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ON A SEPARATION THEOREM FOR DELTA-CONVEX FUNCTIONS

Andrzej Olbryś

Dedicated to Professor Zygfryd Kominek on the occasion of his 75th birthday

Abstract. In the present paper we establish necessary and sufficient conditions under which two functions can be separated by a delta-convex function. This separation will be understood as a separation with respect to the partial order generated by the Lorentz cone. An application to a stability problem for delta-convexity is also given.

1. Introduction

A real function f defined on a convex subset of a real linear space is called *a d.c. function (or a delta-convex function)* if it is a difference of two convex functions. Therefore many properties of f are directly inherited from those of convex functions. The class of d.c. functions is the smallest linear space containing all convex functions, in particular, it contains all $\mathcal{C}^2$ functions. D.c. functions of one real variable were considered by numerous mathematicians (see for instance [3], [5], [7], [10]). The first who considered d.c. functions of several variables was probably A.D. Alexandrov ([1], [2]), in 1949 motivated by geometry. It turns out that many operations preserve delta-convexity of

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functions. In fact, the class of delta-convex functions is not only a linear space but also an algebra and lattice. These facts were generalized in [6] by P. Hartman in 1959. He proved that a composition of two d.c. functions is d.c. He also proved that a function which is locally d.c. on an open and convex subset D of $\mathbb{R}^n$ is globally d.c. on D.

In [11], the notion of a d.c. function was extended to the notion of a d.c. mapping between arbitrary normed spaces in the following manner.

DEFINITION 1. Let X and Y be normed linear spaces, $D \subset X$ be a nonempty open and convex set and let $F\colon D \to Y$ be a mapping. We say that F is a *d.c. mapping* if there exists a continuous and convex function f on D such that $y^\star \circ F + f$ is a continuous convex function for each $y^\star \in Y^\star$, $\|y^\star\| = 1$. Every such f is called a *control function* for F.

Let us observe that the definition of delta-convex maps coincides with the definition of delta-convex functions in the case where $(Y, \|\cdot\|) = (\mathbb{R}, |\cdot|)$. Indeed, because there are only two linear functionals $id, -id \in \mathbb{R}^\star$ (identity and minus identity) from the unite sphere then $f + F$ and $f - F$ are convex functions and consequently, we have the representation

$$F = \frac{f+F}{2} - \frac{f-F}{2}.$$

In the present paper we will consider a separation problem for delta-convex functions. A corresponding problem for convex functions was solved in [4] and we will use the main results from [4] in the next section. The separation in our paper will be considered with respect to the partial order generated by so called *Lorentz cone* which appears in a natural way in the context of delta-convexity and was introduced and examined in [9]. Now, we recall only the necessary definitions and facts concerning the mentioned order. We will consider a very particular case where a normed space is $(\mathbb{R}, |\cdot|)$.

Let consider the linear space $Y := \mathbb{R} \times \mathbb{R}$, where the addition and scalar multiplication are defined coordinate-wise. Let us recall that a convex cone defined by formula

$$\mathcal{K} := \{(x,t) \in Y : \ |x| \le t\}$$

is called the *Lorentz cone*. This cone induces in Y a partial order in the following manner:

$$(X_1, x_1) \preceq_{\mathcal{K}} (X_2, x_2) \Longleftrightarrow |X_2 - X_1| \le x_2 - x_1.$$

This partial order is compatible with the linear structure of Y, i.e.

- $x \preceq_{\mathcal{K}} y \Rightarrow x + z \preceq_{\mathcal{K}} y + z$ for $x, y, z \in Y$,
- $x \preceq_{\mathcal{K}} y \Rightarrow \alpha x \preceq_{\mathcal{K}} \alpha y$ for $x, y \in Y,\ \alpha \geq 0$.

Note that, defining for given maps $F\colon D \to \mathbb{R}$ and $f\colon D \to \mathbb{R}$ (where D stands for a nonempty convex subset of a real vector space) the map $\overline{F}\colon D \to Y$ via the formula

$$\overline{F}(x) := (F(x), f(x)), \quad x \in D,$$

we can rewrite the inequality defining the notion of delta-convexity of the map F with a control function f by the formula

$$\overline{F}(tx + (1-t)y) \preceq_{\mathcal{K}} t\overline{F}(x) + (1-t)\overline{F}(y), \quad x, y \in D,\ t \in [0,1].$$

The above remark shows that the notion of delta-convexity generalizes the notion of usual convexity by replacing the classic inequality by the relation of partial order induced by the Lorentz cone. The results for usual convexity are obtained by putting $F = 0$.

In the sequel for $\overline{X}_1, \overline{X}_2 \in Y$ we will write $\overline{X}_1 \preceq \overline{X}_2$ instead of $\overline{X}_1 \preceq_{\mathcal{K}} \overline{X}_2$; moreover, we will use the following notation

$$\mathcal{C}(D) := \{(F, f) : F\colon D \to \mathbb{R} \text{ is delta-convex} \\ \text{with a control function } f\colon D \to \mathbb{R}\}.$$

A survey of results in the theory of delta-convex functions and mappings can be found in [11], in particular, the following theorem was proved in [11].

Theorem 1. *Let D be a nonempty convex subset of a real vector space and let $F\colon D \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(i) *F is a delta-convex function,*

(ii) *there exists a function $f\colon D \to \mathbb{R}$ such that for all $x, y \in D$ and $t \in [0,1]$ the inequality*

$$|tF(x) + (1-t)F(y) - F(tx + (1-t)y)| \\ \leq tf(x) + (1-t)f(y) - f(tx + (1-t)y),$$

holds,

(iii) *there exists a function $f\colon D \to \mathbb{R}$ such that for each positive integer n, for all vectors $x_1, \dots, x_n \in D$ and reals $t_1, \dots, t_n \in [0,1]$ summing up to 1 the inequality*

$$\left|\sum_{j=1}^{n} t_j F(x_j) - F\left(\sum_{j=1}^{n} t_j x_j\right)\right| \le \sum_{j=1}^{n} t_j f(x_j) - f\left(\sum_{j=1}^{n} t_j x_j\right),$$

holds.

2. Results

Below we quote explicitely two results (Theorem 1a and Theorem 1b, respectively from [4]) which will be applied in the proofs of our main results.

Theorem 2. *Real functions f and g, defined on a convex subset D of an $(n-1)$-dimensional real vector space satisfy*

$$g\left(\sum_{j=1}^{n} t_j x_j\right) \le \sum_{j=1}^{n} t_j f(x_j), \tag{1}$$

for all vectors $x_1, \dots, x_n \in D$ and reals $t_1, \dots, t_n \in [0,1]$ summing up to 1 iff there exists convex function $h\colon D \to \mathbb{R}$ such that

$$g(x) \le h(x) \le f(x), \qquad x \in D. \tag{2}$$

For infinite dimensional real vector space we have the following counterpart of the above theorem.

Theorem 3. *Real functions f and g, defined on a convex subset D of a real vector space, satisfy (1) for each positive integer n, vectors $x_1, \dots, x_n \in D$ and real numbers $t_1, \dots, t_n \in [0,1]$ summing up to 1 iff there exists a convex function $h\colon D \to \mathbb{R}$ satisfying (2).*

Theorem 3 has been generalized in [8] (cf. Theorem 3, p. 108 therein).

Our main result reads as follows.

Theorem 4. *Let D be a nonempty convex subset of an n-dimensional real linear space. Functions $F, f, G, g\colon D \to \mathbb{R}$ satisfy*

$$\left|\sum_{j=1}^{n+1} t_j F(x_j) - G\left(\sum_{j=1}^{n+1} t_j x_j\right)\right| \le \sum_{j=1}^{n+1} t_j f(x_j) - g\left(\sum_{j=1}^{n+1} t_j x_j\right), \tag{3}$$

for all $x_1, \ldots, x_{n+1} \in D,\ t_1, \ldots, t_{n+1} \in [0,1]$ *summing up to* 1 *if and only if there exists* $(H,h) \in \mathcal{C}(D)$ *such that*

$$(G(x), g(x)) \preceq (H(x), h(x)) \preceq (F(x), f(x)), \qquad x \in D. \tag{4}$$

Proof. Assume that inequality (4) holds true. Using (4), the fact that $(H,h) \in \mathcal{C}(D)$ (statement (iii) from Theorem 1) and a triangle inequality we obtain

$$\begin{aligned}
\left| \sum_{j=1}^{n+1} t_j F(x_j) - G\left(\sum_{j=1}^{n+1} t_j x_j\right) \right| &= \left| \sum_{j=1}^{n+1} t_j F(x_j) - \sum_{j=1}^{n+1} t_j H(x_j) + \sum_{j=1}^{n+1} t_j H(x_j) \right. \\
&\qquad \left. - H\left(\sum_{j=1}^{n+1} t_j x_j\right) + H\left(\sum_{j=1}^{n+1} t_j x_j\right) - G\left(\sum_{j=1}^{n+1} t_j x_j\right) \right| \\
&\le \sum_{j=1}^{n+1} t_j |F(x_j) - H(x_j)| + \left| \sum_{j=1}^{n+1} t_j H(x_j) - H\left(\sum_{j=1}^{n+1} t_j x_j\right) \right| \\
&\qquad + \left| H\left(\sum_{j=1}^{n+1} t_j x_j\right) - G\left(\sum_{j=1}^{n+1} t_j x_j\right) \right| \le \sum_{j=1}^{n+1} t_j \big(f(x_j) - h(x_j)\big) \\
&\qquad + \sum_{j=1}^{n+1} t_j h(x_j) - h\left(\sum_{j=1}^{n+1} t_j x_j\right) + h\left(\sum_{j=1}^{n+1} t_j x_j\right) - g\left(\sum_{j=1}^{n+1} t_j x_j\right) \\
&= \sum_{j=1}^{n+1} t_j f(x_j) - g\left(\sum_{j=1}^{n+1} t_j x_j\right).
\end{aligned}$$

Conversely, suppose that inequality (3) holds. Then

$$g\left(\sum_{j=1}^{n+1} t_j x_j\right) - G\left(\sum_{j=1}^{n+1} t_j x_j\right) \le \sum_{j=1}^{n+1} t_j (f(x_j) - F(x_j)),$$

and

$$g\left(\sum_{j=1}^{n+1} t_j x_j\right) + G\left(\sum_{j=1}^{n+1} t_j x_j\right) \le \sum_{j=1}^{n+1} t_j (f(x_j) + F(x_j)).$$

By Theorem 2 there exist convex functions $h_1, h_2 \colon D \to \mathbb{R}$ such that

$$g(x) + G(x) \le h_1(x) \le f(x) + F(x), \quad x \in D,$$

and

$$g(x) - G(x) \le h_2(x) \le f(x) - F(x), \quad x \in D.$$

Let define the functions $H, h \colon D \to \mathbb{R}$ by the formulas

$$H(x) := \frac{h_1(x) - h_2(x)}{2}, \qquad h(x) := \frac{h_1(x) + h_2(x)}{2}, \quad x \in D.$$

We shall show that $(H, h) \in \mathcal{C}(D)$, and, moreover,

$$(G(x), g(x)) \preceq (H(x), h(x)) \preceq (F(x), f(x)), \quad x \in D.$$

Observe that the inequality

$$tH(x) + (1-t)H(y) - H(tx + (1-t)y) \le th(x) + (1-t)h(y) - h(tx + (1-t)y),$$

is equivalent to the following one

$$th_2(x) + (1-t)h_2(y) - h_2(tx + (1-t)y) \ge 0.$$

Analogously, the inequality

$$h(tx + (1-t)y) - th(x) - (1-t)h(y) \le tH(x) + (1-t)H(y) - H(tx + (1-t)y),$$

holds if and only if

$$th_1(x) + (1-t)h_1(y) - h_1(tx + (1-t)y) \ge 0,$$

and consequently,

$$|tH(x) + (1-t)H(y) - H(tx + (1-t)y)| \le th(x) + (1-t)h(y) - h(tx + (1-t)y),$$

for all $x, y \in D, t \in [0, 1]$. Therefore $(H, h) \in \mathcal{C}(D)$.

On the other hand, since

$$g(x) + G(x) \le h_1(x), \quad g(x) - G(x) \le h_2(x), \; x \in D,$$

then by subtracting the expression $(h_1(x)-h_2(x))/2$ from the both sides of the above inequalities after easy calculations we obtain

$$\left|\frac{h_1(x)-h_2(x)}{2}-G(x)\right|\le\frac{h_1(x)+h_2(x)}{2}-g(x),\quad x\in D,$$

whence $(G,g)\preceq(H,h)$. Analogously having in mind the inequalities

$$h_1(x)\le F(x)+f(x),\quad h_2(x)\le f(x)-F(x),\ x\in D,$$

after subtracting the term $(h_1(x)-h_2(x))/2$ from their both sides we get

$$\left|\frac{h_1(x)-h_2(x)}{2}-F(x)\right|\le f(x)-\frac{h_1(x)+h_2(x)}{2},\quad x\in D,$$

and consequently $(H,h)\preceq(F,f)$, which finishes the proof. $\square$

Using similar arguments and Theorem 3 instead of Theorem 2 one can prove the following infinite-dimensional version of the previous theorem.

Theorem 5. *Let D be a nonempty convex subset of a real linear space. Functions $F,f,G,g\colon D\to\mathbb{R}$ satisfy*

$$\left|\sum_{j=1}^n t_jF(x_j)-G\left(\sum_{j=1}^n t_jx_j\right)\right|\le\sum_{j=1}^n t_jf(x_j)-g\left(\sum_{j=1}^n t_jx_j\right),$$

for each integer n and for all $x_1,\dots,x_n\in D$, $t_1,\dots,t_n\in[0,1]$ summing up to 1 if and only if there exists $(H,h)\in\mathcal{C}(D)$ such that

$$(G(x),g(x))\preceq(H(x),h(x))\preceq(F(x),f(x)),\quad x\in D.$$

As a consequence of Theorem 4 we obtain the following stability result for delta-convex functions.

Theorem 6. *Let D be a nonempty convex subset of n-dimensional real linear space and assume that $\overline{E}=(E,e)\in\mathcal{K}$ i.e. $|E|\le e$. If the functions $P,p\colon D\to\mathbb{R}$ satisfy the inequality*

$$\begin{aligned}(5)\qquad &|tP(x)+(1-t)P(y)-P(tx+(1-t)y)+E|\\ &\qquad\le tp(x)+(1-t)p(y)-p(tx+(1-t)y)+e,\end{aligned}$$

for all $x, y \in D, t \in (0,1)$, then there exists a delta-convex function $F\colon D \to \mathbb{R}$ with a control function $f\colon D \to \mathbb{R}$ such that

$$(P(x), p(x)) \preceq (F(x), f(x)) \preceq (P(x), p(x)) + n\overline{E}, \quad x \in D.$$

Proof. First, we show by induction that if P and p satisfy (5), then for all $k \in \mathbb{N}$, $x_1, \dots, x_{k+1} \in D, t_1, \dots, t_{k+1} \in (0,1)$ summing up to 1, they satisfy the inequality

$$\left| \sum_{j=1}^{k+1} t_j P(x_j) - P\left(\sum_{j=1}^{k+1} t_j x_j\right) + kE \right| \leq \sum_{j=1}^{k+1} t_j p(x_j) - p\left(\sum_{j=1}^{k+1} t_j x_j\right) + ke. \tag{6}$$

For $k = 1$ the inequality (6) coincides with (5). Suppose that (5) is true for all convex combinations with at most $k - 1 \geq 1$ points. Fix $x_1, \dots, x_{k+1} \in D$ and $t_1, \dots, t_{k+1} \in (0,1)$ summing up to 1, arbitrarily. Then

$$\begin{aligned}
&\left| \sum_{j=1}^{k+1} t_j P(x_j) - P\left(\sum_{j=1}^{k+1} t_j x_j\right) + kE \right| \qquad (7)\\
&= \left| t_{k+1} P(x_{k+1}) + (1 - t_{k+1}) P\left(\sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} x_j\right) - P\left(\sum_{j=1}^{k+1} t_j x_j\right) \right.\\
&\quad \left. + \sum_{j=1}^{k} t_j P(x_j) - (1 - t_{k+1}) P\left(\sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} x_j\right) + kE \right|\\
&\leq \left| t_{k+1} P(x_{k+1}) + (1 - t_{k+1}) P\left(\sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} x_j\right) - P\left(\sum_{j=1}^{k+1} t_j x_j\right) + E \right|\\
&\quad + (1 - t_{k+1}) \left| \sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} P(x_j) - P\left(\sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} x_j\right) + \frac{(k-1)E}{1 - t_{k+1}} \right|\\
&\leq t_{k+1} p(x_{k+1}) + (1 - t_{k+1}) p\left(\sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} x_j\right) - p\left(\sum_{j=1}^{k+1} t_j x_j\right) + e\\
&\quad + (1 - t_{k+1}) \left(\sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} p(x_j) - p\left(\sum_{j=1}^{k} \frac{t_j}{1 - t_{k+1}} x_j\right) + \frac{(k-1)e}{1 - t_{k+1}} \right)\\
&= \sum_{j=1}^{k+1} t_j p(x_j) - p\left(\sum_{j=1}^{k+1} t_j x_j\right) + ke.
\end{aligned}$$

To finish the proof it remains to apply Theorem 4 for

$$G(x) := P(x), \quad g(x) := p(x), \quad F(x) := P(x) + nE,$$
$$f(x) := p(x) + ne, \quad x \in D. \qquad \square$$

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Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: andrzej.olbrys@us.edu.pl