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OPEN

Annales Mathematicae Silesianae **32** (2018), 215–225
DOI: 10.2478/amsil-2018-0005

STABILITY OF FUNCTIONAL EQUATIONS IN DISLOCATED QUASI-METRIC SPACES

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Abstract. We present a result on the generalized Hyers–Ulam stability of a functional equation in a single variable for functions that have values in a complete dislocated quasi-metric space. Next, we show how to apply it to prove stability of the Cauchy functional equation and the linear functional equation in two variables, also for functions taking values in a complete dislocated quasi-metric space. In this way we generalize some earlier results proved for classical complete metric spaces.

1. Introduction and preliminaries

The stability of functional equations is a problem originating from the following question formulated by S.M. Ulam (see [5, 11]) for group homomorphisms:

Assume that $(G_1,+)$ *and* $(G_2,+)$ *are groups,* (G_2,d) *is a metric space and* $\varepsilon>0$*. Find* $\delta>0$ *such that for any function* $\varphi\colon G_1\to G_2$ *with*

$$d(\varphi(x+y),\varphi(x)+\varphi(y))\leqslant\delta,\ \ x,y\in G_1,$$

there exists a group homomorphism $F\colon G_1\to G_2$ *such that*

$$d(\varphi(x),F(x))\leqslant\varepsilon,\ \ x\in G_1.$$

Received: 30.09.2017. Accepted: 11.04.2018. Published online: 14.05.2018.
(2010) Mathematics Subject Classification: 39B82, 54E50.
Key words and phrases: stability of functional equations, square symmetric groupoid, dislocated quasi-metric space, semigroup, Cauchy equation.

The first partial answer to it was published by D.H. Hyers [5] in 1941 and since that year, a lot of analogous results have been appearing (see, e.g., [3, 6]).

In this paper we consider a similar problem for so called dislocated quasi-metrics (dq-metrics) instead of metrics. Namely, we show that [4, Theorem 2.1], proved for classical metric spaces, is valid also for dq-metric spaces. Moreover, we present some applications of that result, analogously as in [4].

Dislocated quasi-metrics play a crucial role, among others, in computer science and cryptography. Below we recall the definition of them.

Let Y be a nonempty set with a function $d\colon Y \times Y \to [0, +\infty)$, which satisfies the following two conditions:

(Q1) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$,

(Q2) $d(x, y) \leqslant d(x, z) + d(z, y)$

for all $x, y, z \in Y$. In this case d is said to be a *dislocated quasi-metric* (*dq-metric* for short) and (Y, d) is called a *dislocated quasi-metric space* (*dq-metric space* for short) (see [10, 9]). If d is a dq-metric in Y, which is symmetric (i.e., $d(x, y) = d(y, x)$ for $x, y \in Y$), then (Y, d) is called a *metric-like space* (see [1]).

Example 1.1 ([9]). *A function $d\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $d(x, y) = |x|$ for $x, y \in \mathbb{R}$ is a dislocated quasi-metric.*

Example 1.2 ([1]). *If $d\colon \mathbb{R}\times\mathbb{R} \to [0, \infty)$ is given by $d(x, y) = \max\{|x|, |y|\}$ for $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a metric-like space.*

We say that $x \in Y$ is a *limit* of a sequence $(x_n)_{n=1}^{\infty}$ of elements in a dq-metric space (Y, d) iff $d(x_n, x) \to 0$ and $d(x, x_n) \to 0$; we then write $x_n \to x$. It is easy to check that a limit of a sequence is unique if it exists. Indeed, if $x, y \in Y$ are limits of a sequence $(x_n)_{n=1}^{\infty}$, then for all $n \in \mathbb{N}$ we have $0 \leqslant d(x, y) \leqslant d(x, x_n) + d(x_n, y)$. Thus, $d(x, y) = 0$ and similarly, $d(y, x) = 0$, so $x = y$.

A sequence $(x_n)_{n=1}^{\infty}$ is said to be a *Cauchy sequence* if

$$\forall_{\varepsilon>0}\, \exists_{N\in\mathbb{N}}\, \forall_{m,n\geqslant N}\, d(x_n, x_m) < \varepsilon.$$

A dq-metric space (Y, d) is called *complete* if every Cauchy sequence has a limit in this space.

2. Main theorem

Let (Y, d) be a complete dq-metric space. Consider a nonempty set K and functions $\Psi\colon Y \to Y$, $\alpha\colon K \to K$, and $h_1, h_2\colon K \to [0, +\infty)$. Assume that Ψ satisfies the Lipschitz condition with a Lipschitz constant $\lambda \in (0, +\infty)$, i.e., $d(\Psi(x), \Psi(y)) \leqslant \lambda d(x, y)$ for all $x, y \in Y$, and $H_j(x) := \sum_{i=0}^{\infty} \lambda^i h_j(\alpha^i(x))$ is convergent for every $x \in K$ and $j = 1, 2$.

Theorem 2.1. *Let $f\colon K \to Y$ be a function such that*

$$d(\Psi \circ f \circ \alpha(x), f(x)) \leqslant h_1(x), \quad x \in K,$$

$$d(f(x), \Psi \circ f \circ \alpha(x)) \leqslant h_2(x), \quad x \in K.$$

Then the limit

$$F(x) := \lim_{n\to\infty} \Psi^n \circ f \circ \alpha^n(x)$$

exists for every $x \in K$ and the function $F\colon K \to Y$ is a unique solution of the equation $\Psi \circ F \circ \alpha = F$ such that

$$d(F(x), f(x)) \leqslant H_1(x), \quad x \in K,$$

$$d(f(x), F(x)) \leqslant H_2(x), \quad x \in K.$$

Proof. We claim that for all $n \in \mathbb{N}$ the following two inequalities hold:

$$d(\Psi^n \circ f \circ \alpha^n(x), f(x)) \leqslant \sum_{i=0}^{n-1} \lambda^i h_1(\alpha^i(x)), \quad x \in K, \tag{2.1}$$

$$d(f(x), \Psi^n \circ f \circ \alpha^n(x)) \leqslant \sum_{i=0}^{n-1} \lambda^i h_2(\alpha^i(x)), \quad x \in K. \tag{2.2}$$

For $n = 1$ these inequalities hold by the assumption. Let $n \in \mathbb{N}$ and suppose that (2.1) and (2.2) are true. From the triangle inequality (Q2), the Lipschitz condition, the assumption and the induction hypothesis we have

$$\begin{aligned} d(\Psi^{n+1} \circ f \circ \alpha^{n+1}(x), f(x)) &\leqslant d(\Psi^{n+1} \circ f \circ \alpha^{n+1}(x), \Psi \circ f \circ \alpha(x)) \\ &\quad + d(\Psi \circ f \circ \alpha(x), f(x)) \\ &\leqslant \lambda d(\Psi^n \circ f \circ \alpha^n(\alpha(x)), f(\alpha(x))) + h_1(x) \end{aligned}$$

$$\leqslant \lambda \sum_{i=0}^{n-1} \lambda^i h_1(\alpha^i(\alpha(x))) + h_1(x) = \sum_{i=0}^{n-1} \lambda^{i+1} h_1(\alpha^{i+1}(x)) + h_1(x)$$

$$= \sum_{i=1}^{n} \lambda^i h_1(\alpha^i(x)) + h_1(x) = \sum_{i=0}^{n} \lambda^i h_1(\alpha^i(x))$$

for all $x \in K$. Analogously, we can prove that

$$d(f(x), \Psi^{n+1} \circ f \circ \alpha^{n+1}(x)) \leqslant \sum_{i=0}^{n} \lambda^i h_2(\alpha^i(x))$$

for every $x \in K$, so the claim follows.

Now, we show that for all $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ we have

$$d(\Psi^{m+k} \circ f \circ \alpha^{m+k}(x), \Psi^k \circ f \circ \alpha^k(x)) \leqslant \sum_{i=k}^{m+k-1} \lambda^i h_1(\alpha^i(x)),\ x \in K, \tag{2.3}$$

$$d(\Psi^k \circ f \circ \alpha^k(x), \Psi^{m+k} \circ f \circ \alpha^{m+k}(x)) \leqslant \sum_{i=k}^{m+k-1} \lambda^i h_2(\alpha^i(x)),\ x \in K. \tag{2.4}$$

Let $m \in \mathbb{N}$. For $k = 0$ these conditions are simply the inequalities (2.1) and (2.2) with $n = m$. Thus, take any $k \in \mathbb{N}$ and suppose that (2.3) and (2.4) are true. Then

$$\begin{aligned} d(\Psi^{m+k+1} \circ f \circ \alpha^{m+k+1}(x), \Psi^{k+1} \circ f \circ \alpha^{k+1}(x)) & \\ \leqslant \lambda d(\Psi^{m+k} \circ f \circ \alpha^{m+k}(\alpha(x)), \Psi^k \circ f \circ \alpha^k(\alpha(x))) & \\ \leqslant \lambda \sum_{i=k}^{m+k-1} \lambda^i h_1(\alpha^i(\alpha(x))) & \\ = \sum_{i=k}^{m+k-1} \lambda^{i+1} h_1(\alpha^{i+1}(x)) = \sum_{i=k+1}^{m+k} \lambda^i h_1(\alpha^i(x)) & \end{aligned}$$

for all $x \in K$. In the same way we can show that

$$d(\Psi^{k+1} \circ f \circ \alpha^{k+1}(x), \Psi^{m+k+1} \circ f \circ \alpha^{m+k+1}(x)) \leqslant \sum_{i=k+1}^{m+k} \lambda^i h_2(\alpha^i(x))$$

for every $x \in K$. Therefore inequalities (2.3) and (2.4) are proved.

We conclude from this that for all $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$

$$d(\Psi^{m+k} \circ f \circ \alpha^{m+k}(x), \Psi^k \circ f \circ \alpha^k(x)) \leqslant \sum_{i=k}^{\infty} \lambda^i h_1(\alpha^i(x)), \quad x \in K,$$

$$d(\Psi^k \circ f \circ \alpha^k(x), \Psi^{m+k} \circ f \circ \alpha^{m+k}(x)) \leqslant \sum_{i=k}^{\infty} \lambda^i h_2(\alpha^i(x)), \quad x \in K.$$

Consequently, for every $x \in K$ and for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m, l \in \mathbb{N}$ we have

$$\begin{aligned} d(\Psi^{m+k} \circ f \circ \alpha^{m+k}(x), &\Psi^{l+k} \circ f \circ \alpha^{l+k}(x)) \\ &\leqslant d(\Psi^{m+k} \circ f \circ \alpha^{m+k}(x), \Psi^k \circ f \circ \alpha^k(x)) \\ &\quad + d(\Psi^k \circ f \circ \alpha^k(x), \Psi^{l+k} \circ f \circ \alpha^{l+k}(x)) \\ &\leqslant \sum_{i=k}^{\infty} \lambda^i h_1(\alpha^i(x)) + \sum_{i=k}^{\infty} \lambda^i h_2(\alpha^i(x)) < \varepsilon. \end{aligned}$$

It means that $(\Psi^n \circ f \circ \alpha^n(x))_{n=1}^{\infty}$ is a Cauchy sequence for every $x \in K$. The space (Y, d) is complete, so the limit $F(x) := \lim_{n\to\infty} \Psi^n \circ f \circ \alpha^n(x)$ exists for every $x \in K$.

Observe that

$$\begin{aligned} d(F(x), f(x)) &\leqslant d(F(x), \Psi^n \circ f \circ \alpha^n(x)) + d(\Psi^n \circ f \circ \alpha^n(x), f(x)) \\ &\leqslant d(F(x), \Psi^n \circ f \circ \alpha^n(x)) + H_1(x) \end{aligned}$$

for $x \in K, n \in \mathbb{N}$, what implies $d(F(x), f(x)) \leqslant H_1(x)$ for all $x \in K$. Similarly, we can show that $d(f(x), F(x)) \leqslant H_2(x)$ for every $x \in K$. Furthermore, for every $x \in K$ we have

$$\begin{aligned} 0 &\leqslant d(\Psi \circ F \circ \alpha(x), F(x)) \\ &\leqslant d(\Psi(F(\alpha(x))), \Psi^{n+1} \circ f \circ \alpha^{n+1}(x)) + d(\Psi^{n+1} \circ f \circ \alpha^{n+1}(x), F(x)) \\ &\leqslant \lambda d(F(\alpha(x)), \Psi^n \circ f \circ \alpha^n(\alpha(x))) + d(\Psi^{n+1} \circ f \circ \alpha^{n+1}(x), F(x)), \end{aligned}$$

thus, $d(\Psi \circ F \circ \alpha(x), F(x)) = 0$ for $x \in K$. Similarly, $d(F(x), \Psi \circ F \circ \alpha(x)) = 0$ for $x \in K$. This implies that $\Psi \circ F \circ \alpha = F$.

Suppose that $G\colon K \to Y$ is a function such that $\Psi \circ G \circ \alpha = G$ and

$$d(G(x), f(x)) \leqslant H_1(x), \quad x \in K,$$

$$d(f(x), G(x)) \leqslant H_2(x), \quad x \in K.$$

By induction we obtain $\Psi^n \circ G \circ \alpha^n = G$ and $\Psi^n \circ F \circ \alpha^n = F$ for all $n \in \mathbb{N}$, so

$$\begin{aligned}
0 \leqslant d(F(x), G(x)) &= d(\Psi^n \circ F \circ \alpha^n(x), \Psi^n \circ G \circ \alpha^n(x)) \\
&\leqslant \lambda^n d(F(\alpha^n(x)), G(\alpha^n(x))) \\
&\leqslant \lambda^n [d(F(\alpha^n(x)), f(\alpha^n(x))) + d(f(\alpha^n(x)), G(\alpha^n(x)))] \\
&\leqslant \lambda^n \sum_{i=0}^{\infty} \lambda^i h_1(\alpha^{i+n}(x)) + \lambda^n \sum_{i=0}^{\infty} \lambda^i h_2(\alpha^{i+n}(x)) \\
&= \sum_{i=0}^{\infty} \lambda^{i+n} h_1(\alpha^{i+n}(x)) + \sum_{i=0}^{\infty} \lambda^{i+n} h_2(\alpha^{i+n}(x)) \\
&= \sum_{i=n}^{\infty} \lambda^i h_1(\alpha^i(x)) + \sum_{i=n}^{\infty} \lambda^i h_2(\alpha^i(x))
\end{aligned}$$

for all $x \in K$ and $n \in \mathbb{N}$. This implies that $d(F(x), G(x)) = 0$ for $x \in K$. In the same way we obtain $d(G(x), F(x)) = 0$ for $x \in K$, so $F = G$ and the theorem follows. □

3. Applications

If $(G, +)$ is a groupoid (i.e., G is a nonempty set and $+ : G^2 \to G$ is a binary operation, not necessarily commutative), then we say that it is *uniquely divisible* by 2 if for every $x \in G$ there exists a unique $y \in G$ such that $x = y + y =: 2y$. In this case y is denoted by $\frac{1}{2}x$ or $\frac{x}{2}$. A groupoid $(G, +)$ is called *square symmetric* iff $2(x + y) = 2x + 2y$ for all $x, y \in G$.

Consider the following notations: $2^0 x = x$, $2^{n+1}x = 2(2^n x)$ for $x \in G$ and $n \in \mathbb{N}$. Moreover, if $(G, +)$ is uniquely divisible by 2 we write $2^{-1}x = \frac{1}{2}x$, $2^{-n-1}x = 2^{-1}(2^{-n}x)$ for $x \in G$, $n \in \mathbb{N}$.

Let $(X, +)$, $(Y, +)$ be square symmetric groupoids. Assume that $(X, +)$ is uniquely divisible by 2, (Y, d) is a complete dq-metric space and the operation $+\colon Y \times Y \to Y$ is *continuous*, i.e., if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$

for all sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ of elements of the space Y and $x, y \in Y$. Consider also a nonempty set $K \subset X$ and functions $\chi_i \colon K \times K \to [0, \infty)$, $i = 1, 2$.

With this notations we have the following corollary from Theorem 2.1.

Corollary 3.1. *Suppose that* $\frac{1}{2}K \subset K$ *and that there exist constants* $\eta_1, \eta_2, \varepsilon \in (0, \infty)$ *such that* $\eta_i \varepsilon < 1$ *and*

$$\chi_i(\tfrac{x}{2}, \tfrac{y}{2}) \leqslant \eta_i \chi_i(x, y), \quad x, y \in K,\ i = 1, 2,$$

$$d(2x, 2y) \leqslant \varepsilon d(x, y), \quad x, y \in Y.$$

Let $\varphi \colon K \to Y$ *be a function that satisfies the conditions:*

$$d(\varphi(x) + \varphi(y), \varphi(x + y)) \leqslant \chi_1(x, y),$$

$$d(\varphi(x + y), \varphi(x) + \varphi(y)) \leqslant \chi_2(x, y)$$

for all $x, y \in K$, $x + y \in K$. *Then there exists a unique solution* $F \colon K \to Y$ *of the equation*

$$F(x + y) = F(x) + F(y), \quad x, y \in K, x + y \in K$$

such that

$$d(F(x), \varphi(x)) \leqslant \frac{\eta_1 \chi_1(x, x)}{1 - \eta_1 \varepsilon}, \quad x \in K$$

and

$$d(\varphi(x), F(x)) \leqslant \frac{\eta_2 \chi_2(x, x)}{1 - \eta_2 \varepsilon}, \quad x \in K.$$

Proof. By the assumption, for every $x \in K$,

$$d\left(2\varphi\left(\tfrac{x}{2}\right), \varphi(x)\right) = d\left(\varphi\left(\tfrac{x}{2}\right) + \varphi\left(\tfrac{x}{2}\right), \varphi\left(\tfrac{x}{2} + \tfrac{x}{2}\right)\right) \leqslant \chi_1(\tfrac{x}{2}, \tfrac{x}{2})$$

and similarly

$$d\left(\varphi(x), 2\varphi\left(\tfrac{x}{2}\right)\right) \leqslant \chi_2\left(\tfrac{x}{2}, \tfrac{x}{2}\right),\ x \in K.$$

Setting $f = \varphi, \Psi(x) = 2x, \lambda = \varepsilon, h_i(x) = \chi_i\left(\frac{x}{2}, \frac{x}{2}\right), i = 1, 2$, and $\alpha(x) = \frac{x}{2}$ in Theorem 2.1, the limit $F(x) = \lim_{n\to\infty} 2^n \varphi(2^{-n}x)$ exists for every $x \in K$ and it is a unique solution of the equation $2F(2^{-1}x) = F(x)$, $x \in K$, such that for every $x \in K$,

$$d(F(x), \varphi(x)) \leqslant \sum_{i=0}^{\infty} \varepsilon^i \chi_1 \left(2^{-i}\frac{x}{2}, 2^{-i}\frac{x}{2}\right) \leqslant \sum_{i=0}^{\infty} \varepsilon^i \eta_1^i \chi_1 \left(\frac{x}{2}, \frac{x}{2}\right)$$
$$\leqslant \eta_1\chi_1(x,x) \sum_{i=0}^{\infty} (\varepsilon\eta_1)^i = \frac{\eta_1\chi_1(x,x)}{1-\eta_1\varepsilon}$$

and analogously,

$$d(\varphi(x), F(x)) \leqslant \frac{\eta_2\chi_2(x,x)}{1-\eta_2\varepsilon}, \quad x \in K.$$

Let $x, y \in K, x+y \in K$. Observe that for all $n \in \mathbb{N}$ we have

$$\begin{aligned} d(2^n\varphi(2^{-n}x) + 2^n\varphi(2^{-n}y), &2^n\varphi(2^{-n}(x+y))) \\ &\leqslant \varepsilon^n d(\varphi(2^{-n}x) + \varphi(2^{-n}y), \varphi(2^{-n}(x+y))) \\ &\leqslant \varepsilon^n \chi_1(2^{-n}x, 2^{-n}y) \leqslant (\varepsilon\eta_1)^n \chi_1(x,y). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(2^n\varphi(2^{-n}x) + 2^n\varphi(2^{-n}y), 2^n\varphi(2^{-n}(x+y))) = 0.$$

Similarly,

$$\lim_{n\to\infty} d(2^n\varphi(2^{-n}(x+y)), 2^n\varphi(2^{-n}x) + 2^n\varphi(2^{-n}y)) = 0.$$

For all $n \in \mathbb{N}$

$$\begin{aligned} 0 &\leqslant d(F(x)+F(y), F(x+y)) \\ &\leqslant d(F(x)+F(y), 2^n\varphi(2^{-n}x) + 2^n\varphi(2^{-n}y)) \\ &\quad + d(2^n\varphi(2^{-n}x) + 2^n\varphi(2^{-n}y), 2^n\varphi(2^{-n}(x+y))) \\ &\quad + d(2^n\varphi(2^{-n}(x+y)), F(x+y)), \end{aligned}$$

what implies $d(F(x)+F(y), F(x+y)) = 0$. In the same way we can prove that $d(F(x+y), F(x)+F(y)) = 0$, hence $F(x+y) = F(x)+F(y)$.

Suppose that a function $G\colon K \to Y$ satisfies the conditions:

$$G(x+y) = G(x) + G(y), \quad x, y \in K,\ x+y \in K,$$

$$d(G(x), \varphi(x)) \leqslant \frac{\eta_1 \chi_1(x,x)}{1-\eta_1\varepsilon}, \quad x \in K,$$

$$d(\varphi(x), G(x)) \leqslant \frac{\eta_2 \chi_2(x,x)}{1-\eta_2\varepsilon}, \quad x \in K.$$

Then

$$2G\left(\frac{x}{2}\right) = G\left(\frac{x}{2}\right) + G\left(\frac{x}{2}\right) = G\left(\frac{x}{2} + \frac{x}{2}\right) = G(x)$$

for every $x \in K$, so by the uniqueness in Theorem 2.1 we obtain $F = G$. □

Assume that $(X,+)$ and $(Y,+)$ are abelian groups, $c \in X$, $C \in Y$. Let $a, b\colon X \to X$, $A, B\colon Y \to Y$ be homomorphisms such that $a \circ b = b \circ a$ and $A \circ B = B \circ A$. Suppose that the function $\alpha\colon X \ni x \longmapsto a(x) + b(x) + c \in X$ is bijective and (Y, d) is a complete dq-metric space. Assume also that $x_n \to x$ and $y_n \to y$ implies $A(x_n) + B(y_n) + C \to A(x) + B(y) + C$ for all sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ of elements of the space Y and $x, y \in Y$ and consider a nonempty subset $K \subset X$ and functions $\chi_i\colon K \times K \to [0,\infty)$, $i = 1, 2$. The next corollary corresponds in particular to the recent results in [2, 7, 8].

Corollary 3.2. *Suppose that $\alpha^{-1}(K) \subset K$ and that there exist constants $\eta_1, \eta_2, \varepsilon \in (0,\infty)$ such that $\eta_i\varepsilon < 1$ and*

$$\chi_i(x,y) \leqslant \eta_i \chi_i(a(x)+b(x)+c, a(y)+b(y)+c), \quad x, y \in K,\ i = 1, 2, \tag{3.1}$$

$$d(A(x)+B(x)+C, A(y)+B(y)+C) \leqslant \varepsilon d(x,y), \quad x, y \in Y. \tag{3.2}$$

Let $\varphi\colon K \to Y$ be a function that satisfies the conditions:

$$d(A(\varphi(x)) + B(\varphi(y)) + C, \varphi(a(x)+b(y)+c)) \leqslant \chi_1(x,y), \tag{3.3}$$

$$d(\varphi(a(x)+b(y)+c), A(\varphi(x)) + B(\varphi(y)) + C) \leqslant \chi_2(x,y) \tag{3.4}$$

for all $x, y \in K$, $a(x)+b(y)+c \in K$. Then there exists a unique solution $F\colon K \to Y$ of equation

$$F(a(x)+b(y)+c) = A(F(x)) + B(F(y)) + C, \quad x, y \in K, a(x)+b(y)+c \in K$$

such that

$$d(F(x),\varphi(x)) \leqslant \frac{\eta_1\chi_1(x,x)}{1-\eta_1\varepsilon}, \quad x \in K,$$

$$d(\varphi(x),F(x)) \leqslant \frac{\eta_2\chi_2(x,x)}{1-\eta_2\varepsilon}, \quad x \in K.$$

Proof. Define operations $\star\colon X \times X \to X$ and $\diamond\colon Y \times Y \to Y$ by:

$$x \star y = a(x) + b(y) + c, \quad x, y \in X,$$

$$z \diamond w = A(z) + B(w) + C, \quad z, w \in Y.$$

Then $(X,\star)$ and $(Y,\diamond)$ form groupoids that are a square symmetric, because a, b and A, B are homomorphisms such that $a \circ b = b \circ a$ and $A \circ B = B \circ A$. Moreover, since the function α is bijective, $(X,\star)$ is uniquely divisible by 2. Note that $2x = x \star x = \alpha(x)$ for $x \in X$ and $2z = z \diamond z = A(z) + B(z) + C$ for $z \in Y$, so (3.1) and (3.2) take the forms

$$\chi_i(x,y) \leqslant \eta_i\chi_i(2x,2y), \quad x,y \in K,\ i = 1,2,$$

$$d(2x,2y) \leqslant \varepsilon d(x,y), \quad x,y \in Y,$$

and (3.3) and (3.4) can be written as

$$d(\varphi(x) \diamond \varphi(y), \varphi(x \star y)) \leqslant \chi_1(x,y),$$

$$d(\varphi(x \star y), \varphi(x) \diamond \varphi(y)) \leqslant \chi_2(x,y)$$

for all $x, y \in K$, $x \star y \in K$. From these observations and from Corollary 3.1 the statement follows. □

Acknowledgements. The author would like to thank Professor Janusz Brzdęk for useful comments and helpful suggestions concerning this paper.

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