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ON JACOBSTHAL AND JACOBSTHAL-LUCAS HYBRID NUMBERS

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Abstract. The hybrid numbers are generalization of complex, hyperbolic and dual numbers. In this paper we consider special kinds of hybrid numbers, namely the Jacobsthal and the Jacobsthal-Lucas hybrid numbers and we give some their properties.

1. Introduction

Let us consider the set $\mathbb{K}$ of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\epsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ϵ, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1,\ \epsilon^2 = 0,\ \mathbf{h}^2 = 1 \tag{1.1}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \epsilon + \mathbf{i}. \tag{1.2}$$

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Let $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\epsilon + d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\epsilon + d_2\mathbf{h}$ be any two hybrid numbers. We define equality, addition, substraction and multiplication by scalar in the following way:

$$\mathbf{Z}_1 = \mathbf{Z}_2 \text{ only if } a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2,\ d_1 = d_2 \quad \text{(equality)}$$
$$\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\epsilon + (d_1 + d_2)\mathbf{h} \quad \text{(addition)}$$
$$\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\epsilon + (d_1 - d_2)\mathbf{h} \quad \text{(substraction)}$$
$$s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\epsilon + sd_1\mathbf{h} \quad \text{(multiplication by scalar } s \in \mathbb{R}).$$

Using equalities (1.1) and (1.2) we define multiplication of hybrid numbers. Moreover, by formulas (1.1) and (1.2) the product of any two hybrid operators can be calculated. For example, to find $\mathbf{i}\epsilon$ we can multiply $\mathbf{ih} = \epsilon + \mathbf{i}$ by $\mathbf{i}$ from the left. We obtain $\mathbf{i}^2\mathbf{h} = \mathbf{i}\epsilon + \mathbf{i}^2$ and after calculation $\mathbf{i}\epsilon = 1 - \mathbf{h}$. If we proceed in a similar way, we get the following multiplication table.

Table 1. The hybrid numbers multiplication

$\cdot$	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\epsilon+\mathbf{i}$
ϵ	$\mathbf{h}+1$	0	$-\epsilon$
$\mathbf{h}$	$-\epsilon-\mathbf{i}$	ϵ	1

From the above rules the multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. Note that multiplication operation in the hybrid numbers is associative, but not commutative.

The conjugate of a hybrid number $\mathbf{Z}$ is defined by

$$\overline{\mathbf{Z}} = \overline{a + b\mathbf{i} + c\epsilon + d\mathbf{h}} = a - b\mathbf{i} - c\epsilon - d\mathbf{h}.$$

The real number

$$C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2$$

is called the character of the hybrid number $\mathbf{Z}$.

The hybrid numbers were introduced by Özdemir in [5] as a generalization of complex, hyperbolic and dual numbers.

A special kind of hybrid numbers, namely Horadam hybrid numbers, were introduced and studied in [6]. For positive integer n, the nth Horadam number W_n is defined by the recurrence relation of the form $W_n = p \cdot W_{n-1} - q \cdot W_{n-2}$ with the initial values W_0, W_1, where $p,\ q \in \mathbb{Z}$ and $W_0,\ W_1 \in \mathbb{R}$. For $W_0 = 0$, $W_1 = 1$, $p = 1$ and $q = -2$ we obtain the nth Jacobsthal number J_n. For $W_0 = 2$, $W_1 = 1$, $p = 1$ and $q = -2$ we have the nth Jacobsthal-Lucas number j_n. In this paper we describe some properties of Jacobsthal hybrid numbers and Jacobsthal-Lucas hybrid numbers.

2. The Jacobsthal and Jacobsthal-Lucas numbers

Let $n \geq 0$ be an integer. The nth Jacobsthal number J_n is defined recursively by

$$J_n = J_{n-1} + 2J_{n-2}$$

for $n \geq 2$ with $J_0 = 0,\ J_1 = 1$.

Then the direct formula for nth Jacobsthal number has the form

$$J_n = \frac{1}{3}(2^n - (-1)^n). \tag{2.1}$$

The equality (2.1) also is named as the Binet formula for Jacobsthal numbers.

For $n \geq 2$ the nth Jacobsthal-Lucas number j_n is defined also by the second order linear recurrence relation

$$j_n = j_{n-1} + 2j_{n-2}$$

with $j_0 = 2,\ j_1 = 1$. The Binet formula for Jacobsthal-Lucas numbers has the form

$$j_n = 2^n + (-1)^n. \tag{2.2}$$

Jacobsthal sequence and Jacobsthal-Lucas sequence has the first few elements $0, 1, 1, 3, 5, 11, 21, 43, \ldots$ and $2, 1, 5, 7, 17, 31, 65, 127, \ldots$, respectively.

The Jacobsthal numbers and Jacobsthal-Lucas numbers were introduced in [2] and [3], respectively. In [3] many identities for J_n and j_n were given. We list them as follow:

$$j_{n+1} + j_n = 3\left(J_{n+1} + J_n\right) = 3 \cdot 2^n, \tag{2.3}$$

$$j_{n+1} - j_n = 3\left(J_{n+1} - J_n\right) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \tag{2.4}$$

$$\begin{aligned} j_{n+r} + j_{n-r} &= 3\left(J_{n+r} + J_{n-r}\right) + 4(-1)^{n-r} \\ &= 2^{n-r}\left(2^{2r} + 1\right) + 2(-1)^{n-r}, \end{aligned} \tag{2.5}$$

$$j_{n+r} - j_{n-r} = 3\left(J_{n+r} - J_{n-r}\right) = 2^{n-r}\left(2^{2r} - 1\right), \tag{2.6}$$

$$J_n + j_n = 2J_{n+1}, \tag{2.7}$$

$$J_{n+2} + 2J_n = j_{n+1}, \tag{2.8}$$

$$3J_n + j_n = 2^{n+1}, \tag{2.9}$$

$$\sum_{l=0}^{n} J_l = \frac{J_{n+2} - 1}{2}, \tag{2.10}$$

$$\sum_{l=0}^{n} j_l = \frac{j_{n+2} - 1}{2}. \tag{2.11}$$

3. The Jacobsthal and Jacobsthal-Lucas hybrid numbers

Let $n \geq 0$ be an integer. The nth Jacobsthal hybrid number JH_n and nth Jacobsthal-Lucas hybrid number jH_n are defined by

$$JH_n = J_n + J_{n+1}\mathbf{i} + J_{n+2}\epsilon + J_{n+3}\mathbf{h}, \tag{3.1}$$

$$jH_n = j_n + j_{n+1}\mathbf{i} + j_{n+2}\epsilon + j_{n+3}\mathbf{h}, \tag{3.2}$$

respectively. From the above equalities we can write initial Jacobsthal and Jacobsthal-Lucas hybrid numbers, namely

$$\begin{aligned}
JH_0 &= \mathbf{i} + \epsilon + 3\mathbf{h}, \\
JH_1 &= 1 + \mathbf{i} + 3\epsilon + 5\mathbf{h}, \\
JH_2 &= 1 + 3\mathbf{i} + 5\epsilon + 11\mathbf{h}, \\
&\vdots \\
jH_0 &= 2 + \mathbf{i} + 5\epsilon + 7\mathbf{h}, \\
jH_1 &= 1 + 5\mathbf{i} + 7\epsilon + 17\mathbf{h}, \\
jH_2 &= 5 + 7\mathbf{i} + 17\epsilon + 31\mathbf{h}, \\
&\vdots
\end{aligned}$$

The Jacobsthal hybrid numbers were studied in [6]. In particular the character and the Binet formula of the Jacobsthal hybrid number were given.

Theorem 3.1 ([6]). *Let $n \geq 0$ be an integer. Then*

$$C(JH_n) = -3J_n^2 - 10J_{n+1}^2 - 16J_nJ_{n+1}.$$

Theorem 3.2 ([6]). *Let $n \geq 0$ be an integer. Then*

$$JH_n = \frac{1}{3}2^n\left(1 + 2\mathbf{i} + 4\epsilon + 8\mathbf{h}\right) - \frac{1}{3}(-1)^n\left(1 - \mathbf{i} + \epsilon - \mathbf{h}\right).$$

For the Jacobsthal-Lucas hybrid numbers we can prove:

Theorem 3.3. *Let $n \geq 0$ be an integer. Then*

$$C(jH_n) = -3j_n^2 - 10j_{n+1}^2 - 16j_nj_{n+1}.$$

Proof. Let $j_{n+2} = j_{n+1} + 2j_n$, $j_{n+3} = j_{n+2} + 2j_{n+1} = 3j_{n+1} + 2j_n$ and $C(jH_n) = j_n^2 + j_{n+1}^2 - 2j_{n+1}j_{n+2} - j_{n+3}^2$. Then

$$C(jH_n) = j_n^2 + j_{n+1}^2 - 2j_{n+1}\left(j_{n+1} + 2j_n\right) - \left(3j_{n+1} + 2j_n\right)^2$$

and by simple calculations the result follows. □

Theorem 3.4. *Let $n \geq 0$ be an integer. Then*

$$jH_n = 2^n\left(1 + 2\mathbf{i} + 4\epsilon + 8\mathbf{h}\right) + (-1)^n\left(1 - \mathbf{i} + \epsilon - \mathbf{h}\right). \tag{3.3}$$

Proof. Using the definition of Jacobsthal-Lucas hybrid number (3.2) and the Binet formula for the Jacobsthal-Lucas numbers (2.2) we have

$$\begin{aligned} jH_n = {} & (2^n + (-1)^n) + \left(2^{n+1} + (-1)^{n+1}\right)\mathbf{i} \\ & + \left(2^{n+2} + (-1)^{n+2}\right)\epsilon + \left(2^{n+3} + (-1)^{n+3}\right)\mathbf{h} \end{aligned}$$

and after simple calculations we obtain (3.3). □

In [6] the ordinary generating function for the Jacobsthal hybrid numbers also was given.

Theorem 3.5 ([6]). *The generating function for the Jacobsthal hybrid number sequence $\{JH_n\}$ is*

$$\sum_{n=0}^{\infty} JH_nt^n = \frac{JH_0 + t(JH_1 - JH_0)}{1 - t - 2t^2} = \frac{(\mathbf{i} + \epsilon + 3\mathbf{h}) + t(1 + 2\epsilon + 2\mathbf{h})}{1 - t - 2t^2}.$$

THEOREM 3.6. *The generating function for the Jacobsthal-Lucas hybrid number sequence* $\{jH_n\}$ *is*

$$\sum_{n=0}^{\infty} jH_n t^n = \frac{jH_0 + t(jH_1 - jH_0)}{1 - t - 2t^2}$$
$$= \frac{(2 + \mathbf{i} + 5\epsilon + 7\mathbf{h}) + t(-1 + 4\mathbf{i} + 2\epsilon + 10\mathbf{h})}{1 - t - 2t^2}.$$

PROOF. Assuming that the generating function of the Jacobsthal-Lucas hybrid number sequence $\{jH_n\}$ has the form $G(t) = \sum_{n=0}^{\infty} jH_n t^n$, we obtain that

$$\begin{aligned}(1 - t - 2t^2)G(t) &= (1 - t - 2t^2) \cdot (jH_0 + jH_1 t + jH_2 t^2 + \ldots) \\ &= jH_0 + jH_1 t + jH_2 t^2 + \ldots + \\ &\quad - jH_0 t - jH_1 t^2 - jH_2 t^3 - \ldots + \\ &\quad - 2jH_0 t^2 - 2jH_1 t^3 - 2jH_2 t^4 + \ldots \\ &= jH_0 + t(jH_1 - jH_0),\end{aligned}$$

since $jH_n = jH_{n-1} + 2jH_{n-2}$ and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, $jH_0 = 2 + \mathbf{i} + 5\epsilon + 7\mathbf{h}$ and $jH_1 - JH_0 = -1 + 4\mathbf{i} + 2\epsilon + 10\mathbf{h}$. □

4. Properties of Jacobsthal and Jacobsthal-Lucas hybrid numbers

In this section we give some identities for Jacobsthal hybrid numbers and Jacobsthal-Lucas hybrid numbers.

Using (2.3)–(2.4) and (3.1)–(3.2) it immediately follows

THEOREM 4.1. *Let* $n \geq 0$. *Then*

(i) $JH_{n+1} + JH_n = 2^n(1 + 2\mathbf{i} + 4\epsilon + 8\mathbf{h})$,
(ii) $jH_{n+1} + jH_n = 3 \cdot 2^n(1 + 2\mathbf{i} + 4\epsilon + 8\mathbf{h})$,
(iii) $JH_{n+1} - JH_n = \frac{1}{3}\left[2^n(1 + 2\mathbf{i} + 4\epsilon + 8\mathbf{h}) + 2(-1)^n(1 - \mathbf{i} + \epsilon - \mathbf{h})\right]$,
(iv) $jH_{n+1} - jH_n = 2^n(1 + 2\mathbf{i} + 4\epsilon + 8\mathbf{h}) - 2(-1)^n(1 - \mathbf{i} + \epsilon - \mathbf{h})$.

In the same way, using (2.5)–(2.6) and (3.1)–(3.2), we can prove

THEOREM 4.2. *Let $n \geq r \geq 0$. Then*

(i) $JH_{n+r} + JH_{n-r} = \frac{1}{3}\left[2^{n-r}\left(2^{2r}+1\right)(1+2\mathbf{i}+4\epsilon+8\mathbf{h}) - 2(-1)^{n-r}(1-\mathbf{i}+\epsilon-\mathbf{h})\right]$

(ii) $jH_{n+r} + jH_{n-r} = 2^{n-r}\left(2^{2r}+1\right)(1+2\mathbf{i}+4\epsilon+8\mathbf{h}) + 2(-1)^{n-r}(1-\mathbf{i}+\epsilon-\mathbf{h})$

(iii) $JH_{n+r} - JH_{n-r} = \frac{1}{3}\cdot 2^{n-r}\left(2^{2r}-1\right)(1+2\mathbf{i}+4\epsilon+8\mathbf{h})$,

(iv) $jH_{n+r} - jH_{n-r} = 2^{n-r}\left(2^{2r}-1\right)(1+2\mathbf{i}+4\epsilon+8\mathbf{h})$.

Using (2.7)–(2.9) and (3.1)–(3.2) one can easily obtain

THEOREM 4.3. *Let $n \geq 0$. Then*

(i) $JH_n + jH_n = 2JH_{n+1}$,

(ii) $JH_{n+2} + 2JH_n = jH_{n+1}$,

(iii) $3JH_n + jH_n = 2^{n+1}(1+2\mathbf{i}+4\epsilon+8\mathbf{h})$.

Now we give formulas for the sum of Jacobsthal hybrid numbers and Jacobsthal-Lucas hybrid numbers.

THEOREM 4.4. *Let $n \geq 0$. Then*

(i) $\sum_{l=0}^{n} JH_l = \frac{JH_{n+2} - JH_1}{2}$,

(ii) $\sum_{l=0}^{n} jH_l = \frac{jH_{n+2} - jH_1}{2}$.

PROOF. (i) Using (2.10) and (3.1) we have

$$\begin{aligned}
\sum_{l=0}^{n} JH_l &= JH_0 + JH_1 + \ldots + JH_n \\
&= (J_0 + J_1\mathbf{i} + J_2\epsilon + J_3\mathbf{h}) + (J_1 + J_2\mathbf{i} + J_3\epsilon + J_4\mathbf{h}) \\
&\quad + \ldots + (J_n + J_{n+1}\mathbf{i} + J_{n+2}\epsilon + J_{n+3}\mathbf{h}) \\
&= (J_0 + J_1 + \ldots + J_n) + (J_1 + J_2 + \ldots + J_{n+1} + J_0 - J_0)\mathbf{i} \\
&\quad + (J_2 + J_3 + \ldots + J_{n+2} + J_0 + J_1 - J_0 - J_1)\epsilon \\
&\quad + (J_3 + J_4 + \ldots + J_{n+3} + J_0 + J_1 + J_2 - J_0 - J_1 - J_2)\mathbf{h} \\
&= \frac{J_{n+2}-1}{2} + \frac{J_{n+3}-1}{2}\mathbf{i} + \frac{J_{n+4}-3}{2}\epsilon + \frac{J_{n+5}-5}{2}\mathbf{h} \\
&= \frac{JH_{n+2} - (1+\mathbf{i}+3\epsilon+5\mathbf{h})}{2}
\end{aligned}$$

which ends the proof.

(ii) Prooving analogously as above, using (2.11) and (3.2) the formula (ii) follows, which ends the proof. □

5. Concluding

In the literature there are some generalizations of Jacobsthal numbers. In [7] Uygun defined (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences. Distinct properties of k-Jacobsthal and k-Jacobsthal Lucas numbers we can found in [1], [4], [8]. Using these numbers we can define new types of hybrid numbers and their properties can be studied.

Compliance with Ethical Standards. Conflict of Interest: The authors declare that they have no conflict of interest.

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