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A SIMPLE PROOF OF THE POLAR DECOMPOSITION THEOREM

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Abstract. In this expository paper, we present a new and easier proof of the Polar Decomposition Theorem. Unlike in classical proofs, we do not use the square root of a positive matrix. The presented proof is accessible to a broad audience.

1. Introduction

The algebra of all real (or complex) $n \times n$ matrices is denoted by $M_n(\mathbb{K})$. Let us recall a well-known result.

Theorem 1.1 (Polar Decomposition). *Suppose that $A \in M_n(\mathbb{K})$ is a nonzero matrix. Then there are $U, P \in M_n(\mathbb{K})$ such that U is unitary, $P \geq 0$, and $A = UP$.*

This result is called the Polar Decomposition, and its proof uses the square root of a positive matrix (or The Functional Calculus). Different proofs can be found, e.g., in [1, 2, 3]. The aim of this article is to introduce a new proof of the Polar Decomposition. Let us point out that our proof neither uses the square root of a positive operator nor The Functional Calculus.

It should be easier to prove Polar Decomposition Theorem, if we consider operators instead of matrices. Using elementary techniques, Polar Decomposition will be proved. Throughout this paper we assume that the considered

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Hilbert spaces are finite dimensional and their dimensions are not less than 2. Let $\mathcal{B}(\mathcal{H};\mathcal{K})$ denote the Banach space of all bounded linear operators (between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$) and we write $\mathcal{B}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H};\mathcal{H})$. We shall identify $\mathcal{B}(\mathcal{H})$ (where $\dim\mathcal{H} = n$) and $M_n(\mathbb{K})$ in the natural way. Let us denote the unit sphere by $S(\mathcal{H}) := \{x \in \mathcal{H} : \|x\| = 1\}$. Throughout this work, all Hilbert spaces are assumed to be real or complex.

If $P \in \mathcal{B}(\mathcal{H})$, then P is *positive* if $\langle Px|x\rangle \geq 0$ for all $x \in \mathcal{H}$. In symbols this is denoted by $P \geq 0$. If $U \in \mathcal{B}(\mathcal{H})$, then U is an *isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$, or, equivalently, $\langle Ux|Uy\rangle = \langle x|y\rangle$ for all $x, y \in \mathcal{H}$.

2. A new and easy proof

In this section we present an elementary proof of the Polar Decomposition. The method of proof presented here is different from that of [1, 2, 3]. We start with the following lemma.

Lemma 2.1. *Assume that* $\dim\mathcal{H} = 2$. *If* $A \in \mathcal{B}(\mathcal{H};\mathcal{K})$ *and* $A \neq 0$, *then there are vectors* x_1, x_2 *in* $S(\mathcal{H})$ *such that* $x_1 \perp x_2$ *and* $Ax_1 \perp Ax_2$.

Proof. Fix $x, z \in S(\mathcal{H})$ such that $x \perp z$. If $\langle Ax|Az\rangle = 0$, we define $x_1 := x$, $x_2 := z$.

Now, assume that $\langle Ax|Az\rangle = c \neq 0$. Then we define a vector $y := \frac{c}{|c|}z$. It follows that $\langle Ax|Ay\rangle \in \mathbb{R}$ and $y \in S(\mathcal{H})$. Moreover, $x \perp y$ and $\langle Ay|Ax\rangle \in \mathbb{R}$. Then we define a mapping $\varphi\colon [0,1] \to \mathbb{K}$ by

$$\varphi(t) := \left\langle A\left(\frac{(1-t)x + ty}{\|(1-t)x + ty\|}\right) \,\middle|\, A\left(\frac{t(-x) + (1-t)y}{\|t(-x) + (1-t)y\|}\right)\right\rangle.$$

Define now $N_1(t) := \|(1-t)x + ty\|$ and $N_2(t) := \|t(-x) + (1-t)y\|$. It is easy to check that $\varphi(t) \in \mathbb{R}$ for all $t \in [0,1]$. Indeed, we have

$$\varphi(t) = \frac{(1-t)(-t)}{N_1(t)N_2(t)}\langle Ax|Ax\rangle + \frac{(1-t)^2}{N_1(t)N_2(t)}\langle Ax|Ay\rangle + \frac{-t^2}{N_1(t)N_2(t)}\langle Ay|Ax\rangle + \frac{t(1-t)}{N_1(t)N_2(t)}\langle Ay|Ay\rangle \in \mathbb{R}.$$

In fact, we can write $\varphi\colon [0,1] \to \mathbb{R}$. It is easy to see that φ is continuous. Moreover, we have

$$\varphi(0) = \left\langle A\left(\frac{x}{\|x\|}\right) \,\middle|\, A\left(\frac{y}{\|y\|}\right)\right\rangle$$

and

$$\varphi(1) = -\left\langle A\left(\frac{y}{\|y\|}\right) \,\middle|\, A\left(\frac{x}{\|x\|}\right)\right\rangle = -\left\langle A\left(\frac{x}{\|x\|}\right) \,\middle|\, A\left(\frac{y}{\|y\|}\right)\right\rangle,$$

which means $\varphi(0) = -\varphi(1)$. Thus we get $\varphi(0) \le 0 \le \varphi(1)$ or $\varphi(1) \le 0 \le \varphi(0)$. Without loss of generality, we may assume that $\varphi(0) \le 0 \le \varphi(1)$. Using the Darboux property we get $\varphi(t_o) = 0$ for some $t_o \in [0,1]$. Thus for the vectors

$$x_1 := \frac{(1-t_o)x + t_o y}{\|(1-t_o)x + t_o y\|}, \quad x_2 := \frac{t_o(-x) + (1-t_o)y}{\|t_o(-x) + (1-t_o)y\|}$$

we have $x_1 \perp x_2$ and $0 = \varphi(t_o) = \langle A(x_1)\,|\,A(x_2)\rangle$, therefore $Ax_1 \perp Ax_2$. The proof is complete. □

The next result is a consequence of the above lemma.

THEOREM 2.2. *Assume that* $\dim \mathcal{H} = n$. *If* $A \in \mathcal{B}(\mathcal{H};\mathcal{K})$ *and* $A \neq 0$, *then there are vectors* $x_1, \dots, x_n$ *in* $S(\mathcal{H})$ *such that*

$$x_j \perp x_k \quad \textit{and} \quad Ax_j \perp Ax_k, \quad j,k \in \{1,\dots,n\}, j \neq k.$$

PROOF. We proceed by induction (with respect to the dimension of $\mathcal{H}$). For $n = 2$ we have proved that it is true (see Lemma 2.1).

Assume the statement holds for n. We will prove it for $n+1$. Suppose that $\dim \mathcal{H} = n+1$. Obviously $S(\mathcal{H})$ is compact. Therefore there is a y_o in $S(\mathcal{H})$ such that $\|A\| = \|Ay_o\|$. It is clear that $\dim\{y_o\}^\perp = n$. Then, by inductive assumption, there are the vectors $x_1, \dots, x_n \in S\left(\{y_o\}^\perp\right) \subset S(\mathcal{H})$ such that $x_j \perp x_k$ and $Ax_j \perp Ax_k$, for $j,k \in \{1,\dots,n\}$, $j \neq k$. We define a vector $x_{n+1} := y_o$. It is easy to observe that $x_j \perp x_{n+1}$ for all $j \in \{1,\dots,n\}$.

We will show that $Ax_j \perp Ax_{n+1}$ for all $j \in \{1,\dots,n\}$. Assume, contrary to our claim, that $\langle Ax_{j_o} | Ax_{n+1}\rangle = c \neq 0$, for some $x_{j_o} \in \{x_1,\dots,x_n\}$. We define a vector $u := \frac{\bar{c}}{|c|} x_{j_o}$. It follows that $u \perp x_{n+1}$, $\|u\| = 1$ and

$$\langle Au | Ax_{n+1}\rangle = |c| \in \mathbb{R}. \tag{1}$$

Let $\alpha \in (0,1)$. It is easy to check that $\alpha u + \sqrt{1-\alpha^2}x_{n+1} \in S(\mathcal{H})$. Therefore

$$\begin{aligned}\|A\|^2 &\geq \left\| A\left(\alpha u + \sqrt{1-\alpha^2}x_{n+1}\right)\right\|^2 \\ &= \alpha^2\|Au\|^2 + 2\mathfrak{Re}\left(\alpha\sqrt{1-\alpha^2}\,\langle Au|Ax_{n+1}\rangle\right) + (1-\alpha^2)\|Ax_{n+1}\|^2\end{aligned}$$

and making use of (1), we obtain

$$\begin{aligned}\|A\|^2 &\geq \alpha^2\|Au\|^2 + 2\alpha\sqrt{1-\alpha^2}\,\langle Au|Ax_{n+1}\rangle + (1-\alpha^2)\|Ay_o\|^2 \\ &= -\alpha^2\|Au\|^2 + 2\alpha\sqrt{1-\alpha^2}|c| + (1-\alpha^2)\|A\|^2.\end{aligned}$$

It follows from the above inequality that

$$\alpha^2\|A\|^2 \geq \alpha^2\|Au\|^2 + 2\alpha\sqrt{1-\alpha^2}|c|$$

and

$$\alpha^2\left(\|A\|^2 - \|Au\|^2\right) \geq 2\alpha\sqrt{1-\alpha^2}|c|.$$

Thus we have

$$\alpha\left(\|A\|^2 - \|Au\|^2\right) \geq 2\sqrt{1-\alpha^2}|c|.$$

By letting α tend to 0, we get $0 \geq 2|c|$, which is a contradiction. □

As an illustration of the applications of this theorem we prove here the polar decomposition of an operator. The main result of this paper is the following.

THEOREM 2.3 (Polar Decomposition). *Let $\mathcal{H}$ be a Hilbert space such that $\dim\mathcal{H} = n$. If $A \in \mathcal{B}(\mathcal{H})$, then there are $U, P \in \mathcal{B}(\mathcal{H})$ such that U is unitary, $P \geq 0$, and $A = UP$.*

PROOF. Assume that $\dim(\ker A)^\perp = p$. Thus we obtain $\dim\ker A = n-p$. It is clear that $A|_{(\ker A)^\perp} : (\ker A)^\perp \to \mathcal{H}$ is injective. We choose $\{x_1, \ldots, x_p\} \subset S(\mathcal{H}) \cap (\ker A)^\perp$ such that

$$x_j \perp x_k \quad \text{and} \quad Ax_j \perp Ax_k, \quad j,k \in \{1,\ldots,p\}, j \neq k,$$

by Theorem 2.2. By the injectivity of $A|_{(\ker A)^\perp}$, we obtain $Ax_k \neq 0$ for all $k \in \{1, \ldots, p\}$. It is easy to see that $\{x_1, \ldots, x_p\}$ and $\left\{\frac{1}{\|Ax_1\|}Ax_1, \ldots, \frac{1}{\|Ax_p\|}Ax_p\right\}$ are two orthonormal bases for $(\ker A)^\perp$ and $A\left((\ker A)^\perp\right)$, respectively.

Let $\{e_1, \ldots, e_{n-p}\}$ be an orthonormal basis for $\ker A$ and let $\{y_1, \ldots, y_{n-p}\}$ be an orthonormal basis for $A\left((\ker A)^\perp\right)^\perp$. Then we define a positive operator $P \in \mathcal{B}(\mathcal{H})$ by

$$Px_k := \|Ax_k\|x_k, \quad k \in \{1, \ldots, p\}; \quad Pe_t := 0, \quad t \in \{1, \ldots, n-p\}.$$

We can now define an isometry $U \in \mathcal{B}(\mathcal{H})$ by

$$Ux_k := \frac{1}{\|Ax_k\|}Ax_k, \quad k \in \{1, \ldots, p\}; \quad Ue_t := y_t, \quad t \in \{1, \ldots, n-p\}.$$

We have

$$UPx_k = U\left(\|Ax_k\|x_k\right) = \|Ax_k\|U\left(x_k\right) = \|Ax_k\|\frac{1}{\|Ax_k\|}Ax_k = Ax_k,$$

and $UPe_t = U(0) = 0 = Ae_t$. We have shown that UP and A coincide on the basis, thus they are equal: $UP = A$. This completes the proof. $\square$

3. Remark

Now, we are going to present one more application of Theorem 2.2. Namely, we will prove that any injective operator can restrict to a similarity (a scalar multiple of an isometry).

THEOREM 3.1. *Assume that* $\dim \mathcal{H} = n = 2m \geq 4$. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be injective. Then there is a subspace* $\mathcal{M} \subset \mathcal{H}$ *such that* $\dim \mathcal{M} = \frac{1}{2}n = m$ *and* $A|_{\mathcal{M}}$ *is a similarity (a scalar multiple of an isometry).*

PROOF. We choose $\{x_1, x_2, \ldots, x_{2m}\} \subset S(\mathcal{H})$ such that

$$x_j \perp x_k \quad \text{and} \quad Ax_j \perp Ax_k, \quad j, k \in \{1, 2, \ldots, 2m\}, j \neq k;$$

see Theorem 2.2. Without loss of generality, we may assume that

$$\|Ax_1\| \leq \|Ax_2\| \leq \ldots \leq \|Ax_{2m}\|.$$

Choose $\gamma \in \mathbb{R}$ such that

$$\|Ax_1\| \leq \ldots \leq \|Ax_m\| \leq \gamma \leq \|Ax_{m+1}\| \leq \ldots \leq \|Ax_{2m}\|.$$

We consider the following collection of subspaces:

$$\begin{aligned} X_1 &:= \operatorname{span}\{x_1, x_{2m}\}, \\ X_2 &:= \operatorname{span}\{x_2, x_{2m-1}\}, \\ &\vdots \\ X_m &:= \operatorname{span}\{x_m, x_{m+1}\}. \end{aligned}$$

It is easy to observe that $X_j \perp X_k$ for $j,k \in \{1,\ldots,m\}$, $j \neq k$. Since $S(X_1) = X_1 \cap S(\mathcal{H})$, the unit sphere $S(X_1)$ is an arcwise connected subset of $\mathcal{H}$. Moreover, we have $\|Ax_1\| \leq \gamma \leq \|Ax_{2m}\|$. Hence there is a vector $w_1 \in S(X_1)$ such that $\gamma = \|Aw_1\|$.

In a similar way we obtain a vector $w_2 \in S(X_2)$ such that $\gamma = \|Aw_2\|$. Indeed, since $S(X_2) = X_2 \cap S(\mathcal{H})$, the unit sphere $S(X_2)$ is an arcwise connected subset of $\mathcal{H}$. Moreover, we have $\|Ax_2\| \leq \gamma \leq \|Ax_{2m-1}\|$. Hence there is a vector $w_2 \in S(X_2)$ such that $\gamma = \|Aw_2\|$.

This and similar reasoning shows that there are vectors $w_1, \ldots, w_m$ such that

$$w_j \in S(X_j), \quad \gamma = \|Aw_j\|, \quad \text{where } j \in \{1,\ldots,m\}.$$

It is easy to check that $\{w_1,\ldots,w_m\}$ is an orthonormal set in $\mathcal{H}$.

It is not hard to see that $A(X_j) \perp A(X_k)$ for $j,k \in \{1,\ldots,m\}$, $j \neq k$. Therefore $\{\frac{1}{\gamma}Aw_1,\ldots,\frac{1}{\gamma}Aw_m\}$ is also an orthogonal set in $\mathcal{H}$. We define a subspace $\mathcal{M} := \operatorname{span}\{w_1,\ldots,w_m\}$. Thus we have $\dim\mathcal{M} = m = \frac{1}{2}n$. Now, we define an operator $T \in \mathcal{B}(\mathcal{M};\mathcal{H})$ as follows:

$$Tw_j := \frac{1}{\gamma}Aw_j, \quad j \in \{1,2,\ldots,m\}.$$

It follows that T is an isometry. Finally, we get $A|_{\mathcal{M}} = \gamma T$. The proof is complete. □

Theorem 3.2. *Assume that* $\dim\mathcal{H} = n = 2m+1 \geq 3$. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be injective. Then there is a subspace* $\mathcal{M} \subset \mathcal{H}$ *such that* $\dim\mathcal{M} = \frac{1}{2}(n+1) = m+1$ *and* $A|_{\mathcal{M}}$ *is a similarity.*

The proof of Theorem 3.2 runs similarly.

References

[1] Conway J.B., *A course in functional analysis*, Springer-Verlag, New York, 1985.
[2] Gantmacher F.R., Brenner J.L., *Applications of the theory of matrices*, Dover Publications, New York, 2005.
[3] Pedersen G.K., *Analysis now*, Springer-Verlag, New York, 1989.

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