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A LEVI–CIVITA EQUATION ON MONOIDS, TWO WAYS

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Abstract. We consider the Levi–Civita equation

$$f(xy) = g_1(x)h_1(y) + g_2(x)h_2(y)$$

for unknown functions $f, g_1, g_2, h_1, h_2 \colon S \to \mathbb{C}$, where S is a monoid. This functional equation contains as special cases many familiar functional equations, including the sine and cosine addition formulas. In a previous paper we solved this equation on groups and on monoids generated by their squares under the assumption that f is central. Here we solve the equation on monoids by two different methods. The first method is elementary and works on a general monoid, assuming only that the function f is central. The second way uses representation theory and assumes that the monoid is commutative. The solutions are found (in both cases) with the help of the recently obtained solution of the sine addition formula on semigroups. We also find the continuous solutions on topological monoids.

1. Introduction

Let S be a semigroup and $\mathbb{C}$ the set of complex numbers. Our main focus is the Levi–Civita functional equation

$$f(xy) = g_1(x)h_1(y) + g_2(x)h_2(y), \quad x, y \in S, \tag{1.1}$$

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for unknown functions $f, g_1, g_2, h_1, h_2 \colon S \to \mathbb{C}$. This equation contains as special cases both the sine and cosine addition formulas, respectively

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in S, \tag{1.2}$$

and

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in S, \tag{1.3}$$

which play important roles in the solution of (1.1).

General methods have been developed to describe the forms of the solutions of Levi–Civita functional equations

$$f(xy) = \sum_{j=1}^{n} g_j(x)h_j(y), \quad x, y \in S,$$

of which (1.1) is the case $n = 2$. These methods use representation theory (see [6], [7, Chapter 5]) or, if S is an Abelian group, spectral synthesis (see [9, Chapter 10]). For either method, further calculations are needed to get explicit solution formulas. For small values of n it may be more efficient to solve the given equation by an ad hoc method.

In [1] we solved (1.1) by elementary methods, assuming that S is a group or a monoid generated by its squares and that f is central. Here we solve (1.1) in two different ways. The first method is elementary, as in [1], solving (1.1) on a general monoid assuming that the function f is central. The second method uses representation theory, applying results from the author's joint paper [5] with Che Tat Ng, and solves (1.1) on a general commutative monoid.

The outline is as follows. Sections 2 and 3 introduce notation and terminology, as well as stating known results about (1.2) and (1.3) and making a small extension of one of them. In section 4 we use elementary methods to find the solutions of (1.1) for central f on monoids. The process is simplified relative to the method used in [1]. In section 5 we use representation theory to get the solutions of (1.1) on commutative monoids. The main results are Theorems 4.4 and 5.3. Both methods rely on a recent result (Proposition 3.1) giving the solutions of the sine addition formula (1.2) on semigroups. Some examples applying the results are given in section 6.

Generally our results are presented in their topological versions, but one may choose the discrete topology.

2. Notation and terminology

Throughout this paper S denotes a semigroup. We call S a *monoid* (with identity e) if there exists an $e \in S$ such that $ex = xe = x$ for all $x \in S$.

An *additive function* on S is a homomorphism from S into $(\mathbb{C}, +)$.

A *multiplicative function* on S is a homomorphism from S into $(\mathbb{C}, \cdot)$. If $\chi \colon S \to \mathbb{C}$ is multiplicative and $\chi \neq 0$, then we say χ is an *exponential* on S. We define the *nullspace* of a multiplicative χ by

$$I_\chi := \{x \in S \mid \chi(x) = 0\}.$$

If $I_\chi \neq \emptyset$ then it is a two-sided ideal of S and is called the *null ideal* of χ.

An ideal $I \subset S$ is said to be a *prime ideal* if $I \neq S$ and whenever $xy \in I$ it follows that either $x \in I$ or $y \in I$ (so $S \setminus I$ is a nonempty subsemigroup of S). There is a very close connection between prime ideals and exponentials on semigroups. For any exponential χ it is easy to see that if $I_\chi \neq \emptyset$ then I_χ is a prime ideal. Conversely, if I is any prime ideal of S and we define $\chi(x) = 0$ for $x \in I$, $\chi(x) = 1$ for $x \in S \setminus I$, then $\chi \colon S \to \mathbb{C}$ is an exponential with null ideal $I_\chi = I$.

For any subset $T \subseteq S$ we define $T^2 := \{t_1 t_2 \mid t_1, t_2 \in T\}$. In addition to the nullspace I_χ for a multiplicative $\chi \colon S \to \mathbb{C}$, the subset

$$P_\chi := \{p \in I_\chi \setminus I_\chi^2 \mid up, pv, upv \in I_\chi \setminus I_\chi^2 \text{ for all } u, v \in S \setminus I_\chi\}$$

also plays an important role in our story.

A function F on S is said to be *central* if $F(xy) = F(yx)$ for all $x, y \in S$.

That a function F is *nonzero* means $F \neq 0$.

For a topological space X, let $C(X)$ denote the algebra of continuous functions from X into $\mathbb{C}$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

3. Preliminary results

The following is [4, Theorem 3.1].

Proposition 3.1. *Let S be a topological semigroup, and suppose $f, g \colon S \to \mathbb{C}$ satisfy the sine addition law (1.2) with $f \neq 0$ and $f \in C(S)$. Then the pair (f, g) belongs to one of the following families, where $\chi_1, \chi_2 \in C(S)$ are multiplicative functions.*

(a) *For* $\chi_1 \neq \chi_2$ *there exists* $b \in \mathbb{C}^*$ *such that* $f = b(\chi_1 - \chi_2)$ *and* $g = (\chi_1 + \chi_2)/2$.
(b) *For* $\chi_1 = \chi_2 =: \chi$, *we have* $g = \chi$ *and*

$$f(x) = \begin{cases} A(x)\chi(x) & \text{for } x \in S \setminus I_\chi, \\ \rho(x) & \text{for } x \in P_\chi, \\ 0 & \text{for } x \in I_\chi \setminus P_\chi, \end{cases} \tag{3.1}$$

where $A \in C(S \setminus I_\chi)$ *is additive and* $\rho \in C(P_\chi)$. *In addition we have the following conditions.*

(i) $f(qt) = f(tq) = 0$ *for all* $q \in I_\chi \setminus P_\chi$ *and* $t \in S \setminus I_\chi$.
(ii) *If* $x \in \{up, pv, upv\}$ *for* $p \in P_\chi$ *and* $u, v \in S \setminus I_\chi$, *then* $x \in P_\chi$ *and we have respectively* $\rho(x) = \rho(p)\chi(u)$, $\rho(x) = \rho(p)\chi(v)$, *or* $\rho(x) = \rho(p)\chi(uv)$.

(c) *For* $\chi_1 = \chi_2 = 0$, *we have* $g = 0$, $S \neq S^2$, *and*

$$f(x) = \begin{cases} f_0(x) & \text{for } x \in S \setminus S^2, \\ 0 & \text{for } x \in S^2, \end{cases} \tag{3.2}$$

where $f_0 \in C(S \setminus S^2)$ *is an arbitrary nonzero function.*

Conversely, if the pair (f, g) *is given by the formulas in* (a), (b) *with conditions* (i) *and* (ii), *or* (c), *then* (f, g) *satisfies* (1.2).

Since functions having the form of f in part (b) above will occur repeatedly in this article, we introduce the following notation for convenience.

Notation 3.2. Let $\chi \colon S \to \mathbb{C}$ be an exponential. We denote by $\Phi \colon S \to \mathbb{C}$ any solution of the special sine addition law

$$\Phi(xy) = \Phi(x)\chi(y) + \chi(x)\Phi(y), \quad x, y \in S. \tag{3.3}$$

Usage of this notation means that any function labeled Φ has the form of f given in (3.1) and carries all the properties attached to f in part (b) of Proposition 3.1.

Note that if S has no prime ideals then $\chi(x) \neq 0$ for all $x \in S$. In this case, dividing (3.3) by $\chi(xy)$ yields that Φ/χ is an additive function A on S and we have $\Phi = A\chi$. This is the case not only on groups but also on other semigroups such as $((0,1), \cdot)$ and $(\mathbb{N}, +)$.

We will need the following small extension of Proposition 3.1.

COROLLARY 3.3. *Let S be a topological semigroup, and suppose $f, g\colon S \to \mathbb{C}$ satisfy*

$$f(xy) = f(x)g(y) + g(x)f(y) + cf(x)f(y), \quad x, y \in S, \tag{3.4}$$

with $f \neq 0$, $f \in C(S)$, and $c \in \mathbb{C}$. Then f, g have the following forms, where $\chi, \chi_1, \chi_2 \in C(S)$ are multiplicative functions, $b \in \mathbb{C}^*$, $\Phi\colon S \to \mathbb{C}$ (as defined in Notation 3.2*) is continuous, and $f_0 \in C(S \setminus S^2)$ is an arbitrary nonzero function.*

(a) *For $\chi_1 \neq \chi_2$,*

$$f = b(\chi_1 - \chi_2), \quad g = \frac{1}{2}(\chi_1 + \chi_2) - \frac{c}{2}f.$$

(b) *For $\chi \neq 0$ we have $f = \Phi$ and $g = \chi - \frac{c}{2}\Phi$.*

(c) *For $S \neq S^2$, f has the form* (3.2) *and $g = -\frac{c}{2}f$.*

Conversely, any pair (f, g) given by the formulas in (a), (b), *or* (c) *satisfies* (3.4).

PROOF. We begin by re-writing (3.4) as

$$f(xy) = f(x)g'(y) + g'(x)f(y), \quad x, y \in S,$$

where $g'\colon S \to \mathbb{C}$ is defined by $g' := g + \frac{c}{2}f$. Thus (f, g') is a solution of the sine addition formula (1.2). Conversely, if $f, g'\colon S \to \mathbb{C}$ satisfy (1.2) and $g = g' - \frac{c}{2}f$ then f, g satisfy (3.4). The rest is Proposition 3.1. □

The next result is essentially [3, Theorem 3.2] (with a small technical improvement in part (c) as stated in [4]).

PROPOSITION 3.4. *Let S be a topological semigroup, and suppose $g, f \in C(S)$ satisfy the cosine addition law* (1.3)*. Then the pair (g, f) belongs to one of the following families, where $\chi, \chi_1, \chi_2 \in C(S)$ are multiplicative functions with $\chi \neq 0$, $\chi_1 \neq \chi_2$, $\Phi \in C(S)$, and $f_0 \in C(S \setminus S^2)$ is an arbitrary nonzero function.*

(a) $g = f = 0$.

(b) $g = \dfrac{c^{-1}\chi_1 + c\chi_2}{c^{-1} + c}$ *and* $f = \dfrac{\chi_1 - \chi_2}{i(c^{-1} + c)}$, *where $c \in \mathbb{C}^* \setminus \{\pm i\}$.*

(c) $g = \chi \pm \Phi$ *and* $f = \Phi$.

(d) *If $S \neq S^2$, then $g = \pm f$ and f has the form* (3.2).

Conversely, the pair (g, f) in each family is a solution of (1.3).

The following observation will be useful.

Remark 3.5. Let S be a semigroup, let $\chi\colon S \to \mathbb{C}$ be an exponential, and let $\Phi\colon S \to \mathbb{C}$ be as defined in Notation 3.2 with $\Phi \neq 0$. Then it is not difficult to see that $\{\Phi, \chi\}$ is linearly independent (see [4, Lemma 5.1(b)]).

4. Solution of (1.1) by elementary methods

We follow the general plan of sections 4 and 5 in [1]. The first step is to consider the functional equation

$$f(xy) = f(x)h(y) + g(x)f(y), \quad x, y \in S, \tag{4.1}$$

with $g \neq h$ so that it differs from the sine addition law. As pointed out in [8, Example 1], even when S is a group the functional equation (4.1) may have non-central solutions. Thus there may exist solutions which cannot be expressed in terms of multiplicative functions and functions of the form Φ as defined in Notation 3.2. That is why we assume f is central here.

Lemma 4.1. *Let S be a topological semigroup, and let $f, g, h\colon S \to \mathbb{C}$ satisfy* (4.1) *with $f \neq 0$, $f \in C(S)$, $g \neq h$, and f central. The solutions are given by the following families, where $\chi, \chi_1, \chi_2 \in C(S)$ are multiplicative, $\Phi \in C(S)$, $f_0 \in C(S \setminus S^2)$ is an arbitrary nonzero function, and $b, c \in \mathbb{C}^*$.*
(a) *For $\chi_1 \neq \chi_2$,*

$$f = b(\chi_1 - \chi_2), \quad h = \frac{1}{2}(\chi_1 + \chi_2) - \frac{c}{2}f, \quad g = \frac{1}{2}(\chi_1 + \chi_2) + \frac{c}{2}f.$$

(b) *For $\chi \neq 0$, we have $f = \Phi$, $h = \chi - \frac{c}{2}\Phi$, and $g = \chi + \frac{c}{2}\Phi$.*
(c) *For $S \neq S^2$, f has the form* (3.2), *$h = -\frac{c}{2}f$, and $g = \frac{c}{2}f$.*

Proof. As in the proof of [1, Corollary 4.1], the interchange of x and y in (4.1) and comparison of the results leads to $g = h + cf$ for some $c \in \mathbb{C}^*$, since $f \neq 0$ and $g \neq h$. Consequently the pair (f, h) is a solution of (3.4). Applying Corollary 3.3 we get the solutions shown above, which clearly satisfy (4.1). $\square$

The next step is to treat the partial Pexiderization

$$g(xy) = g(x)g(y) + h(x)k(y), \quad x, y \in S, \tag{4.2}$$

of the cosine addition law.

Lemma 4.2. *Let S be a topological semigroup. The solutions $g, h, k \in C(S)$ of (4.2) with g central and $g \neq 0$ are the following, where $\chi, \chi_1, \chi_2 \in C(S)$ are multiplicative functions, $\Phi \in C(S)$, and $b \in \mathbb{C}^*$.*

(i) *$g = \chi \neq 0$, $h = 0$, k arbitrary in $C(S)$.*

(ii) *$g = \chi \neq 0$, $k = 0$, h arbitrary in $C(S)$.*

(iii) *For $\chi_1 \neq \chi_2$ and $c \in \mathbb{C}^* \setminus \{\pm i\}$,*

$$g = \frac{c^{-1}\chi_1 + c\chi_2}{c^{-1} + c}, \quad h = \frac{b(\chi_1 - \chi_2)}{c^{-1} + c}, \quad k = \frac{\chi_1 - \chi_2}{b(c^{-1} + c)}.$$

(iv) *For $\chi \neq 0$ we have*

$$g = \chi \pm \Phi, \quad h = ib\Phi, \quad k = i\Phi/b.$$

(v) *If $S \neq S^2$, then $g \in C(S)\setminus\{0\}$ has the form (3.2), $h = ibg$, and $k = ig/b$.*

Proof. As in the proof of [1, Corollary 4.2], the interchange of x and y in (4.2) and comparison of the results yields $h(x)k(y) = h(y)k(x)$ for all $x, y \in S$. This leads to solution (i) or (ii) if $h = 0$ or $k = 0$. Henceforth we assume that $h \neq 0$ and $k \neq 0$, so that $h = ak$ for some constant $a \in \mathbb{C}^*$. Now (4.2) becomes

$$g(xy) = g(x)g(y) + ak(x)k(y), \quad x, y \in S.$$

Putting $f = -ibk$ where $b^2 = a$, we arrive at the cosine addition formula (1.3) and take the solutions for f, g from Proposition 3.4. In each case note that the solutions for k, h are given by $k = if/b$ and $h = iaf/b = ibf$.

Case (a) of Proposition 3.4 does not occur here since $g \neq 0$. In case (b) we get g of the form shown in (iii) and $f = (\chi_1 - \chi_2)/[i(c^{-1} + c)]$ for $c \in \mathbb{C}^* \setminus \{\pm i\}$. The forms shown for h, k in (iii) follow. Cases (c) and (d) lead respectively to the solution forms in (iv) and (v). All solutions are easily verified by substitution. □

Our third lemma (generalizing [1, Lemma 5.1]) concerns a functional equation which is a partial Pexiderization of both the sine and cosine addition formulas.

Lemma 4.3. *Let S be a topological semigroup, and suppose $f, g, h, k \in C(S)$ where $f \neq 0$, $g \neq 0$, f is central, and $\{h, k\}$ is linearly independent. Then the solutions of*

$$f(xy) = f(x)h(y) + g(x)k(y), \quad x, y \in S, \tag{4.3}$$

are given by the following families, where $\chi, \chi_1, \chi_2 \in C(S)$ are exponentials with $\chi_1 \neq \chi_2$, and $\Phi \in C(S)$ is nonzero.

(a) $f, g, h, k \in \operatorname{span}\{\chi_1, \chi_2\}$, *specifically*

$$f = a_1\chi_1 + a_2\chi_2, \quad g = b_1\chi_1 + b_2\chi_2, \quad h = c_1\chi_1 + c_2\chi_2, \quad k = d_1\chi_1 + d_2\chi_2,$$

where the constants $a_i, b_i, c_i, d_i \in \mathbb{C}$ satisfy $(a_1, a_2) \neq (0,0) \neq (b_1, b_2)$, $c_1 d_2 \neq d_1 c_2$, and

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}. \tag{4.4}$$

(b) $f, g, h, k \in \operatorname{span}\{\chi, \Phi\}$, *specifically*

$$f = a_1\chi + a_2\Phi, \quad g = b_1\chi + b_2\Phi, \quad h = c_1\chi + c_2\Phi, \quad k = d_1\chi + d_2\Phi,$$

where the constants $a_i, b_i, c_i, d_i \in \mathbb{C}$ satisfy $(a_1, a_2) \neq (0,0) \neq (b_1, b_2)$, $c_1 d_2 \neq d_1 c_2$, and

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix}. \tag{4.5}$$

(c) $f = a\chi$, $g = b\chi$, $h = \chi - \frac{b}{a}k$, *where $a, b \in \mathbb{C}^*$ and $k \in C(S) \setminus \{0\}$ is any function that is not a scalar multiple of χ.*

Proof. Suppose $f, g, h, k \in C(S)$ satisfy (4.3) with $f \neq 0$, $g \neq 0$, f central, and $\{h, k\}$ linearly independent. Then

$$f(x)h(y) + g(x)k(y) = f(xy) = f(yx) = f(y)h(x) + g(y)k(x) \tag{4.6}$$

for all $x, y \in S$. Since $h \neq 0$ this shows that $f \in \operatorname{span}\{g, h, k\}$, say $f = a'g + b'h + c'k$. Putting this into (4.6) we find after some rearrangement that

$$g(x)[a'h(y) + k(y)] = h(x)[a'g(y) + c'k(y)] + k(x)[g(y) - c'h(y)].$$

By the linear independence of $\{h, k\}$ this shows that $g \in \operatorname{span}\{h, k\}$, so $f \in \operatorname{span}\{h, k\}$. Thus we can write

$$f = ah + bk \tag{4.7}$$

for some $a, b \in \mathbb{C}$ with $(a, b) \neq (0, 0)$. Here we divide the proof into two cases.

Case 1: Suppose $a = 0$. Then (4.7) shows that $f = bk$, and since $b \neq 0$ we have $k = f/b$. With this (4.3) becomes

$$f(xy) = f(x)h(y) + g'(x)f(y)$$

where $g' := g/b$. If $g' = h$ then this is the sine addition law. By Proposition 3.1 we have either $f, h(= g') \in \operatorname{span}\{\chi_1, \chi_2\}$ for multiplicative functions $\chi_j \in C(S)$ with $\chi_1 \neq \chi_2$, or $f, h \in \operatorname{span}\{\chi, \Phi\}$ for $\Phi \in C(S)$ and exponential $\chi \in C(S)$. By (4.7) and $g = bh$ we see that $g, k \in \operatorname{span}\{f, h\}$. If on the other hand $g' \neq h$ then Lemma 4.1 leads to the conclusion that f, g', h belong to either $\operatorname{span}\{\chi_1, \chi_2\}$ or $\operatorname{span}\{\chi, \Phi\}$ (case (c) of Lemma 4.1 is eliminated because $\{f, h\}$ is linearly independent). Again it follows that $g, k \in \operatorname{span}\{f, h\}$. Thus we have the general forms for f, g, h, k displayed in (a) and (b). Moreover χ_1, χ_2, Φ must all be nonzero since $\{h, k\}$ is linearly independent.

Case 2: Suppose $a \neq 0$. Then from (4.7) we get $h = (f - bk)/a$, and (4.3) gives

$$\begin{aligned} af(xy) =& f(x)[f(y) - bk(y)] + ag(x)k(y) \\ =& f(x)f(y) + [ag(x) - bf(x)]k(y). \end{aligned}$$

Defining $f' := f/a$, $g' := (ag - bf)/a^2$ this equation transforms into

$$f'(xy) = f'(x)f'(y) + g'(x)k(y),$$

with solutions given by Lemma 4.2. Case (i) of Lemma 4.2 yields $f' = \chi$, $g' = 0$, and k arbitrary (subject to $\{h, k\}$ linearly independent), where $\chi \in C(S)$ is an exponential since $f \neq 0$. This is solution family (c). Case (ii) of Lemma 4.2 can be eliminated since $k = 0$ contradicts the linear independence of $\{h, k\}$. Case (iii) of Lemma 4.2 yields $f', g', k \in \operatorname{span}\{\chi_1, \chi_2\}$ for distinct multiplicative functions $\chi_1, \chi_2 \in C(S)$. It follows that $f, g, h \in \operatorname{span}\{\chi_1, \chi_2\}$ also. Moreover χ_1, χ_2 are again exponentials by the linear independence of $\{h, k\}$. Similarly, case (iv) of Lemma 4.2 gives $f', g', k \in \operatorname{span}\{\chi, \Phi\}$ for some $\Phi \in C(S)$ and exponential χ, and it follows that $f, g, h \in \operatorname{span}\{\chi, \Phi\}$ too. Moreover $\Phi \neq 0$ since $\{h, k\}$ is linearly independent. So from cases (iii),(iv) we again arrive at the solution forms for f, g, h, k seen in (a), respectively (b).

Next we show that the coefficients must fulfill the stated constraints in (a) and (b). In family (a), constraint (4.4) is verified by substituting the forms of f, g, h, k into (4.3) and using the linear independence of distinct exponentials on a semigroup (see [7, Theorem 3.18]). The other constraints in (a) follow from $f \neq 0$, $g \neq 0$, and $\{h, k\}$ linearly independent.

For family (b), constraint (4.5) is verified using the linear independence of $\{\chi, \Phi\}$ (Remark 3.5) and the fact that Φ satisfies (3.3): $\Phi(xy) = \Phi(x)\chi(y) + \chi(x)\Phi(y)$. The other constraints in (b) follow as for (a).

The converse is a straightforward verification. $\square$

Now we arrive at our prime objective, which is the Levi–Civita equation (1.1):

$$f(xy) = g_1(x)h_1(y) + g_2(x)h_2(y), \qquad x, y \in S,$$

for unknown functions $f, g_1, g_2, h_1, h_2 \colon S \to \mathbb{C}$. If either $\{g_1, g_2\}$ or $\{h_1, h_2\}$ is linearly dependent, then (1.1) reduces to the Pexider equation $f(xy) = g(x)h(y)$ for some functions g, h. The solutions of this equation are known in a very general setting. (Note that in this case at least one of the original functions g_1, g_2, h_1, h_2 is arbitrary.) We omit that simple case.

The following result generalizes [1, Theorem 5.2].

THEOREM 4.4. *Let S be a topological monoid, and let $f, g_1, g_2, h_1, h_2 \in C(S)$ be a solution of* (1.1) *with f central, and with both $\{g_1, g_2\}$ and $\{h_1, h_2\}$ linearly independent. The solutions are given by the following families, where $\chi, \chi_1, \chi_2 \in C(S)$ are exponentials with $\chi_1 \neq \chi_2$, and $\Phi \in C(S)$ is nonzero.*

(i) $f, g_1, g_2, h_1, h_2 \in \operatorname{span}\{\chi_1, \chi_2\}$, *specifically*

$$f = a_1\chi_1 + a_2\chi_2, \quad g_1 = b_1\chi_1 + b_2\chi_2, \quad g_2 = d_1\chi_1 + d_2\chi_2,$$

$$h_1 = c_1\chi_1 + c_2\chi_2, \quad h_2 = e_1\chi_1 + e_2\chi_2,$$

where the constants $a_i, b_i, c_i, d_i, e_i \in \mathbb{C}$ satisfy

$$\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ e_1 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

in which all three matrices have (full) rank 2.

(ii) $f, g_1, g_2, h_1, h_2 \in \operatorname{span}\{\chi, \Phi\}$, *specifically*

$$f = a_1\chi + a_2\Phi, \quad g_1 = b_1\chi + b_2\Phi, \quad g_2 = d_1\chi + d_2\Phi,$$

$$h_1 = c_1\chi + c_2\Phi, \quad h_2 = e_1\chi + e_2\Phi,$$

where the constants $a_i, b_i, c_i, d_i, e_i \in \mathbb{C}$ satisfy

$$\begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ e_1 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix}$$

in which all three matrices have rank 2.

Proof. First note that $f \neq 0$ by the linear independence of $\{g_1, g_2\}$ and $\{h_1, h_2\}$. Next, putting $y = e$ into (1.1) we see that

$$f = h_1(e)g_1 + h_2(e)g_2. \tag{4.8}$$

We cannot have $h_1(e) = h_2(e) = 0$ since $f \neq 0$, so without loss of generality we assume that $h_2(e) \neq 0$. It follows that $g_2 = f/h_2(e) - h_1(e)g_1/h_2(e)$, and putting this into (1.1) we find that

$$f(xy) = f(x)\frac{h_2(y)}{h_2(e)} + g_1(x)\left[h_1(y) - \frac{h_1(e)h_2(y)}{h_2(e)}\right].$$

Defining new functions $h, k \in C(S)$ by

$$h := h_2/h_2(e) \quad \text{and} \quad k := h_1 - h_1(e)h_2/h_2(e), \tag{4.9}$$

we arrive at the functional equation

$$f(xy) = f(x)h(y) + g_1(x)k(y)$$

which was treated in Lemma 4.3. Moreover (4.9) shows that $\{h, k\}$ is linearly independent, so we read the solutions from Lemma 4.3.

Starting with case (a) of Lemma 4.3 we have $f, h, g_1, k \in \operatorname{span}\{\chi_1, \chi_2\}$ for exponentials $\chi_1 \neq \chi_2 \in C(S)$. By (4.8) and the definitions of h, k we also have $g_2, h_2, h_1 \in \operatorname{span}\{\chi_1, \chi_2\}$. Thus our functions have the forms seen in family (i). Substituting them into (1.1) and using the linear independence of $\{g_1, g_2\}, \{h_1, h_2\}$, and $\{\chi_1, \chi_2\}$, we find that the constants must fulfill the stated conditions.

Next, case (b) of Lemma 4.3 yields that $f, h, g_1, k \in \operatorname{span}\{\chi, \Phi\}$ for an exponential $\chi \in C(S)$ and a nonzero function $\Phi \in C(S)$. As before we also get $h_2, h_1, g_2 \in \operatorname{span}\{\chi, \Phi\}$. Inserting the forms shown for f, g_j, h_j in family (ii) into (1.1) and using the linear independence of $\{g_1, g_2\}, \{h_1, h_2\}, \{\chi, \Phi\}$ together with (3.3), we get the constraints stated for family (ii).

Finally, consider case (c) from Lemma 4.3: $f = a\chi$, $g_1 = b\chi$, $h = \chi - bk/a$, where $a, b \in \mathbb{C}^*$. From this and equation (4.9) we find that $h_2 = c(\chi - bk/a)$ and $h_1 = d\chi + (1 - bd/a)k$, where $c = h_2(e) \neq 0$ and $d = h_1(e)$. But then from (4.8) we get $g_2 = (a - db)\chi/c$, so the condition that $\{g_1, g_2\}$ is linearly independent is violated and we must discard this family.

□

5. Solution of (1.1) using representation theory

Now we use a second way to solve (1.1), assuming that our monoid is commutative. We start with two results that establish the theory behind this method.

Consider $\mathbb{C}^n$ as a vector space of column vectors, and let $\mathcal{M}_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over $\mathbb{C}$. The following is [5, Lemma 2.4]. Contrary to common usage we denote the $n \times n$ *identity matrix* by E_n in order to avoid confusion with null ideals labeled as I_r below.

Lemma 5.1. *Let $n \in \mathbb{N}$, let S be a topological commutative monoid, and suppose $f, g_j, h_j \in C(S)$ for $1 \le j \le n$ satisfy the Levi–Civita equation*

$$f(xy) = \sum_{i=1}^{n} g_i(x)h_i(y), \quad x, y \in S, \tag{5.1}$$

with $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ linearly independent. Let $V = \mathrm{span}\{g_1, \dots, g_n\}$ and $g = [g_1, \dots, g_n]^t$.

There exists an associative and commutative algebra $(\mathbb{C}^n, +, *)$ with identity element $g(e)$ and regular representation $R\colon \mathbb{C}^n \to \mathcal{M}_n(\mathbb{C})$ such that

$$R(g(xy)) = R(g(x))R(g(y)), \quad x, y \in S,$$

with $R(g(e)) = E_n$ and $g(x) = R(g(x))g(e)$ for all $x \in S$.

There exists a similarity matrix $D \in \mathcal{M}_n(\mathbb{C})$ simultaneously transforming the family $\{R(g(x)) \mid x \in S\}$ of commuting matrices into block diagonal form

$$D^{-1}R(g(x))D = diag\{M_1(x), \dots, M_s(x)\},$$

$$M_r(x) \in \mathcal{M}_{d_r}(\mathbb{C}), \quad d_1 + \cdots + d_s = n,$$

where each M_r is lower triangular of the form

$$M_r(x) = \chi_r(x)E_{d_r} + \left(\rho_r^{i,j}(x)\right), \quad \rho_r^{i,j}(x) = 0 \text{ for } i \le j \in \{1, \dots, d_r\}, \tag{5.2}$$

for all $x \in S$, where $\chi_r, \rho_r^{i,j} \in V$ for all $r \in \{1, \dots, s\}$ and $i, j \in \{1, \dots, d_r\}$.

Furthermore, each χ_r is an exponential with $\chi_r(e) = 1$, and we have for each block M_r the system of equations

$$\rho_r^{i,j}(xy) = \rho_r^{i,j}(x)\chi_r(y) + \rho_r^{i,j}(y)\chi_r(x) + \sum_{k=j+1}^{i-1} \rho_r^{i,k}(x)\rho_r^{k,j}(y) \tag{5.3}$$

for $r \in \{1, \dots, s\}$ and $1 \le i, j \le d_r$. Moreover $\chi_1, \dots, \chi_s$ are distinct.

Note that if $d_r = 1$ for some r then the corresponding system (5.3) *is trivial, since $\rho_r^{1,1} = 0$ by* (5.2) *hence the block M_r is just the exponential χ_r.*

In the next result, which is [5, Theorem 2.5], a *pure polynomial* is a polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ such that $P(0, \dots, 0) = 0$, so P has constant term 0.

Proposition 5.2. *Let $n \in \mathbb{N}$, let S be a topological commutative monoid, and suppose $f, g_j, h_j \in C(S)$ for $1 \le j \le n$ satisfy* (5.1) *with $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ linearly independent. Let $V = \operatorname{span}\{g_1, \dots, g_n\}$. Then there exist positive integers $s, d_1, \dots, d_s$ with $d_1 + \dots + d_s = n$, distinct exponentials $\chi_r \in V$ for each $1 \le r \le s$ with corresponding nullspaces I_r, additive functions $A_{r,j} \in C(S \setminus I_r)$ and pure polynomials $P_{r,j}$ of degree at most $d_r - 1$ for each $1 \le r \le s$ and $1 \le j \le d_r - 1$, and functions $q_{r,j} \in C(S)$ for each $1 \le r \le s$ and $1 \le j \le d_r - 1$, such that*

$$q_{r,j}(x) = P_{r,j}(A_{r,1}(x), \dots, A_{r,d_r-1}(x))\chi_r(x), \qquad \text{for all } x \in S \setminus I_r,$$

and $B = B_1 \cup \dots \cup B_s$ is a basis for V, where

$$B_r = \{\chi_r, q_{r,1}, \dots, q_{r,d_r-1}\}.$$

Moreover $f, h_1, \dots, h_n \in V$.
Note that the form of $q_{r,j}$ on I_r is unspecified.

Now we have the following companion result to Theorem 4.4, but by a completely different proof.

Theorem 5.3. *Let S be a topological commutative monoid, and let $f, g_1, g_2, h_1, h_2 \in C(S)$ be a solution of* (1.1) *with both $\{g_1, g_2\}$ and $\{h_1, h_2\}$ linearly independent. The solutions are exactly the same as in Theorem 4.4, where $\chi, \chi_1, \chi_2 \in C(S)$ are exponentials with $\chi_1 \ne \chi_2$ and $\Phi \in C(S)$ is nonzero.*

Proof. By Proposition 5.2 for $n = 2$ we have $f, h_1, h_2 \in V = \operatorname{span}\{g_1, g_2\}$. There are two cases to consider, either $s = 2$ with $d_1 = d_2 = 1$, or $s = 1$ with $d_1 = 2$.

Case (i): Suppose $s = 2$ with $d_1 = d_2 = 1$. Then there are distinct exponentials $\chi_1, \chi_2 \in C(S)$ such that $B = \{\chi_1, \chi_2\}$ is a basis for V. Putting the formulas of f, g_1, g_2, h_1, h_2 shown in Theorem 4.4(i) into (1.1) we get the constraints on the coefficients as before.

Case (ii): Suppose $s = 1$ with $d_1 = 2$. Then there exists an exponential $\chi \in C(S)$ and a function $q \in C(S)$ such that $B = \{\chi, q\}$ is a basis for V. We are given that

$$q(x) = P(A(x))\chi(x), \quad x \in S \setminus I_\chi,$$

where $A \in C(S)$ is additive and $P \in \mathbb{C}[x]$ is a polynomial of the form $P(x) = cx$ for some $c \in \mathbb{C}$. Since cA is again a continuous additive function we thus have

$$q(x) = A(x)\chi(x), \quad x \in S \setminus I_\chi.$$

To get some information about the form of q on I_χ we look to Lemma 5.1. Applying that lemma to the case at hand we have $D^{-1}R(g(x))D = M(x)$, where (5.2) takes the form

$$M = \begin{pmatrix} \chi & 0 \\ \rho & \chi \end{pmatrix}$$

with $\rho \in C(S)$ satisfying (5.3). Here the latter functional equation is simply the special sine addition law (3.3):

$$\rho(xy) = \rho(x)\chi(y) + \rho(y)\chi(x), \quad x, y \in S,$$

so $\rho = \Phi$. Since the matrix coefficients of $R \circ g$ are $\chi, \Phi, 0$, and since $dim(V) = 2$, we must have $\Phi \neq 0$. Now since $\{\chi, \Phi\}$ is linearly independent by Remark 3.5, we can replace q in the basis by Φ to get $V = \operatorname{span}\{\chi, \Phi\}$, and we arrive at the forms of solution functions shown in Theorem 4.4(ii). Substitution of these forms into (1.1) yields the constraints on the coefficients as before. $\square$

In conclusion, the trade-off for using the powerful tool of representation theory (to get the short proof of Theorem 5.3) is that we had to assume S is commutative. Using elementary methods we can substitute the assumption that f is central to get the same solutions on non-commutative S.

6. Examples

We include two examples to illustrate the results. Since both examples are on commutative monoids, either Theorem 4.4 or Theorem 5.3 can be applied. For each monoid it suffices to identify the forms of $\chi, \Phi \in C(S)$ to be substituted into the solution formulas. The monoids below are neither regular (so not groups) nor generated by their squares, thus the results of [1] cannot be applied.

Let $\Re(z)$ denote the real part of $z \in \mathbb{C}$.

Example 6.1. Let $S = [-1, 1]$ under multiplication and the usual topology. The continuous exponentials χ on S come in three forms,

$$\chi_0 := 1, \quad \chi_\alpha(x) := \begin{cases} |x|^\alpha & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

$$\text{or} \quad \hat{\chi}_\alpha(x) := \begin{cases} |x|^\alpha \operatorname{sgn}(x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

where $\Re(\alpha) > 0$.

Now we identify the possible forms of continuous Φ satisfying (3.3), supposing that an exponential χ is given. Note that $P_\chi = \emptyset$ for each form of χ, so the form (3.1) for Φ reduces to

$$\Phi(x) = \begin{cases} A(x)\chi(x) & \text{for } x \in S \setminus I_\chi, \\ 0 & \text{for } x \in I_\chi, \end{cases}$$

where $A \in C(S \setminus I_\chi)$ is additive.

For $\chi = \chi_0$ we have $I_\chi = \emptyset$, so the continuous Φ on S have the form $\Phi = A\chi$. But since S contains a zero, the only additive function on S is $A = 0$, therefore $\Phi = 0$.

For $\chi \in \{\chi_\alpha, \hat{\chi}_\alpha\}$ with $\Re(\alpha) > 0$, we have $I_\chi = \{0\}$ and $A \in C(S \setminus \{0\})$. Such additive A have the form $A(x) = c \log |x|$ for some $c \in \mathbb{C}$.

The final example deals with the monoid $S = (\mathbb{N}, \cdot)$, which contains infinitely many prime ideals. Letting P denote the set of prime numbers, the set $p\mathbb{N}$ is a prime ideal of S for each $p \in P$. The nullspace I_χ of a multiplicative function $\chi \colon S \to \mathbb{C}$ is the union of all $p\mathbb{N}$ for primes p such that $\chi(p) = 0$. For each $p \in P$ define the function $C_p \colon S \to \mathbb{C}$ for each $x \in S$ by

$$C_p(x) := \text{ the number of copies of } p \text{ in the prime factorization of } x.$$

We note that C_p is additive for each $p \in P$, and the (unique) prime factorization of $x \in S$ can be written as $x = \prod_{p \in P} p^{C_p(x)}$.

Example 6.2. Let $S = (\mathbb{N}, \cdot)$ equipped with the discrete topology. The multiplicative functions $\chi \colon S \to \mathbb{C}$ are generated by their values on P, specifically $\chi(x) = \prod_{p \in P} \chi(p)^{C_p(x)}$ with the convention $0^0 := 1$. The value of $\chi(p)$ for each $p \in P$ can be chosen arbitrarily. Such χ is an exponential if and only if there exists some prime p_0 such that $\chi(p_0) \neq 0$.

As shown in [2, Corollary 6.3], the functions Φ satisfying the special sine addition formula (3.3) with an exponential χ have the form

$$\Phi(x) = \begin{cases} A(x)\chi(x) & \text{for } x \in S \setminus I_\chi, \\ \phi(p)\chi(w) & \text{for } x = pw \text{ with } p \in P \cap I_\chi, w \in S \setminus I_\chi, \\ 0 & \text{for } x \in I_\chi^2, \end{cases}$$

where $A\colon S \setminus I_\chi \to \mathbb{C}$ is additive and $\phi\colon P \cap I_\chi \to \mathbb{C}$ is an arbitrary function. Such additive A are generated by their values on $P \setminus I_\chi$, namely $A(x) = \sum_{p\in P\setminus I_\chi} C_p(x)A(p)$, where the values of A on $P\setminus I_\chi$ may be chosen arbitrarily.

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