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OUTER MEASURES ON A COMMUTATIVE RING INDUCED BY MEASURES ON ITS SPECTRUM

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Abstract. On a commutative ring R we study outer measures induced by measures on $\operatorname{Spec}(R)$. The focus is on examples of such outer measures and on subsets of R that satisfy the Carathéodory condition.

1. Preliminaries and introduction

Throughout the note, R stands for a nonzero commutative ring with identity and $R^{\times}$ denotes the set of invertible elements of R. We define $\operatorname{Spec}(R)$ to be the spectrum of R, i.e., the family of all prime ideals $\wp \subset R$. The family of all maximal ideals of R will be denoted by $\operatorname{Max}(R)$. Recall that $\operatorname{Max}(R) \subseteq \operatorname{Spec}(R)$ and

$$\bigcup \operatorname{Spec}(R) = R \setminus R^{\times} = \bigcup \operatorname{Max}(R).$$

By "measure" we always mean a "non-negative σ-additive measure". The power set of a set X is denoted by 2^X. We use the following definition of an outer measure.

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Definition 1.1. A function $\varphi\colon 2^X \to [0,+\infty]$ is said to be an *outer measure* on a set X, if it satisfies two conditions:

(i) $\varphi(A) \leqslant \sum_{n=1}^{\infty} \varphi(B_n)$ for every sequence $\{B_n\}_{n=1}^{\infty}$ of subsets of X and every set A such that $A \subseteq \bigcup_{n=1}^{\infty} B_n$,

(ii) $\varphi(\emptyset) = 0$.

We refer to [1] for more information about commutative rings and to [3, 4] for elements of measure theory.

Consider a family $\mathcal{P} \subseteq \operatorname{Spec}(R)$ such that $\bigcup \mathcal{P} = R \setminus R^{\times}$. Consider also a σ-algebra $\mathfrak{M}$ of subsets of $\mathcal{P}$. Let $\mu\colon \mathfrak{M} \to [0,+\infty]$ be a measure. Given any set $A \subseteq R$, we define

$$\Omega(A) = \left\{ \mathcal{S} \in \mathfrak{M} : \bigcup \mathcal{S} \supseteq A \setminus R^{\times} \right\}.$$

In [2] we proved that

$$\mu^*\colon 2^R \ni A \mapsto \inf_{\mathcal{S} \in \Omega(A)} \mu(\mathcal{S}) \in [0,+\infty]$$

is an outer measure on the ring R. This outer measure will be referred to as the outer measure induced by μ. The main theorem of [2] shows that μ^* behaves well with respect to elementwise multiplication of sets.

The present note is a continuation of [2]. Our purpose is twofold: to characterize subsets of R that satisfy the Carathéodory condition with respect to μ^* and to discuss some quite general examples of the outer measures induced by measures on spectra.

2. Some measures on spectra and the outer measures induced by them

For any element $a \in R$, we define (a) to be the principal ideal of the ring R generated by a. Suppose that R is a unique factorization domain and is not a field. Let E be the set of all irreducible elements of R. Notice that, by the definition of an irreducible element, $E \subseteq R \setminus (R^{\times} \cup \{0\})$. Moreover, since R is not a field, we have $E \neq \emptyset$. Let us define $\mathcal{P}_{\mathrm{irr}}(R) = \{(a) : a \in E\}$. Then $\mathcal{P}_{\mathrm{irr}}(R) \subseteq \operatorname{Spec}(R)$ and

$$\bigcup \mathcal{P}_{\mathrm{irr}}(R) = R \setminus R^{\times}$$

(because R is a unique factorization domain).

Consider now a nonempty set $F \subseteq E$ and the map $\Phi\colon F \ni a \mapsto (a) \in \mathcal{P}_{\mathrm{irr}}(R)$. If $\mathfrak{N}$ is a σ-algebra of subsets of F and $\nu\colon \mathfrak{N} \to [0,+\infty]$ is a measure, then $\mathfrak{M} = \{\mathcal{S} \subseteq \mathcal{P}_{\mathrm{irr}}(R) : \Phi^{-1}(\mathcal{S}) \in \mathfrak{N}\}$ is a σ-algebra of subsets of $\mathcal{P}_{\mathrm{irr}}(R)$ and

$$\mu\colon \mathfrak{M} \ni \mathcal{S} \mapsto \nu(\Phi^{-1}(\mathcal{S})) \in [0,+\infty]$$

is a measure. Let us take a closer look on the outer measure $\mu^*\colon 2^R \to [0,+\infty]$.

Proposition 2.1. *In the situation described above, assume additionally that the map Φ is bijective. Then, for any $A \subseteq R$, we have $\mu^*(A) = \inf_{G \in \widetilde{\mathfrak{N}}} \nu(G)$, where*

$$\widetilde{\mathfrak{N}} = \big\{G \in \mathfrak{N} \mid \forall\, x \in A \setminus R^{\times}\, \exists\, g \in G : g \text{ is a divisor of } x\big\}.$$

Proof. By the surjectivity of Φ and the definition of a principal ideal, a set $\mathcal{S} \in \mathfrak{M}$ belongs to $\Omega(A)$ if and only if

$$\forall\, x \in A \setminus R^{\times}\, \exists\, g \in \Phi^{-1}(\mathcal{S}) : g \text{ is a divisor of } x.$$

Pick an arbitrary $G \subseteq F$. The injectivity of Φ yields that $\Phi^{-1}(\Phi(G)) = G$, and hence $\Phi(G) \in \mathfrak{M}$ if and only if $G \in \mathfrak{N}$. We thus obtain

$$\begin{aligned}
&\{\mu(\mathcal{S}) : \mathcal{S} \in \Omega(A)\}\\
&\quad= \big\{\nu(\Phi^{-1}(\mathcal{S})) \mid \mathcal{S} \in \mathfrak{M},\ \forall\, x \in A \setminus R^{\times}\, \exists\, g \in \Phi^{-1}(\mathcal{S}) : g \text{ is a divisor of } x\big\}\\
&\qquad= \big\{\nu(G) \mid G \in \mathfrak{N},\ \forall\, x \in A \setminus R^{\times}\, \exists\, g \in G : g \text{ is a divisor of } x\big\}\\
&\qquad\qquad= \{\nu(G) : G \in \widetilde{\mathfrak{N}}\}.
\end{aligned}$$

In view of the definition of μ^*, the proof is complete. □

In fact, the bijectivity assumption above means that the set F contains precisely one element from each class of associate elements of the set E.

Example 2.2. Let $R = \mathbb{Z}$, the ring of integers, $F = \mathbb{P}$, the set of prime numbers, and ν be the counting measure on $\mathbb{P}$. Observe that

$$\mathcal{P}_{\mathrm{irr}}(\mathbb{Z}) = \{(p) : p \in \mathbb{P}\} = \mathrm{Max}(\mathbb{Z}).$$

Therefore, $\Phi\colon \mathbb{P} \ni p \mapsto (p) \in \mathcal{P}_{\mathrm{irr}}(\mathbb{Z})$ is a bijection and the measure μ coincides with the counting measure on $\mathcal{P}_{\mathrm{irr}}(\mathbb{Z})$.

Consider next the set $A = \{-14, -5, -1, 0, 6, 9, 15, 28\}$. Define

$$\widetilde{\mathfrak{N}} = \big\{G \subseteq \mathbb{P} \,|\, \forall\, x \in A \setminus \mathbb{Z}^\times \,\exists\, g \in G:\, g \text{ is a divisor of } x\big\}$$

and recall that $\mathbb{Z}^\times = \{-1, 1\}$. Since $3 \in \mathbb{P}$, $5 \in \mathbb{P}$ and $9 = 3^2$, we get that $\{3, 5\} \subseteq G$ for any $G \in \widetilde{\mathfrak{N}}$. However, neither 3 nor 5 is a divisor of 28, and hence $\{3, 5\} \notin \widetilde{\mathfrak{N}}$. It is evident that $\{2, 3, 5\} \in \widetilde{\mathfrak{N}}$ and $\{3, 5, 7\} \in \widetilde{\mathfrak{N}}$. By Proposition 2.1, we obtain

$$\mu^*(A) = \inf_{G \in \widetilde{\mathfrak{N}}} \nu(G) = \nu(\{2, 3, 5\}) = 3.$$

Let us turn to function rings. Consider a nonempty set X and a field $\mathbb{F}$. We denote by $\mathbb{F}^X$ the ring of all functions $f\colon X \to \mathbb{F}$ (pointwise operations). Let R be a subring of $\mathbb{F}^X$ such that every constant function belongs to R (in other words, $\mathbb{F} \subseteq R$) and $R^\times = \{f \in R:\, f(x) \neq 0 \text{ for all } x \in X\}$. If $x \in X$, then $\wp_x = \{f \in R:\, f(x) = 0\}$ is a maximal ideal of the ring R. Notice also that

$$\bigcup_{x \in X} \wp_x = R \setminus R^\times.$$

We now define $\mathcal{P}_X(R) = \{\wp_x : x \in X\}$. Consider the map $\Psi\colon X \ni x \mapsto \wp_x \in \mathcal{P}_X(R)$. If $\mathfrak{N}$ is a σ-algebra of subsets of X and $\nu\colon \mathfrak{N} \to [0, +\infty]$ is a measure, then identically to the previous part of the section, $\mathfrak{M} = \{\mathcal{S} \subseteq \mathcal{P}_X(R):\, \Psi^{-1}(\mathcal{S}) \in \mathfrak{N}\}$ is a σ-algebra of subsets of $\mathcal{P}_X(R)$ and

$$\mu\colon \mathfrak{M} \ni \mathcal{S} \mapsto \nu(\Psi^{-1}(\mathcal{S})) \in [0, +\infty]$$

is a measure.

Proposition 2.3. *In the situation described above, assume additionally that Ψ is an injection. Then, for any $A \subseteq R$, we have $\mu^*(A) = \inf_{Y \in \mathfrak{N}_0} \nu(Y)$, where*

$$\mathfrak{N}_0 = \big\{Y \in \mathfrak{N} \,|\, \forall\, f \in A \setminus R^\times :\, Y \cap f^{-1}(0) \neq \emptyset\big\}.$$

Proof. The proof is completely analogous to the proof of Proposition 2.1. The crucial point is that a set $\mathcal{S} \in \mathfrak{M}$ belongs to $\Omega(A)$ if and only if

$$\forall\, f \in A \setminus R^\times \,\exists\, x \in \Psi^{-1}(\mathcal{S}):\, f(x) = 0. \qquad \square$$

The injectivity of Ψ means that R separates the points in the set X. Hence, if X is a normal topological space and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then $\mathcal{C}(X, \mathbb{F})$, the ring of all continuous functions $f \colon X \to \mathbb{F}$, satisfies the assumptions of Proposition 2.3. If n is a positive integer and $\mathbb{F}$ is an algebraically closed field, then so does the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$. Proposition 2.3 can also be applied to various rings of differentiable or holomorphic functions. The proposition generalizes [2, Proposition 3].

Let us finally discuss an example showing that $\mathcal{P}_X(R)$ does not have to coincide with $\operatorname{Max}(R)$.

Example 2.4. Suppose that X is an infinite set. Define I to be the family of all functions $f \colon X \to \mathbb{F}$ with the property that

$$\exists\, Y \subseteq X : \begin{cases} Y \text{ is finite,} \\ f(x) = 0 \text{ for any } x \in X \setminus Y. \end{cases}$$

Then I is a proper ideal of the ring $R = \mathbb{F}^X$. Let Z be an infinite subset of X such that $X \setminus Z$ is also infinite. Consider the function $h \in R$ defined by

$$h(x) = \begin{cases} 1, & \text{if } x \in Z, \\ 0, & \text{if } x \notin Z. \end{cases}$$

Since $h \notin I$ and $I + Rh$ is a proper ideal of R, we get $I \notin \operatorname{Max}(R)$. However, obviously, $I \subseteq \wp$ for some $\wp \in \operatorname{Max}(R)$. Observe that

$$\forall\, x \in X\, \exists\, f \in I : \ f(x) \neq 0.$$

Consequently, no point $x \in X$ has the property that $f(x) = 0$ for all $f \in \wp$. This yields $\wp \notin \mathcal{P}_X(R)$.

3. μ^*-measurable sets

We begin with a very brief recapitulation of the Carathéodory condition. Let φ be an outer measure on a set X.

Definition 3.1. A set $A \subseteq X$ is said to be *φ-measurable* (or to satisfy the *Carathéodory condition with respect to φ*), if

$$\forall\, T \subseteq X : \ \varphi(T) = \varphi(T \cap A) + \varphi(T \setminus A).$$

THEOREM 3.2 (Carathéodory). *The totality $\mathfrak{M}$ of φ-measurable subsets of X is a σ-algebra. Moreover,*

(i) *the restriction $\varphi|_{\mathfrak{M}}$ is a measure,*
(ii) *if $\varphi(A) = 0$ for some $A \subseteq X$, then $A \in \mathfrak{M}$.*

It is obvious that $\mathfrak{M} \supseteq \{A \in 2^X : \varphi(X \setminus A) = 0\}$. The class of "obviously φ-measurable" sets is, in fact, a bit larger. We will say that a set $A \subseteq X$ satisfies condition $(\bullet)$ with respect to φ, if $\varphi(B) \in \{0, +\infty\}$ for any $B \subseteq A$.

PROPOSITION 3.3. *Let $A \subseteq X$. Suppose that either A or $X \setminus A$ satisfies condition $(\bullet)$ with respect to φ. Then A is a φ-measurable set.*

PROOF. Pick an arbitrary $T \subseteq X$. By the definition and the monotonicity of an outer measure,

$$\max\{\varphi(T \cap A), \varphi(T \setminus A)\} \leqslant \varphi(T) \leqslant \varphi(T \cap A) + \varphi(T \setminus A).$$

Hence, $\varphi(T) = \varphi(T \setminus A)$ whenever $\varphi(T \cap A) = 0$, and $\varphi(T) = \varphi(T \cap A)$ whenever $\varphi(T \setminus A) = 0$. It is obvious that $\varphi(T) = +\infty$ whenever $\varphi(T \cap A) = +\infty$ or $\varphi(T \setminus A) = +\infty$. Condition $(\bullet)$ therefore implies that $\varphi(T) = \varphi(T \cap A) + \varphi(T \setminus A)$. □

Let us also recall some properties of the outer measure $\mu^* : 2^R \to [0, +\infty]$ induced by a measure μ on a suitable set $\mathcal{P} \subseteq \operatorname{Spec}(R)$.

PROPOSITION 3.4. *If $A, B \subseteq R \setminus R^\times$ and $C \subseteq R$, then*

(i) $\mu^*(AB) = \min\{\mu^*(A), \mu^*(B)\}$,
(ii) $\mu^*(C) = \mu^*(C \setminus R^\times)$.

The above proposition is a part of [2, Theorem 1]. Notice that the set C is μ^*-measurable whenever $C \subseteq R^\times$ or $C \supseteq R \setminus R^\times$.

We are now ready to state and prove the main result of the note.

THEOREM 3.5. *Let $A \subseteq R \setminus R^\times$ be such that $0 < \mu^*(A) < +\infty$. Assume moreover that $0 < \mu^*(B) < +\infty$ for some $B \subseteq R \setminus (A \cup R^\times)$. Then the set A is not μ^*-measurable.*

PROOF. Suppose, in order to derive a contradiction, that A is μ^*-measurable. Pick arbitrary sets $E \subseteq A$ and $Z \subseteq R$. Define $T = EZ \cup A$. Then

$$\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \setminus A) = \mu^*(A) + \mu^*(EZ \setminus A).$$

If $\mathcal{S} \in \Omega(A)$, $a \in E$ and $b \in Z$, then $a \in \wp$ for some $\wp \in \mathcal{S}$ (because $E \subseteq A$ and $A \subseteq R \setminus R^{\times}$), and hence $ab \in \wp$. This proves that $\Omega(A) \subseteq \Omega(EZ)$. Consequently, $\Omega(A) \subseteq \Omega(EZ \cup A) = \Omega(T)$. We therefore obtain that

$$\mu^*(A) \geqslant \mu^*(T) = \mu^*(A) + \mu^*(EZ \setminus A).$$

Since $\mu^*(A) < +\infty$, the above inequality yields $\mu^*(EZ \setminus A) = 0$. Thus we have proved the following property:

$$\forall E \subseteq A \, \forall Z \subseteq R : \; \mu^*(EZ \setminus A) = 0.$$

Define now $W = AB \cup B$. Recall that $B \cap A = \emptyset$. Combining the property we have just proved with the μ^*-measurability of A, we get

$$\mu^*(AB) = \mu^*(AB \cap A) + \mu^*(AB \setminus A) = \mu^*(AB \cap A).$$

Consequently,

$$\mu^*(W \cap A) = \mu^*((AB \cap A) \cup (B \cap A)) = \mu^*(AB \cap A) = \mu^*(AB).$$

Notice also that by the monotonicity of μ^*,

$$\mu^*(W \setminus A) = \mu^*((AB \setminus A) \cup (B \setminus A)) \geqslant \mu^*(B \setminus A) = \mu^*(B).$$

The same argument as in the previous part of the proof shows that $\Omega(B) \subseteq \Omega(W)$, and hence $\mu^*(B) \geqslant \mu^*(W)$. It follows therefore from the μ^*-measurability of A that

$$\mu^*(B) \geqslant \mu^*(W) = \mu^*(W \cap A) + \mu^*(W \setminus A) \geqslant \mu^*(AB) + \mu^*(B).$$

Since $\mu^*(B) < +\infty$, the above inequalities yield $\mu^*(AB) = 0$. But by Proposition 3.4 (i) we have $\mu^*(AB) = \min\{\mu^*(A), \mu^*(B)\} > 0$, a contradiction. □

It seems worth noting that a set $A \subseteq R$ is μ^*-measurable if and only if so is $A \setminus R^{\times}$ (this follows from the fact that every subset of $R^{\times}$ is μ^*-measurable). Recall also that A is μ^*-measurable if and only if so is $R \setminus A$. In view of these two equivalences and Proposition 3.4 (ii), our main theorem implies the following corollary.

Corollary 3.6. *If $A \subseteq R$ is a μ^*-measurable set and neither A nor $R \setminus A$ satisfies condition $(\bullet)$ with respect to μ^*, then $\mu^*(A) = +\infty = \mu^*(R \setminus A)$.*

As an easy consequence of the main theorem, we also obtain a complete characterization of μ^*-measurable sets in the case where $\mu^*(R) < +\infty$.

Corollary 3.7. *Suppose that μ^* is finite (i.e., $\mu^*(R) < +\infty$). Then $A \subseteq R$ is a μ^*-measurable set if and only if either $\mu^*(A) = 0$ or $\mu^*(R \setminus A) = 0$.*

Let us conclude the note with an example concerning the case where $\mu^*(R) = +\infty$.

Example 3.8. Consider the measure $\mu\colon 2^{\mathrm{Max}(\mathbb{Z})} \to [0, +\infty]$ defined by

$$\mu(\{(p)\}) = \begin{cases} +\infty, & \text{if } p = 2, \\ 1/p, & \text{if } p \in \mathbb{P} \setminus \{2\}. \end{cases}$$

We can apply Proposition 2.1 to the outer measure $\mu^*\colon 2^{\mathbb{Z}} \to [0, +\infty]$. If $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$ is neither a power of 2 nor the opposite of a power of 2, then $\mu^*(\{k\}) = 1/p_k$, where p_k stands for the largest prime divisor of k. Notice also that $\mu^*(\{0\}) = 0$. Therefore, a set $A \in 2^{\mathbb{Z}}$ satisfies condition ($\bullet$) with respect to μ^* if and only if

$$A \setminus \{-1, 0, 1\} \subseteq \bigcup_{n=1}^{\infty} \{-2^n, 2^n\}.$$

Observe now that $\mu^*(\mathbb{P}) = +\infty = \mu^*(\mathbb{Z} \setminus \mathbb{P})$. It is obvious that $\mathbb{P}$ and $\mathbb{Z} \setminus \mathbb{P}$ do not satisfy condition ($\bullet$) with respect to μ^*. Define $T = \{3, 9\}$. Then

$$\mu^*(T) = \mu^*(T \cap \mathbb{P}) = \mu^*(T \setminus \mathbb{P}) = 1/3.$$

Consequently, $\mathbb{P}$ is not a μ^*-measurable set.

References

[1] Atiyah M.F., MacDonald I.G., *Introduction to Commutative Algebra*, Addison–Wesley Publishing Company, Reading, 1969.

[2] Dudzik D., Skrzyński M., *An outer measure on a commutative ring*, Algebra Discrete Math. **21** (2016), no. 1, 51–58.

[3] Federer H., *Geometric Measure Theory*, Springer, Berlin, 1969.

[4] Halmos P.R., *Measure Theory*, Springer, New York, 1976.

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