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D-HOMOTHETICALLY DEFORMED KENMOTSU METRIC AS A RICCI SOLITON

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Abstract. In this paper we study the nature of Ricci solitons in D-homothetically deformed Kenmotsu manifolds. We prove that η-Einstein Kenmotsu metric as a Ricci soliton remains η-Einstein under D-homothetic deformation and the scalar curvature remains constant.

1. Introduction

One of the important topics in the study of almost contact metric manifolds is the study of Ricci flow and Ricci solitons. A Ricci soliton is a Riemannian metric g on a manifold M together with a vector field V such that

$$(\mathscr{L}_{\mathrm{V}} g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

where $\mathscr{L}_{\mathrm{V}}$, S and λ denote the Lie derivative along V, Ricci tensor and a constant. A Ricci soliton is said to be shrinking or steady or expanding if λ is negative, zero or positive, respectively. A Ricci soliton is said to be a gradient Ricci soliton if the vector field V is gradient of some smooth function f on M.

Sharma ([11]) initiated the study of Ricci solitons in contact Riemannian geometry. Ghosh and Sharma ([5], [6]), Sharma ([11]) established results by considering K-contact, Kenmotsu, Sasakian and (κ,μ)-contact metrics as

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Ricci solitons. Bejan and Crasmareanu ([1]) extended the study of Ricci solitons to paracontact manifolds. De and others ([15], [8], [9]) studied Ricci solitons in f-Kenmotsu manifolds and Kenmotsu manifolds. In [10] authors analyze the behaviour of trans-Sasakian manifolds under D-homothetic deformations. Several authors, e.g. Nagaraja and Premalatha ([7]), De and Ghosh ([4]) studied the behaviour of K-contact, normal almost contact metric manifolds under D-homothetic deformations. We make use of the invariance of certain contact structures under D-homothetic deformations to study Ricci solitons.

This paper is structured as follows: after a brief review of Kenmotsu manifolds in section 2, we study D-homothetically deformed Kenmotsu metrics as Ricci solitons in section 3.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad g(X,\xi) = \eta(X), \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0,$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.1}$$

An almost contact metric manifold is said to be a Kenmotsu manifold ([2]) if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.2}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.3}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y),$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold the following relations hold ([3]):

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.4}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.5}$$

$$S(\phi X,\phi Y) = S(X,Y) + 2n\eta(X)\eta(Y),$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of type $(1,3)$ on M.

A vector field V on a Kenmotsu manifold is said to be conformal Killing vector field ([14]) if

$$(\mathscr{L}_{\mathrm{V}}g)(X,Y) = 2\rho g(X,Y), \tag{2.6}$$

where ρ is a function on the manifold.

Let (g, V, λ) be a Ricci soliton in a 3 dimensional Kenmotsu manifold M. Then from (2.6) and (1.1), we have

$$S(X,Y) = -(\lambda+\rho)g(X,Y), \tag{2.7}$$

which yields

$$QX = -(\lambda+\rho)X, \tag{2.8}$$

$$S(X,\xi) = -(\lambda+\rho)\eta(X),$$

$$r = -3(\lambda+\rho), \tag{2.9}$$

where Q is the Ricci operator and r is the scalar curvature on M.

3. Ricci solitons in Kenmotsu manifolds under D-homothetic deformations

Let (M,ϕ,ξ,η,g) be an almost contact metric manifold, where g is a Ricci soliton. The D-homothetic deformation ([13] on M is given by

$$\phi^* = \phi, \quad \xi^* = \frac{1}{a}\xi, \quad \eta^* = a\eta, \quad g^* = ag + a(a-1)\eta\otimes\eta \tag{3.1}$$

for a positive constant a. If (M,ϕ,ξ,η,g) is an almost contact metric structure with contact form η, then $(M,\phi^*,\xi^*,\eta^*,g^*)$ is also an almost contact metric structure ([13]). Now we recall the Ricci tensor of a Kenmotsu manifold transforms under a D-homothetic deformation ([10]) as

$$S^*(X,Y) = S(X,Y) + \frac{2n(a-1)}{a}\{g(X,Y) + (a-a^2-1)\eta(X)\eta(Y)\}. \tag{3.2}$$

Taking the Lie derivative of $g^* = ag + a(a-1)\eta\otimes\eta$ along V and using (3.1) and (3.2), we obtain

$$\begin{aligned}(\mathscr{L}_V g^*)(X,Y) &+ 2S^*(X,Y) + 2\lambda g^*(X,Y) \\ &= a(\mathscr{L}_V g)(X,Y) + a(a-1)\{(\mathscr{L}_V\eta)(X)\eta(Y) + \eta(X)\mathscr{L}_V\eta)(Y)\} \\ &\quad + 2S(X,Y) + \frac{4n(a-1)}{a}\{g(X,Y) + (a-a^2-1)\eta(X)\eta(Y)\} \\ &\quad + 2\lambda a\{g(X,Y) + (a-1)\eta(X)\eta(Y)\}.\end{aligned} \tag{3.3}$$

We Lie-differentiate $\eta(\xi) = 1$ along V to get

$$(\mathscr{L}_V\eta)(\xi) + \eta(\mathscr{L}_V\xi) = 0. \tag{3.4}$$

Also Lie-differentiation of $g(\xi,\xi) = 1$ along V gives

$$(\mathscr{L}_V g)(\xi,\xi) + 2\eta(\mathscr{L}_V\xi) = 0. \tag{3.5}$$

Further, setting $X = Y = \xi$ in (1.1) and using (2.5), we obtain

$$(\mathscr{L}_V g)(\xi,\xi) = 4n - 2\lambda. \tag{3.6}$$

Using (3.6), equation (3.5) yields

$$\eta(\mathscr{L}_V\xi) = \lambda - 2n. \tag{3.7}$$

Now, (3.4) yields

$$(\mathscr{L}_V\eta)(\xi) = 2n - \lambda.$$

By putting $Y = \xi$ in (1.1), we obtain

$$(\mathscr{L}_V\eta)(X) = g(X,\mathscr{L}_V\xi) - 2S(X,\xi) - 2\lambda\eta(X). \tag{3.8}$$

We know that $\mathscr{L}_V\xi = \eta(\mathscr{L}_V\xi)\xi$ ([12]) and using (2.5), (3.7) in (3.8), we get

$$(\mathscr{L}_V\eta)(X) = (2n-\lambda)\eta(X). \tag{3.9}$$

By hypothesis $(\mathscr{L}_{\mathrm{V}} g)(X,Y) = -2(S(X,Y) + \lambda g(X,Y))$ and with the use of (3.9), (3.3) reduces to

$$(\mathscr{L}_{\mathrm{V}} g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) \\ = -2(a-1)[S(X,Y) - \frac{2n}{a}\{g(X,Y) + (a-1)\eta(X)\eta(Y)\}],$$

i.e g^* is a Ricci soliton if and only if

$$S(X,Y) = \frac{2n}{a}\{g(X,Y) + (a-1)\eta(X)\eta(Y)\}. \tag{3.10}$$

Therefore, we have the following theorem.

THEOREM 3.1. *Under D-homothetic deformation, a Kenmotsu metric which is η-Einstein Ricci soliton remains η-Einstein Ricci soliton.*

Contracting (3.10), we have

$$r = \frac{2n}{a}\{2n + a\}. \tag{3.11}$$

Let us now use the formula ([11])

$$\mathscr{L}_V r = -\triangle r + 2R_{ij}R^{ij} + 2\lambda r. \tag{3.12}$$

As r is a constant, we get

$$R_{ij}R^{ij} = -\lambda r. \tag{3.13}$$

On contracting (3.2), we obtain

$$r^* = r + \frac{2n(a-1)}{a}\{2n + a - a^2\}. \tag{3.14}$$

By substituting (3.11) in (3.14), we have

$$r^* = 2n(2n + 2a - a^2). \tag{3.15}$$

Thus, we state the following:

THEOREM 3.2. *An η-Einstein Kenmotsu metric as a Ricci soliton remains η-Einstein Ricci soliton and in this case the scalar curvature of a D-homothetically deformed Kenmotsu manifold is constant.*

Using (3.11), (3.13) becomes

$$R_{ij}R^{ij} = -\frac{2n\lambda}{a}\{2n+a\}. \tag{3.16}$$

Analogously to the formula (3.12), we write

$$\mathscr{L}_V r^* = -\triangle r^* + 2R^*_{ij}(R^*)^{ij} + 2\lambda r^*.$$

From (3.15), r^* is a constant, so we get

$$R^*_{ij}(R^*)^{ij} = -\lambda r^*. \tag{3.17}$$

By making use of (3.14) and (3.11), (3.17) becomes

$$R^*_{ij}(R^*)^{ij} = -\lambda r - \frac{2n\lambda(a-1)}{a}\{2n+a-a^2\}. \tag{3.18}$$

Comparing the above with (3.2), we get

$$\begin{aligned} R^*_{ij}(R^*)^{ij} &= R_{ij}R^{ij} \\ &+ \frac{4n^2(a-1)^2}{a^2}[\{g_{i,j} + (a-a^2-1)\eta_i\eta_j\}\{g^{i,j} + (a-a^2-1)\eta^i\eta^j\}]. \end{aligned} \tag{3.19}$$

After simplification, equation (3.19) gives

$$R^*_{ij}(R^*)^{ij} = R_{ij}R^{ij} + \frac{4n^2(a-1)^2}{a^2}\{2n + a^2(a-1)^2\}. \tag{3.20}$$

In view of (3.18) and (3.20) , using (3.16), we obtain

$$\lambda = \frac{2n(1-a)\{2n+a^2(a-1)^2\}}{a\{2n+a(1-a)\}}.$$

Thus, we can state the following:

Theorem 3.3. *A Ricci soliton in a D-homothetically deformed Kenmotsu manifold is expanding for $a < 1$.*

Since in a three-dimensional Riemannian manifold the conformal curvature tensor C vanishes, we have

$$(3.21)\quad R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y],$$

where R is Riemannian curvature tensor of type (1,3).

Using (2.7), (2.8), (2.9) in (3.21) and by putting $Z = \xi$, we get

$$(3.22)\quad R(X,Y)\xi = \frac{(\lambda+\rho)}{2}\{\eta(X)Y - \eta(Y)X\}.$$

By comparing (2.4) and (3.22), we obtain

$$\lambda + \rho = 2.$$

Thus, we have

THEOREM 3.4. *If the generating vector field V is a conformal Killing vector field with associated function ρ, then the Ricci soliton in a three-dimensional Kenmotsu manifold is shrinking or expanding or steady if $\rho > 2$ or $\rho < 2$ or $\rho = 2$, respectively.*

EXAMPLE 3.1. We consider the three-dimensional manifold

$$M = \{(x,y,z) \in R^3; z \neq 0\},$$

where (x,y,z) are the standard coordinates in R^3. The vector fields

$$E_1 = e^z\frac{\partial}{\partial x},\quad E_2 = e^z\frac{\partial}{\partial y},\quad E_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, E_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_2, \phi E_2 = -E_1, \phi E_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_3) = 1, \quad \phi^2(Z) = -Z + \eta(Z)E_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus, for $E_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By Koszul's formula, we get

$$\begin{aligned} &\nabla_{E_1} E_3 = -E_1, && \nabla_{E_2} E_3 = -E_2, && \nabla_{E_3} E_3 = 0, \\ &\nabla_{E_1} E_2 = 0, && \nabla_{E_2} E_2 = E_3, && \nabla_{E_3} E_2 = 0, \\ &\nabla_{E_1} E_1 = E_3, && \nabla_{E_2} E_1 = 0, && \nabla_{E_3} E_1 = 0. \end{aligned}$$

From the above expressions it follows that the manifold satisfies (2.1), (2.2) and (2.3) for $\xi = E_3$. Hence, the manifold is a Kenmotsu manifold. With the help of the above results we can verify the following results:

$$\begin{aligned} &R(E_1, E_1)E_1 = 0, && R(E_1, E_2)E_2 = -E_1, && R(E_1, E_3)E_3 = -E_1, \\ &R(E_2, E_1)E_1 = -E_2, && R(E_2, E_2)E_2 = 0, && R(E_2, E_3)E_3 = -E_2, \\ &R(E_3, E_1)E_1 = -E_3, && R(E_3, E_2)E_2 = -E_3, && R(E_3, E_3)E_3 = 0. \end{aligned}$$

From the above expressions of the curvature tensor, we obtain the non-zero components of Ricci tensor S as follows:

$$S(E_1, E_1) = g(R(E_1, E_2)E_2, E_1) + g(R(E_1, E_3)E_3, E_1) = -2.$$

Similarly, we have

$$S(E_2, E_2) = S(E_3, E_3) = -2.$$

For $V = e^{-z}E_3$, we have

$$(\mathscr{L}_{\rm V} g)(E_i, E_i) = -2e^{-z}. \tag{3.23}$$

Now, by taking $X = Y = E_i$ in (1.1), where $i = 1, 2, 3$, and by virtue of the above equations, we have that g is a Ricci soliton for $\lambda = e^{-z} + 2$. Here λ is positive for all z. Hence, the soliton is expanding.

Equation (3.23) can be written as $(\mathscr{L}_{\rm V} g)(E_i, E_i) = 2\rho g(E_i, E_i)$, where $\rho = -e^{-z}$, i.e. $\lambda + \rho = 2$.

In this example $\rho < 2$ for all values of z. This verifies Theorem 3.4.

Suppose (g^*, V, λ) is a Ricci soliton, where g^* is obtained by D-homothetic change of a three-dimensional Kenmotsu metric g. Then

$$(\mathscr{L}_{\rm V} g^*)(X, Y) + 2S^*(X, Y) + 2\lambda g^*(X, Y) = 0.$$

Now, by taking the Lie derivative of $g^* = ag + a(a-1)\eta \otimes \eta$ along V and using (3.9), we obtain

$$\begin{aligned} a\{(\mathscr{L}_{\rm V} g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y)\} + 4a(a-1)\eta(X)\eta(Y) \\ + 2(1-a)S(X, Y) + \frac{4(a-1)}{a}\{g(X, Y) + (a - a^2 - 1)\eta(X)\eta(Y)\} = 0. \end{aligned} \tag{3.24}$$

By using (1.1) and (2.7), (3.24) becomes

$$\{\lambda + \rho + \frac{2}{a}\}g(X, Y) + \{2 - \frac{2}{a}\}\eta(X)\eta(Y) = 0. \tag{3.25}$$

Putting $X = Y = \xi$ in (3.25), we get

$$\lambda + \rho = -2.$$

THEOREM 3.5. *Under D-homothetic deformation, Ricci soliton in a three-dimensional Kenmotsu manifold with the generating vector field V as a conformal Killing vector field and ρ as associated function is expanding or shrinking or steady if $\rho < -2$ or $\rho > -2$ or $\rho = -2$, respectively.*

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