 0$, $\rho > 0$, $\beta > 0$ *and* $\eta > 0$, *then*

$$\begin{aligned} f\left(\frac{a^{\rho(\eta+1)} + b^{\rho(\eta+1)}}{2}\right) &\leq \frac{(\eta+1)\Gamma(\alpha+1)}{2\rho^{-\beta}(b^{\rho(\eta+1)} - a^{\rho(\eta+1)})^{\alpha}} \\ &\times \left[\frac{1}{b^{\rho^2\eta(\eta+1)}}\,{}^{\rho}I^{\alpha,\beta}_{a^{\rho(\eta+1)}+,\eta,\rho\eta} f(b^{\rho(\eta+1)}) + \frac{1}{a^{k\rho(\eta+1)}}\,{}^{\rho}I^{\alpha,\beta}_{b^{\rho(\eta+1)}-,\eta,\kappa} f(a^{\rho(\eta+1)})\right] \\ &\leq \frac{f(a^{\rho(\eta+1)}) + f(b^{\rho(\eta+1)})}{2}. \end{aligned}$$

Remark 2.3. In view of Theorem 2.1, we make the following observations:

1. If we set $\beta = \alpha$ and $\kappa = \eta = 0$ in Theorem 2.1, then we recover Theorem 1.7.
2. By setting $\beta = \alpha$, $\kappa = \eta = 0$ and taking limit $\rho \to 0^+$ in Theorem 2.1, we get the Hermite–Hadamard inequality involving the Hadamard fractional integral operators.
3. Let $\beta = 0$ and $\kappa = -\rho(\alpha + \eta)$ in Theorem 2.1, we obtain the Hermite–Hadamard inequality involving the Erdélyi–Kober fractional integral operators.
4. If we let $a = 0$, $\kappa = \eta = 0$ and taking limit $\rho \to 1$ in Theorem 2.1, then we have the Hermite–Hadamard inequality involving the Liouville fractional integral operators.
5. Substituting $\beta = \alpha$, $\kappa = \eta = 0$ and taking limit $\rho \to 1$ in Theorem 2.1, we get the following Hermite–Hadamard inequality involving the Riemann–Liouville fractional integral operators:

$$\begin{aligned} X\left(\frac{a+b}{2}, \cdot\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big(J^{\alpha}_{b-}X(a,\cdot) + J^{\alpha}_{a+}X(b,\cdot)\Big) \\ &\leq \frac{X(a,\cdot) + X(b,\cdot)}{2}, \end{aligned} \tag{2.5}$$

where $X(t,\cdot) \in X_c^p(a,b)$, and J^{α}_{b-} and J^{α}_{a+} are the left and right Riemann--Liouville fractional integral operators, respectively.

In 2013, Sarikaya et al. [8] introduced the (k,r)-fractional integral operators (which are generalizations of the Riemann–Liouville fractional integral operators) as follows:

$$I^{\alpha,r}_{a^+,k}f(t)=\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t (t^{r+1}-s^{r+1})^{\frac{\alpha}{k}-1}s^r f(s)ds,\quad t\in[a,b]$$

and

$$I^{\alpha,r}_{b^-,k}f(t)=\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_t^b (s^{r+1}-t^{r+1})^{\frac{\alpha}{k}-1}s^r f(s)ds,\quad t\in[a,b],$$

with $I^{\alpha,0}_{a,1}f(t)=I^{\alpha}_a f(t)$ where Γ_k is the Euler gamma k-function given by

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt,\quad \mathcal{R}e(x)>0,\, k>0$$

and $\Gamma_1(x)=\Gamma(x)$ the usual gamma function, satisfying the following properties: $\Gamma_k(x+k)=x\Gamma_k(x)$ and $\Gamma_k(k)=1$.

By following a similar approach as outlined in the proof of Theorem 2.1, we state (without proof) a generalization of (2.5) involving the (k,r)-Riemann--Liouville fractional integral operators in the following theorem:

THEOREM 2.4. *Let $\alpha>0$, $k>0$ and $r\geq 0$. Let $X\colon[a^{r+1},b^{r+1}]\times\Omega\to\mathbb{R}$ be a positive stochastic with $0\leq a<b$ and $X(t,.)\in X^p_c(a^{r+1},b^{r+1})$. If $X(t,.)$ is convex, then the following inequality holds almost everywhere*

$$\begin{aligned}X\left(\frac{a^{r+1}+b^{r+1}}{2},\cdot\right)&\leq\frac{\Gamma_k(\alpha+k)}{2(r+1)^{-\frac{\alpha}{k}}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\\&\quad\times\left(I^{\alpha,r}_{b^{(r+1)^-},k}X(a^{r+1},\cdot)+I^{\alpha,r}_{a^{(r+1)^+},k}X(b^{r+1},\cdot)\right)\\&\leq\frac{X(a^{r+1},\cdot)+X(b^{r+1},\cdot)}{2}.\end{aligned}$$

2.2. Moment Estimates

Let $X\colon I\times\Omega\to\mathbb{R}$ be a random variable given by $X=X(t,\omega)$, $t\in I$, $\omega\in\Omega$. For convenience and simplicity, we will denote $X=X(t)=X(t,\cdot)$ for a fixed $\omega\in\Omega$. Here and throughout this section, we assume $X(t)$ to be an $\{\mathcal{F}_t\}$-adapted, positive, non-decreasing convex stochastic process, where $\mathcal{F}_t=\sigma\{X(s),0\leq s\leq t\}$ for each $t\in[0,T]$, is the (its natural) filtration.

Remark 2.5. If $X(t)$ is a positive convex stochastic process, then $X^2(t)$ is also convex. Since the composition of non-decreasing convex functions is still convex and non-decreasing, it follows therefore that for any two convex, non-decreasing functions $\varphi(t)$ and $X^2(t)$, their composition $\varphi(X^2(t))$ is a convex (and non-decreasing) stochastic process.

Theorem 2.6. *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a convex, non-decreasing function and X a one dimensional adapted, positive, non-decreasing convex process such that $\mathbf{E}\big[\varphi\big(X^2(t)\big)\big] < \infty$ for all $a \le t \le b$. Then*

$$\varphi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\mathbf{E}\left[\int_a^b \varphi\big(X^2(t)\big)dt\right] \le \frac{\mathbf{E}\big[\varphi\big(X^2(a)\big)\big] + \mathbf{E}\big[\varphi\big(X^2(b)\big)\big]}{2}.$$

Proof. Following (1.1) for any convex function φ, we have

$$\varphi\left(X^2\left(\frac{a+b}{2}\right)\right) \le \frac{1}{b-a}\int_a^b \varphi\big(X^2(t)\big)dt \le \frac{\varphi\big(X^2(a)\big) + \varphi\big(X^2(b)\big)}{2}.$$

Taking expectation of all sides and applying Jensen's inequality, we obtain

$$\begin{aligned}\varphi\left(\mathbf{E}\left[X^2\left(\frac{a+b}{2}\right)\right]\right) &\le \mathbf{E}\left[\varphi\left(X^2\left(\frac{a+b}{2}\right)\right)\right] \le \frac{1}{b-a}\mathbf{E}\left[\int_a^b \varphi\big(X^2(t)\big)dt\right]\\ &\le \frac{\mathbf{E}\big[\varphi\big(X^2(a)\big)\big] + \mathbf{E}\big[\varphi\big(X^2(b)\big)\big]}{2},\end{aligned}$$

and the result follows. $\square$

Corollary 2.7. *If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a convex function and $\varphi(X^2(t)) = X^2(t)$, a convex process (X – a one dimensional process), then we have*

$$\frac{a+b}{2} \le \frac{1}{b-a}\mathbf{E}\left[\int_a^b X^2(t)dt\right] \le \frac{a+b}{2},$$

and it follows that

$$\mathbf{E}\left[\int_a^b X^2(t)dt\right] = \frac{b^2-a^2}{2}.$$

We now use Theorem 1.2 to compute the second moment of integral of a convex, positive and mean square continuous process (in particular one dimensional process) as follows:

Theorem 2.8. *If $X\colon I \to \mathbb{R}$ is a convex, positive and mean square continuous process in the interval I, then*

$$\mathbf{E}\bigg[\bigg(\int_a^b X(t)dt\bigg)^2\bigg] = \frac{(a+b)(b-a)^2}{2}$$

holds almost everywhere.

Proof. Square all sides of (1.2) and take expectations, to get:

$$\begin{aligned}\mathbf{E}\bigg[X^2\bigg(\frac{a+b}{2}\bigg)\bigg] &\leq \frac{1}{(b-a)^2}\mathbf{E}\bigg[\bigg(\int_a^b X(t)dt\bigg)^2\bigg]\\ &\leq \frac{\mathbf{E}\big[X^2(a)\big]+2\mathbf{E}\big[X(a)X(b)\big]+\mathbf{E}\big[X^2(b)\big]}{4}.\end{aligned}$$

Hence, since X is adapted,

$$\begin{aligned}\frac{a+b}{2} &\leq \frac{1}{(b-a)^2}\mathbf{E}\bigg[\bigg(\int_a^b X(t)dt\bigg)^2\bigg] \leq \frac{a+2\min\{a,b\}+b}{4}\\ &= \frac{a+2a+b}{4} \leq \frac{b+2a+b}{4} = \frac{a+b}{2}.\end{aligned}$$

□

Theorem 1.7 can be applied to estimate the moment of Katugampola fractional integral mean square continuous process:

Theorem 2.9. *If $X(t)$ is a one dimensional adapted, positive, convex stochastic process satisfying statement of Theorem 1.7, then*

$$\bigg(\mathbf{E}\Big[{}^\rho I^\alpha_{b^\rho -}X^2(a^\rho)\Big] + \mathbf{E}\Big[{}^\rho I^\alpha_{a^\rho +}X^2(b^\rho)\Big]\bigg) = \frac{(a^\rho+b^\rho)(b^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}.$$

Proof. Following Theorem 1.7, since X^2 is positive and convex, we have from (1.6) that:

$$\begin{aligned}X^2\bigg(\frac{a^\rho+b^\rho}{2}\bigg) &\leq \frac{\Gamma(\alpha+1)}{2\rho^{-\alpha}(b^\rho-a^\rho)^\alpha}\Big({}^\rho I^\alpha_{b^\rho -}X^2(a^\rho) + {}^\rho I^\alpha_{a^\rho +}X^2(b^\rho)\Big)\\ &\leq \frac{X^2(a^\rho)+X^2(b^\rho)}{2}.\end{aligned}$$

Now, take expectation of both sides to obtain

$$\mathbf{E}\left[X^2\left(\frac{a^\rho+b^\rho}{2}\right)\right] \le \frac{\Gamma(\alpha+1)}{2\rho^{-\alpha}(b^\rho-a^\rho)^\alpha}\left(\mathbf{E}\left[{}^\rho I^\alpha_{b^\rho-}X^2(a^\rho)\right]+\mathbf{E}\left[{}^\rho I^\alpha_{a^\rho+}X^2(b^\rho)\right]\right)$$
$$\le \frac{\mathbf{E}\left[X^2(a^\rho)\right]+\mathbf{E}\left[X^2(b^\rho)\right]}{2},$$

and thus, by assuming that X is adapted we obtain

$$\frac{a^\rho+b^\rho}{2} \le \frac{\Gamma(\alpha+1)}{2\rho^{-\alpha}(b^\rho-a^\rho)^\alpha}\left(\mathbf{E}\left[{}^\rho I^\alpha_{b^\rho-}X^2(a^\rho)\right]+\mathbf{E}\left[{}^\rho I^\alpha_{a^\rho+}X^2(b^\rho)\right]\right)$$
$$\le \frac{a^\rho+b^\rho}{2}. \qquad \square$$

THEOREM 2.10. *Let $X\colon I\subset(0,\infty)\to\mathbb{R}$ be a p-convex, positive stochastic process and mean-square integrable on $[a,b]$ where $a,b\in I$ and $a<b$. Then*

$$\frac{(a^p+b^p)^{\frac{1}{p}}(b^p-a^p)^2}{p^2 2^{\frac{1}{p}}} \le \mathbf{E}\left[\left(\int_a^b \frac{X(t)}{t^{1-p}}dt\right)^2\right] \le \frac{(a+b)(b^p-a^p)^2}{2p^2}.$$

PROOF. Since X is a one dimensional p-convex, positive stochastic process, we have by taking square of all sides of (1.3) that:

$$X^2\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p^2}{(b^p-a^p)^2}\left(\int_a^b \frac{X(t)}{t^{1-p}}dt\right)^2$$
$$\le \frac{X^2(a)+2X(a)X(b)+X^2(b)}{4}.$$

By taking expectations of all sides and following the proof of Theorem 2.8,

$$\mathbf{E}\left[X^2\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right] \le \frac{p^2}{(b^p-a^p)^2}\mathbf{E}\left[\left(\int_a^b \frac{X(t)}{t^{1-p}}dt\right)^2\right] \le \frac{a+b}{2}.$$

We therefore obtain

$$\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}} \le \frac{p^2}{(b^p-a^p)^2}\mathbf{E}\left[\left(\int_a^b \frac{X(t)}{t^{1-p}}dt\right)^2\right] \le \frac{a+b}{2}. \qquad \square$$

THEOREM 2.11. *Let $X\colon I \subset (0,\infty) \to \mathbb{R}$ be a p-convex, positive stochastic process and mean-square integrable on $[a,b]$ where $a,b \in I$ and $a < b$. Then*

$$\frac{(a^p+b^p)^{\frac{1}{p}}(b^p-a^p)}{p2^{\frac{1}{p}}} \le \mathbf{E}\left[\int_a^b \frac{X^2(t)}{t^{1-p}}dt\right] \le \frac{(a+b)(b^p-a^p)}{2p}.$$

PROOF. Since X^2 is a one dimensional p-convex, positive stochastic process, then we have from (1.3) that

$$X^2\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p-a^p}\int_a^b \frac{X^2(t)}{t^{1-p}}dt \le \frac{X^2(a)+X^2(b)}{2}.$$

Taking expectation of all sides yields:

$$\mathbf{E}\left[X^2\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right] \le \frac{p}{b^p-a^p}\mathbf{E}\left[\int_a^b \frac{X^2(t)}{t^{1-p}}dt\right] \le \frac{\mathbf{E}[X^2(a)]+\mathbf{E}[X^2(b)]}{2}.$$

We obtain

$$\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}} \le \frac{p}{b^p-a^p}\mathbf{E}\left[\int_a^b \frac{X^2(t)}{t^{1-p}}dt\right] \le \frac{a+b}{2}. \qquad \square$$

THEOREM 2.12. *Let $X\colon I \subset (0,\infty) \to \mathbb{R}$ be a one dimensional exponentially p-convex, positive stochastic process and integrable on $[a,b]$ with $a,b \in I$ and $a < b$. Then for $p \in \mathbb{R}\setminus\{0\}$ and $\alpha \in \mathbb{R}$,*

$$\frac{(a^p+b^p)^{\frac{1}{p}}(b^p-a^p)}{p2^{\frac{1}{p}}} \le \mathbf{E}\left[\int_a^b \frac{X^2(s)}{s^{1-p}e^{\alpha s}}ds\right] \le \frac{\left(A_1(\alpha)\frac{a}{e^{\alpha a}}+A_2(\alpha)\frac{b}{e^{\alpha b}}\right)(b^p-a^p)}{p}.$$

PROOF. If $\alpha \neq 0$, then

$$X^2\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p-a^p}\int_a^b \frac{X^2(s)}{s^{1-p}e^{\alpha s}}ds \le A_1(\alpha)\frac{X^2(a)}{e^{\alpha a}}+A_2(\alpha)\frac{X^2(b)}{e^{\alpha b}}.$$

Taking expectation of all sides we obtain

$$\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}} \le \frac{p}{b^p-a^p}\mathbf{E}\left(\int_a^b \frac{X^2(s)}{s^{1-p}e^{\alpha s}}ds\right) \le A_1(\alpha)\frac{a}{e^{\alpha a}}+A_2(\alpha)\frac{b}{e^{\alpha b}},$$

for constant numbers $A_1(\alpha)$, $A_2(\alpha)$. $\square$

THEOREM 2.13. *Let $X\colon I \subset (0,\infty) \to \mathbb{R}$ be a one dimensional exponentially p-convex, positive stochastic process and $a, b \in I$ with $a < b$. If X is mean-square integrable on $[a,b]$, then for $p \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$,*

$$\frac{(a^p+b^p)^{\frac{1}{p}}(b^p-a^p)^2}{p^2 2^{\frac{1}{p}}} \le \mathbf{E}\left[\left(\int_a^b \frac{X(s)}{s^{1-p}e^{\alpha s}}ds\right)^2\right]$$

$$\le \frac{(b^p-a^p)^2}{p^2}\left(\left(A_1^2(\alpha)+A_2^2(\alpha)\right)\frac{b}{e^{2\alpha a}}+2A_1(\alpha)A_2(\alpha)\frac{a}{e^{\alpha(a+b)}}\right).$$

PROOF. Square (1.4) and take expectations of all sides to obtain:

$$\mathbf{E}X^2\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p^2}{(b^p-a^p)^2}\mathbb{E}\left(\int_a^b \frac{X(s)}{s^{1-p}e^{\alpha s}}ds\right)^2$$

$$\le \mathbf{E}\left(A_1(\alpha)\frac{X(a)}{e^{\alpha a}}+A_2(\alpha)\frac{X(b)}{e^{\alpha b}}\right)^2,$$

and the result follows. □

3. Conclusion

New fractional inequalities of the Hermite–Hadamard type for positive–convex stochastic processes have been established. The first result generalizes and unifies some known results in the literature. Loads of estimates can be deduced as special cases of our main theorems. For related results, we invite the interested reader to the following papers [4, 9, 10, 11, 12, 13] and the references cited therein.

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