Annales Mathematicae Silesianae **34** (2020), no. 1, 123–132
DOI: 10.2478/amsil-2020-0017

OHLIN AND LEVIN–STEČKIN-TYPE RESULTS FOR STRONGLY CONVEX FUNCTIONS

Kazimierz Nikodem, Teresa Rajba

Dedicated to Professor Zygfryd Kominek on his 75th birthday

Abstract. Counterparts of the Ohlin and Levin–Stečkin theorems for strongly convex functions are proved. An application of these results to obtain some known inequalities related with strongly convex functions in an alternative and unified way is presented.

1. Introduction

In 1969, J. Ohlin [9] proved the following interesting and very useful result on convex functions in a probabilistic context (as usual, $\mathbb{E}[X]$ denotes the expectation of the random variable X):

Received: 06.07.2019. Accepted: 28.06.2020. Published online: 16.07.2020.
(2010) Mathematics Subject Classification: 26A51, 39B62, 26D15.
Key words and phrases: Ohlin's lemma, stochastic ordering, Levin–Stečkin theorem, strongly convex functions, Hermite–Hadamard inequalities.

Lemma 1 ([9]). *Let X, Y be two real valued random variables such that $\mathbb{E}[X] = \mathbb{E}[Y]$. If the distribution functions F_X, F_Y crosses one time, i.e. there exists $t_0 \in \mathbb{R}$ such that*

$$F_X(t) \leq F_Y(t) \quad \text{if } t < t_0 \quad \text{and} \quad F_X(t) \geq F_Y(t) \quad \text{if } t > t_0,$$

then

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \tag{1}$$

for every convex function $f\colon \mathbb{R} \to \mathbb{R}$.

For two real valued random variables X, Y with finite expectations, we say that X *is dominated by* Y *in convex stochastic ordering sense*, if condition (1) holds for all convex functions $f\colon \mathbb{R} \to \mathbb{R}$ (see [1]). Thus Ohlin's lemma gives sufficient conditions for X to be dominated by Y in such ordering. It is interesting that earlier, in 1960, V.I. Levin and S.B. Stečkin [4] (see also [6, Theorem 4.2.7 and Lemma 4.2.9]) proved a more general result giving a necessary and sufficient condition for convex stochastic ordering. However, their result was clearly unknown for Ohlin. For years the Ohlin lemma also was not well-known in the mathematical community. It has been rediscovered by T. Rajba [12], who found its various applications to the theory of functional inequalities (cf. also [10, 13, 16, 17, 18]). In [12], the authoress used the Ohlin lemma to get a very simple proof of known Hermite–Hadamard type inequalities, as well as to obtain new Hermite–Hadamard type inequalities. In the papers [10, 13, 17, 18], furthermore, the Levin–Stečkin theorem [4] is used to examine the Hermite–Hadamard type inequalities. Let us mention also the recent paper by M. Niezgoda [7], in which an extension of the Levin–Stečkin theorem to uniformly convex and superquadratic functions is presented.

In this note we prove counterparts of the Ohlin and Levin–Stečkin theorems for strongly convex functions. We present also applications of these results to obtain some inequalities connected with strongly convex functions.

Let us recall that a function $f\colon I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is called *strongly convex with modulus* $c > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all $x, y \in I$ and $t \in (0,1)$. Strongly convex functions have been introduced by Polyak [11] and they play an important role in optimization theory and mathematical economics. Many properties of them can be found, among others, in [2, 3, 5, 8, 15].

2. Main results

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Assume that $I \subset \mathbb{R}$ is an interval and $c > 0$. Given a random variable $X\colon \Omega \to \mathbb{R}$ we denote by $\mathbb{D}^2[X]$ the variance of X. The following result is a counterpart of Ohlin's lemma for strongly convex functions.

THEOREM 2. *Let $X, Y : \Omega \to I$ be square integrable random variables such that $\mathbb{E}[X] = \mathbb{E}[Y]$. If there exists $t_0 \in \mathbb{R}$ such that*

$$F_X(t) \le F_Y(t) \quad \text{if } t < t_0 \quad \text{and} \quad F_X(t) \ge F_Y(t) \quad \text{if } t > t_0,$$

then

$$\mathbb{E}[f(X)] - c\mathbb{D}^2[X] \le \mathbb{E}[f(Y)] - c\mathbb{D}^2[Y] \tag{2}$$

for every continuous function $f\colon I \to \mathbb{R}$ strongly convex with modulus c.

PROOF. Let $f\colon I \to \mathbb{R}$ be continuous and strongly convex with modulus c. By the characterization of strongly convex functions (see [2, 5, 15]), the function $g\colon I \to \mathbb{R}$ defined by $g(x) = f(x) - cx^2$, $x \in I$, is convex. Therefore, by the Ohlin lemma applied for g, we have

$$\mathbb{E}[g(X)] \le \mathbb{E}[g(Y)],$$

and hence

$$\mathbb{E}[f(X)] - c\mathbb{E}[X^2] \le \mathbb{E}[f(Y)] - c\mathbb{E}[Y^2].$$

Since $\mathbb{E}[X] = \mathbb{E}[Y]$, we have also

$$\mathbb{E}[f(X)] - c\mathbb{E}[X^2] + c(\mathbb{E}[X])^2 \le \mathbb{E}[f(Y)] - c\mathbb{E}[Y^2] + c(\mathbb{E}[Y])^2,$$

which gives

$$\mathbb{E}[f(X)] - c\mathbb{D}^2[X] \le \mathbb{E}[f(Y)] - c\mathbb{D}^2[Y]$$

and finishes the proof. □

REMARK 3. Note that condition (2) is stronger than (1). Indeed, by the Ohlin lemma applied for the function $f(x) = x^2$, we have $\mathbb{E}[X^2] \le \mathbb{E}[Y^2]$, and hence $\mathbb{D}^2[X] \le \mathbb{D}^2[Y]$, because $\mathbb{E}[X] = \mathbb{E}[Y]$.

Let us recall now the theorem proved by V.I. Levin and S.B. Stečkin [4].

THEOREM 4 ([4]). *Let $a, b \in \mathbb{R}$, $a < b$ and let $F_1, F_2\colon [a, b] \to \mathbb{R}$ be functions with bounded variation such that $F_1(a) = F_2(a)$. Then, in order that*

$$\int_a^b f(x)dF_1(x) \leq \int_a^b f(x)dF_2(x)$$

for all continuous convex functions $f\colon [a, b] \to \mathbb{R}$, it is necessary and sufficient that F_1 and F_2 verify the following three conditions:

$$F_1(b) = F_2(b),$$

$$\int_a^x F_1(t)dt \leq \int_a^x F_2(t)dt \quad \text{for every} \quad x \in (a, b),$$

$$\int_a^b F_1(t)dt = \int_a^b F_2(t)dt.$$

The next result is a version of the above Levin–Stečkin theorem for strongly convex functions (cf. [7] where a similar result for uniformly convex functions is obtained).

THEOREM 5. *Let $a, b \in \mathbb{R}$, $a < b$ and let $F_1, F_2\colon [a, b] \to \mathbb{R}$ be functions with bounded variation such that $F_1(a) = F_2(a)$. Then, in order that*

$$\int_a^b f(t)dF_1(t) - c\int_a^b t^2 dF_1(t) \leq \int_a^b f(t)dF_2(t) - c\int_a^b t^2 dF_2(t) \tag{3}$$

for every continuous function $f\colon [a, b] \to \mathbb{R}$ strongly convex with modulus c, it is necessary and sufficient that F_1 and F_2 satisfy the following three conditions:

$$F_1(b) = F_2(b), \tag{4}$$

$$\int_a^x F_1(t)dt \leq \int_a^x F_2(t)dt \quad \text{for every} \quad x \in (a, b), \tag{5}$$

$$\int_a^b F_1(t)dt = \int_a^b F_2(t)dt. \tag{6}$$

PROOF. By the characterization of strongly convex functions, a function $f\colon [a, b] \to \mathbb{R}$ is strongly convex with modulus c if and only if the function $g(x) = f(x) - cx^2$, $x \in [a, b]$, is convex. Therefore condition (3) holds for all continuous functions $f\colon [a,b] \to \mathbb{R}$ strongly convex with modulus c, if and only if

$$\int_a^b g(t)dF_1(t) \le \int_a^b g(t)dF_2(t) \tag{7}$$

holds for all continuous convex functions $g\colon [a,b] \to \mathbb{R}$. Since, by the Levin–Stečkin theorem, condition (7) is equivalent to conditions (4)–(6), the proof is finished. $\square$

As a consequence of the above theorem, we get the following necessary and sufficient condition for random variables X, Y to satisfy (2).

THEOREM 6. *Let $X, Y : \Omega \to [a,b]$ be square integrable random variables such that $\mathbb{E}[X] = \mathbb{E}[Y]$ and let F_X, F_Y be the distribution functions of X, Y, respectively. Then*

$$\mathbb{E}[f(X)] - c\mathbb{D}^2[X] \le \mathbb{E}[f(Y)] - c\mathbb{D}^2[Y] \tag{8}$$

for every continuous function $f\colon [a,b] \to \mathbb{R}$ strongly convex with modulus c, if and only if F_X and F_Y satisfy the following condition:

$$\int_a^x F_X(t)dt \le \int_a^x F_Y(t)dt \quad \textit{for every} \quad x \in [a,\infty).$$

PROOF. Since $X, Y : \Omega \to [a,b]$, we have

$$F_X(a) = F_Y(a) = 0 \quad \text{and} \quad F_X(b_1) = F_Y(b_1) = 1 \quad \text{for any } b_1 > b. \tag{9}$$

We have also

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} tdF_X(t) = \int_a^{b_1} tdF_X(t) \quad \text{and} \quad \mathbb{E}[Y] = \int_a^{b_1} tdF_Y(t). \tag{10}$$

By the integration by parts formula and (9), (10), we obtain

$$\begin{aligned}\mathbb{E}[X] - \mathbb{E}[Y] &= \int_a^{b_1} td(F_X(t) - F_Y(t)) \\ &= t(F_X(t) - F_Y(t))\Big|_a^{b_1} - \int_a^{b_1} (F_X(t) - F_Y(t))dt \\ &= -\int_a^{b_1} (F_X(t) - F_Y(t))dt.\end{aligned}$$

Since $\mathbb{E}[X] = \mathbb{E}[Y]$, this implies

$$\int_a^{b_1} F_X(t)dt = \int_a^{b_1} F_Y(t)dt. \tag{11}$$

Now, using the equalities

$$\mathbb{E}[f(X)] = \int_a^{b_1} f(t)dF_X(t) \quad \text{and} \quad \mathbb{E}[f(Y)] = \int_a^{b_1} f(t)dF_Y(t),$$

and

$$\mathbb{E}[X^2] = \int_a^{b_1} t^2 dF_X(t) \quad \text{and} \quad \mathbb{E}[Y^2] = \int_a^{b_1} t^2 dF_Y(t),$$

and once more the assumption $\mathbb{E}[X] = \mathbb{E}[Y]$, we can rewrite condition (8) in the form

$$\int_a^{b_1} f(t)dF_X(t) - c\int_a^{b_1} t^2 dF_X(t) \leq \int_a^{b_1} f(t)dF_Y(t) - c\int_a^{b_1} t^2 dF_Y(t).$$

Therefore, by Theorem 4, condition (8) is equivalent to

$$\int_a^{x} F_X(t)dt \leq \int_a^{x} F_Y(t)dt \quad \text{for every} \quad x \in [a, \infty),$$

because the remaining conditions (4), (6) in Theorem 5 are already fulfilled as (9) and (11). This finishes the proof. □

3. Applications

In this section we present an application of the Ohlin-type lemma to obtain some known inequalities related with strongly convex functions in an alternative and unified way. The first result is a counterpart of the classical Jensen inequality.

Corollary 7 ([5]). *If $f\colon I\to\mathbb{R}$ is strongly convex with modulus c then*

$$f\left(\sum_{i=1}^{n} t_i x_i\right)\le \sum_{i=1}^{n} t_i f(x_i) - c\sum_{i=1}^{n} t_i (x_i-\bar{x})^2 \tag{12}$$

for all $x_1,\dots,x_n\in I$, $t_1,\dots,t_n>0$ with $t_1+\dots+t_n=1$ and $\bar{x}=t_1x_1+\dots+t_nx_n$.

Proof. Fix $x_1,\dots,x_n\in I$ and $t_1,\dots,t_n>0$ such that $t_1+\dots+t_n=1$ and put $\bar{x}=t_1x_1+\dots+t_nx_n$. Take random variables $X,Y\colon\Omega\to I$ with the distributions $\mu_X=\delta_{\bar{x}}$ and $\mu_Y=t_1\delta_{x_1}+\dots+t_n\delta_{x_n}$. Then the distribution functions F_X,F_Y satisfy the assumption of Theorem 2,

$$\mathbb{E}[X]=\bar{x}=\sum_{i=1}^{n} t_i x_i=\mathbb{E}[Y]$$

and

$$\mathbb{D}^2[X]-\mathbb{D}^2[Y]=\mathbb{E}[X^2]-\mathbb{E}[Y^2]=\bar{x}^2-\sum_{i=1}^{n} t_i x_i^2=-\sum_{i=1}^{n} t_i(x_i-\bar{x})^2.$$

Moreover

$$\mathbb{E}[f(X)]=f(\bar{x})\quad\text{and}\quad\mathbb{E}[f(Y)]=\sum_{i=1}^{n} t_i f(x_i).$$

Therefore, by Theorem 2 we obtain (12). □

We have also the following converse Jensen inequality for strongly convex functions.

Corollary 8 ([3]). *Let $m,M\in I$, $m<M$. If $f\colon I\to\mathbb{R}$ is strongly convex with modulus c, then*

$$\begin{aligned}\sum_{i=1}^{n} t_i f(x_i) - c\sum_{i=1}^{n} t_i(x_i-\bar{x})^2 \\ \le \frac{M-\bar{x}}{M-m}f(m)+\frac{\bar{x}-m}{M-m}f(M)-c(M-\bar{x})(\bar{x}-m)\end{aligned} \tag{13}$$

for all $x_1,\dots,x_n\in[m,M]$, $t_1,\dots,t_n>0$ with $t_1+\dots+t_n=1$ and $\bar{x}=t_1x_1+\dots+t_nx_n$.

Proof. Take random variables $X, Y\colon \Omega \to I$ with the distributions

$$\mu_X = \sum_{i=1}^{n} t_i \delta_{x_i} \quad \text{and} \quad \mu_Y = \frac{M-\bar{x}}{M-m}\delta_m + \frac{\bar{x}-m}{M-m}\delta_M.$$

Then the distribution functions F_X, F_Y satisfy the assumption of Theorem 2,

$$\mathbb{E}[X] = \sum_{i=1}^{n} t_i x_i = \bar{x} = \frac{M-\bar{x}}{M-m}m + \frac{\bar{x}-m}{M-m}M = \mathbb{E}[Y]$$

and

$$\begin{aligned}
\mathbb{D}^2[X] &= \mathbb{E}[(X-\mathbb{E}X)^2] = \sum_{i=1}^{n} t_i (x_i - \bar{x})^2, \\
\mathbb{D}^2[Y] &= \mathbb{E}[(Y-\mathbb{E}Y)^2] = \frac{M-\bar{x}}{M-m}(m-\bar{x})^2 + \frac{\bar{x}-m}{M-m}(M-\bar{x})^2 \\
&= (M-\bar{x})(\bar{x}-m).
\end{aligned}$$

Moreover

$$\mathbb{E}[f(X)] = \sum_{i=1}^{n} t_i f(x_i) \quad \text{and} \quad \mathbb{E}[f(Y)] = \frac{M-\bar{x}}{M-m}f(m) + \frac{\bar{x}-m}{M-m}f(M).$$

Therefore, by Theorem 2 we obtain (13). □

The next result gives a probabilistic characterization of strong convexity obtained by Rajba and Wąsowicz [14].

Corollary 9 ([14]). *A function $f\colon I \to \mathbb{R}$ is strongly convex with modulus c if and only if*

$$f(\mathbb{E}[Y]) \leq \mathbb{E}[f(Y)] - c\mathbb{D}^2[Y] \tag{14}$$

for any square integrable random variable Y taking values in I.

Proof. Let $f\colon I \to \mathbb{R}$ be strongly convex with modulus c and Y be a random variable with values in I. Take a random variable $X\colon \Omega \to I$ with the distributions $\mu_X = \delta_{\mathbb{E}[Y]}$. Then X and Y satisfy the assumption of Theorem 2. Moreover, $\mathbb{E}[f(X)] = f(\mathbb{E}[Y])$ and $\mathbb{D}^2[X] = 0$. Therefore, by Theorem 2,

$$f(\mathbb{E}[Y]) = \mathbb{E}[f(X)] - c\mathbb{D}^2[X] \le \mathbb{E}[f(Y)] - c\mathbb{D}^2[Y].$$

Conversely, assume that $f\colon I \to \mathbb{R}$ satisfy (14) for any random variable Y taking values in I. Fix arbitrary $x_1, x_2 \in I$, $t \in (0,1)$ and take a random variable Y with the distribution $\mu_Y = t\delta_{x_1} + (1-t)\delta_{x_2}$. Then $f(\mathbb{E}[Y]) = f(tx_1 + (1-t)x_2)$, $\mathbb{E}[f(X)] = tf(x_1) + (1-t)f(x_2)$ and $\mathbb{D}^2[Y] = t(1-t)(x_1 - x_2)^2$. Thus condition (14) shows that $f\colon I \to \mathbb{R}$ is strongly convex with modulus c. □

The next corollary is a version of the Hermite–Hadamard inequalities for strongly convex functions.

Corollary 10 ([5]). *If a function $f\colon I \to \mathbb{R}$ is strongly convex with modulus c then*

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2 \tag{15}$$

for all $a, b \in I$, $a < b$.

Proof. Let $X_1, X_2\colon \Omega \to I$ be random variables with the distributions $\mu_{X_1} = \delta_{(a+b)/2}$, $\mu_{X_2} = \frac{1}{2}(\delta_a + \delta_b)$ and let $Y\colon \Omega \to I$ has the uniform distribution on $[a,b]$. Then the pairs X_1, Y and Y, X_2 satisfy the assumptions of Theorem 2 and for every $f\colon I \to \mathbb{R}$

$$\mathbb{E}[f(X_1)] = f\Big(\frac{a+b}{2}\Big), \quad \mathbb{E}[f(X_2)] = \frac{f(a)+f(b)}{2}$$

and

$$\mathbb{E}[f(Y)] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Moreover,

$$\mathbb{D}^2[X_1] = 0, \quad \mathbb{D}^2[X_2] = \frac{(b-a)^2}{4},$$

and

$$\mathbb{D}^2[Y] = \frac{1}{b-a}\int_a^b x^2 dx - \Big(\frac{1}{b-a}\int_a^b x\,dx\Big)^2 = \frac{(b-a)^2}{12}.$$

Therefore, by Theorem 2, we obtain (15). □

References

[1] M. Denuit, C. Lefevre, and M. Shaked, *The s-convex orders among real random variables, with applications*, Math. Inequal. Appl. **1** (1998), 585–613.

[2] J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer–Verlag, Berlin–Heidelberg, 2001.

[3] M. Klaričić Bakula and K. Nikodem, *On the converse Jensen inequality for strongly convex functions*, J. Math. Anal. Appl. **434** (2016), 516–522.

[4] V.I. Levin and S.B. Stečkin, *Inequalities*, Amer. Math. Soc. Transl. (2) **14** (1960), 1–29.

[5] N. Merentes and K. Nikodem, *Remarks on strongly convex functions*, Aequationes Math. **80** (2010), 193–199.

[6] C.P. Niculescu and L.-E. Persson, *Convex Functions and theirApplications. A Contemporary Approach*, CMS Books in Mathematics, Vol. **23**, Springer, New York, 2006.

[7] M. Niezgoda, *An extension of Levin–Stečkin's theorem to uniformly convex and superquadratic functions*, Aequationes Math. **94** (2020), 303–321.

[8] K. Nikodem, *On strongly convex functions and related classes of functions*, in: Th.M. Rassias (ed.), *Handbook of Functional Equations. Functional Inequalities*, Springer Optimization and Its Applications, Vol. **95**, Springer, New York, 2014, Chpt.16, pp. 365–405.

[9] J. Ohlin, *On a class of measures of dispersion with application to optimal reinsurance*, ASTIN Bulletin **5** (1969), 249–266.

[10] A. Olbryś and T. Szostok, *Inequalities of the Hermite–Hadamard type involving numerical differentiation formulas*, Results Math. **67** (2015), 403–416.

[11] B.T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. **7** (1966), 72–75.

[12] T. Rajba, *On the Ohlin lemma for Hermite–Hadamard–Fejér type inequalities*, Math. Inequal. Appl. **17** (2014), 557–571.

[13] T. Rajba, *On some recent applications of stochastic convex ordering theorems to some functional inequalities for convex functions: a survey*, in: J. Brzdęk, K. Ciepliński, Th.M. Rassias (eds.), *Developments in Functional Equations and Related Topics*, Springer Optimization and Its Applications, Vol. **124**, Springer, Cham, 2017, Chpt. 11, pp. 231–274.

[14] T. Rajba and Sz. Wąsowicz, *Probabilistic characterization of strong convexity*, Opuscula Math. **31** (2011), 97–103.

[15] A.W. Roberts and D.E. Varberg, *Convex Functions*, Academic Press, New York–London, 1973.

[16] T. Szostok, *Ohlin's lemma and some inequalities of the Hermite–Hadamard type*, Aequationes Math. **89** (2015), 915–926.

[17] T. Szostok, *Inequalities for convex functions via Stieltjes integral*, Lith. Math. J. **58** (2018), 95–103.

[18] T. Szostok, *Levin Stečkin theorem and inequalities of the Hermite–Hadamard type*, arXiv preprint. Available at arXiv:1411.7708v1.

University of Bielsko-Biala
Department of Mathematics
Willowa 2
43-309 Bielsko-Biała
Poland
e-mail: knikodem@ath.bielsko.pl
e-mail: trajba@ath.bielsko.pl