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Report of Meeting

The Seventeenth Katowice–Debrecen Winter Seminar Zakopane (Poland), February 1–4, 2017

The Seventeenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held in Hotel Geovita in Zakopane, Poland, from February 1 to 4, 2017. The meeting was organized by the Institute of Mathematics of the University of Silesia.

14 participants came from the University of Debrecen (Hungary), 13 from the University of Silesia in Katowice (Poland) and one from each of the universities: Łódź University of Technology (Poland) and University of Miskolc (Hungary).

Professor Maciej Sablik opened the Seminar and welcomed the participants to Zakopane.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iteration theory, equations on abstract algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity. Interesting discussions were generated by the talks.

There was also a Problem Session and a festive dinner.

The closing address was given by Professor Zsolt Páles. His invitation to the Eighteenth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in February 2018 in Hungary was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in Section 1, problems and remarks in chronological order in Section 2, and the list of participants in the final section.

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1. Abstracts of talks

Roman Badora: *Remarks on Farah's theorem, II* (Joint work with Barbara Przebieracz)

In the talk we present a proposal on how to free themselves from the assumption of the finiteness of groups which occurs in theorems proved by Farah and Przebieracz.

Karol Baron: *On some sets of orthogonally additive functions*

We examine density of some subsets of the space of all orthogonally additive functions mapping a real inner product space of dimension at least 2 into a linear topological space and equipped with the Tychonoff topology. Among others we consider sets of orthogonally additive functions which are bijective, injective, surjective, and that with big graphs, respectively.

Mihály Bessenyei: *Separation problems in context of h-convexity* (Joint work with Evelin Pénzes)

The concept of h-convexity extends the notion of classical convexity, using some nonnegative function h of the weights on the right-hand side in the defining inequality. The aim of the talk is to show that nonconstant h-affine functions appear only in the classical convexity case. Affine and convex separation problems are also studied. The obtained results suggest that, in view of Convex Geometry or Convex Analysis, the notion of h-convexity may have only particular importance.

Zoltán Boros: *A characterization of affine differences on intervals*

In his presentation at the Conference on Inequalities and Applications 2016, Mirosław Adamek established a sandwich type theorem for convexity on an interval I with a control function $G\colon [0,1]\times I\times I \to \mathbb{R}$ under the assumption that there exists an affine function with control function G. According to his hypothesis, there exists a function $f\colon I \to \mathbb{R}$ satisfying the functional equation

$$f\big(tx+(1-t)y\big) = tf(x)+(1-t)f(y)+G(t,x,y) \tag{1}$$

for every $t \in [0,1]$ and $x, y \in I$. Adamek posed the problem to describe control functions G such that the functional equation (1) admits a solution f. In this presentation such functions G are characterized by the functional equation

$$G\big(s, rx+(1-r)y, tx+(1-t)y\big) = G\big(sr+(1-s)t, x, y\big) - s\,G\big(r, x, y\big) - (1-s)G\big(t, x, y\big)$$

fulfilled for all $s, r, t \in [0,1]$ and $x, y \in I$.

Pál Burai: *An extension theorem and its possible applications*

Let $\varphi\colon I \to \mathbb{R}$ be an invertible function such that the domain of its inverse is diadically closed. In this case we can define the following two-place function:

$$A_\varphi(x,y) := \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \quad x, y \in I.$$

We consider the equality problem in this class. For this, we need an extension theorem of additive functions.

Szymon Draga: *Notes on the polynomial-like iterative equations order*

We will discuss a new case where the order of a homogenous polynomial-like iterative equation, namely functional equation of the form

$$a_n f^n(x) + \ldots + a_1 f(x) + a_0 x = 0,$$

can be lowered. We will also review the existing results.

Włodzimierz Fechner: *Systems of inequalities related to quadratic-multiplicative mappings* (Joint work with Szymon Głąb)

C. Hammer and P. Volkmann [2] described real-to-real quadratic-multiplicative functions.

Theorem (Hammer–Volkmann). *Assume that $Q\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary mapping. Then Q is a quadratic and multiplicative function if and only if there exists an additive and multiplicative function $T\colon \mathbb{C} \to \mathbb{C}$ such that Q is of the form*

$$Q(x) = |T(x)|^2, \quad x \in \mathbb{R}.$$

A generalization of this theorem is due to Z. Gajda [1]. If K is a field, then we denote by $\overline{K}$ the algebraic closure of K and if $\zeta \in \overline{K}$, then $K(\zeta)$ is the smallest field such that $K \subseteq K(\zeta) \subseteq \overline{K}$.

Theorem (Gajda). *Assume that X is a commutative unitary ring, K is a field with characteristic different from 2 and $Q\colon X \to K$ is an arbitrary mapping. Then Q is a quadratic and multiplicative function if and only if there exist an element $\zeta \in \overline{K}$ such that $\zeta^2 \in K$ and additive and multiplicative mappings $u\colon X \to K(\zeta)$ and $v\colon X \to K(\zeta)$ such that*

$$u(x) + v(x) \in K, \quad u(x) - v(x) \in \zeta K, \quad x \in X,$$

and Q is of the form

$$Q(x) = u(x)v(x), \quad x \in X.$$

The purpose of the talk is to discuss various systems of inequalities related to quadratic-multiplicative operators in the spirit of M. Rădulescu's result [3].

References

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[3] Rădulescu M., *On a supra-additive and supra-multiplicative operator of $C(X)$*, Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S.) **24(72)** (1980), no. 3, 303–305.

Żywilla Fechner: *m-sine functions on hypergroups* (Joint work with László Székelyhidi)

We consider a hypergroup K and an exponential function $m\colon K \to \mathbb{C}$ i.e.

$$m(x * y) = m(x)m(y), \quad x, y \in K.$$

A comprehensive study of functional equations on hypergroups can be found in the monograph [2] of L. Székelyhidi. For a given exponential a function $f\colon K \to \mathbb{C}$ is called an m-sine function if

$$f(x * y) = m(x)f(y) + m(y)f(x), \quad x, y \in K.$$

In this talk we are going to discuss some aspects of so called m-sine functions on special type of hypergroups. These functions have been introduced in the paper [1]. Motivated by the paper of M. Voit [3] we discuss some applications of m-sine functions to the random walks on commutative hypergroups.

References

[1] Fechner Ż., Székelyhidi L., *Sine functions on hypergroups*, Arch. Math. (Basel) **106** (2016), no. 4, 371–382.
[2] Székelyhidi L., *Functional Equations on Hypergroups*, World Scientific Publishing Co. Pte. Ltd., Hackensack, 2013.
[3] Voit M., *Sine functions on compact commutative hypergroups*, Arch. Math. (Basel) **107** (2016), no. 3, 259–263.

György Gát: *Subsequences of partial sums of trigonometric Fourier series, Zygmunt Zalcwasser's problem*

In the theory of trigonometric Fourier series it is of main interest how to reconstruct the function from the partial sums of its Fourier series. In 1966 Carleson [1] showed that if $f \in L^2$, then the partial sums converge to the function almost everywhere. It is also a fundamental question, how to reconstruct a function in L^1 from the partial sums of its Fourier series. Lebesgue showed that for each integrable function we have the almost everywhere convergence of Fejér means $\sigma_n f = \frac{1}{n+1}\sum_{m=0}^{n} S_m f \to f$.

It is also of prior interest, what can be said - with respect to this reconstruction issue - if we have only a subsequence of the partial sums. In 1982 Totik showed [2] that for each subsequence (n_j) of the sequence of natural numbers there exists an integrable function f such that $\sup_j |S_{n_j} f| = +\infty$ everywhere.

In 1936 Zygmunt Zalcwasser [3] asked how "rare" can a sequence of integers (n_j) be such that

$$\frac{1}{N}\sum_{j=1}^{N} S_{n_j} f \to f$$

a.e. for every function $f \in L^1$. In this talk we give an answer for this question.

References

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[2] Totik V., *On the divergence of Fourier series*, Publ. Math. Debrecen **29** (1982), no. 3–4, 251–264.
[3] Zalcwasser Z., *Sur la sommabilité des séries de Fourier*, Studia Math. **6** (1936), 82–88.

Roman Ger: *Delta-convexity with given weights*

Some differentiability results from the paper of D.S. Marinescu & M. Monea [3] on delta-convex mappings, obtained for real functions, are extended for mappings with values in a reflexive normed linear space. In this way, applying Lemma 1 from [4], we are nearing the completion of studies established in papers [1], [2] and [3].

References

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Attila Gilányi: *A computer assisted approach to m-convexity* (Joint work with Nelson Merentes and Roy Quintero)

According to the definition of Gheorghe Toader [1], a set $H \subseteq \mathbb{R}^2$ is called m-convex if $tx + m(1-t)y \in H$ for all $x, y \in H$ and $t \in [0,1]$, where $m \in [0,1]$ is a fixed real number. The m-convex hull of a nonempty set $S \subseteq \mathbb{R}^2$ is defined as the intersection of all m-convex subsets of $\mathbb{R}^2$ containing S. Connected to these concepts, we present an animation of the m-convex hulls of sets consisting of finitely many points in the plane. Our results are also contained in the recent papers [2] and [3].

References

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[2] Gilányi A., Merentes N., Quintero R., *Mathability and an animation related to a convex-like property*, 7^{th} IEEE Conference on Cognitive Infocommunications (CogInfoCom) 2016, pp. 227–231.

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Eszter Gselmann: *The Lie symmetry group of first order ODE's and the rectification theorem* (Joint work with Gábor Horváth)

A flow in most small patches of the phase space can be made very simple. If x^* is a point where the vector field v is nonzero, then there is a change of coordinates for a region around x^* where the vector field becomes a series of parallel vectors of the same magnitude. This is known as the *rectification theorem*. In other words this means that away from singular points the dynamics of a point in a small patch is a straight line. In general this patch cannot be extended to the entire phase space. There may be singular points in the vector field (e.g. where $v(x) = 0$) or the patches may become smaller and smaller as some point is approached. The main aim of this talk is to prove the 'global' counterpart of the rectification theorem.

Furthermore, in the second part of our talk we will show that the Lie group of symmetries of the trivial equation, that is,

$$\dot{\mathbf{x}}(t) = 0$$

is

$$G = \{(t, x_1, \dots, x_n) \mapsto (f(t, x_1, \dots, x_n), g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n))\},$$

where $f\colon I \times M \to I$, $(g_1, \dots, g_n)\colon M \to M$ are arbitrary smooth functions. This is the (continuously differentiable) wreath product of the full symmetry group of M (all diffeomorphisms of M) and the full symmetry group of I (all diffeomorphisms of I):

$$G = \mathcal{SYM}(M) \wr_{C^1} \mathcal{SYM}(I).$$

In particular, if the system of ODE's fulfils some local-global Lipschitz condition, then the *isomorphism class* of the symmetry group depends only on (the diffeomorphism class of) M, rather than the differential equation itself, since $\mathcal{SYM}(I)$ are isomorphic for arbitrary open intervals. In the special case where $M = \mathbb{R}^n$ and $I = \mathbb{R}$, we obtain

$$G = \mathcal{SYM}(\mathbb{R}^n) \wr_{C^1} \mathcal{SYM}(\mathbb{R}).$$

TIBOR KISS: *On a functional equation related to an equality problem for two-variable means* (Joint work with Zsolt Páles)

In order to solve a particular case of the equality problem of two-variable quasi-arithmetic means and arithmetic means with weight function, in the paper [2], the authors investigated and solved the functional equation

$$\text{(1)} \qquad \varphi\Big(\frac{x+y}{2}\Big)(f(x) + f(y)) = \varphi(x)f(x) + \varphi(y)f(y), \quad x, y \in I.$$

They proved that, under

(a) strict monotonicity and continuity of φ and
(b) positivity of f,

the functions φ and f are infinitely many times differentiable on I, provided that they satisfy (1). These conditions concerning the functions were natural, because the main problem was strongly related to means, although the investigation of (1) in general is interesting itself.

The aim of the talk is to investigate (1) under different conditions. More precisely, we only assume that the functions φ and f are continuous. In some particular cases we also obtain that the conditions (a) and (b) are consequences of the equation (1).

References

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[4] Páles Zs., *Problems in the regularity theory of functional equations*, Aequationes Math. **63** (2002), no. 1–2, 1–17.

Judit Kosztur: *Conditionally polynomial functions with asymmetric conditions* (Joint work with Katarzyna Chmielewska and Attila Gilányi)

A function f mapping the real line $\mathbb{R}$ into a linear space Y is called a polynomial function of degree n if it satisfies the functional equation

$$\Delta_y^{n+1} f(x) = 0$$

for all $(x, y) \in \mathbb{R}^2$, where n is a fixed non-negative integer. In this talk, we consider situations, when the equation above is valid for some special elements only, i.e., for pairs $(x, y) \in A \times B$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. An important feature of our investigations is that the sets A and B are different from each other.

Radosław Łukasik: *K-spherical functions on abelian semigroups*

We present the form of the solution $f\colon S \to \mathbb{C}^*$ of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = f(x)f(y), \quad x, y \in S,$$

where $(S, +)$ is an abelian semigroup, K is a subgroup of the group of automorphisms of S, $\mathbb{C}^*$ is the multiplicative group of complex numbers.

Judit Makó: *On Hermite-Hadamard type inequalities*

In this talk, a connection between approximate lower Hermite-Hadamard type inequality and an approximate Jensen convexity type inequality will be examined. The key for the proof of the main result is a Korovkin type theorem.

Gyula Maksa: *On the alienation of the logarithmic and exponential functional equations*

In [1], the authors investigated the functional equation

$$f(xy) - f(x) - f(y) = g(x+y) - g(x)g(y) \tag{1}$$

and formulated the following problem (see Problem 1 in [1]): Find all functions $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R},\ g\colon \mathbb{R} \to \mathbb{R}$ satisfying the conditions

$$f(1) = 0, \quad g(1) = 1 \tag{2}$$

and such that (1) holds for all $x, y \in \mathbb{R}\setminus\{0\}$. In this talk, we present the solution of this problem by showing that these functions are

$$f(x) = a(\ln|x|),\ x \in \mathbb{R}\setminus\{0\}, \quad g(x) = \exp A(x),\ x \in \mathbb{R},$$

where $a, A\colon \mathbb{R} \to \mathbb{R}$ are additive functions with $A(1) = 0$. This result can be interpreted also in the way that, under the additional supposition (2), the logarithmic and the exponential Cauchy equations are alien.

Reference

[1] Kominek Z., Sikorska J., *Alienation of the logarithmic and exponential functional equations*, Aequationes Math. **90** (2016), no. 1, 107–121.

Fruzsina Mészáros: *Some results on the generalized Hosszú functional equation*

In this talk we consider some functional equations connected to the generalized Hosszú equation.

Janusz Morawiec: *On a problem of Janusz Matkowski and Jacek Wesołowski* (Joint work with Tomasz Kania, András M'athé, Martin Rmoutil and Thomas Zürcher)

In 1985 Janusz Matkowski posed a problem asking if equation

$$\varphi(x) = \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{1}{2}\right) \tag{1}$$

has a non-linear monotonic and continuous solution $\varphi\colon [0,1] \to \mathbb{R}$ (see [1]). During the 47th International Symposium on Functional Equations in 2009

Jacek Wesołowski asked whether the identity on $[0,1]$ is the only increasing and continuous solution $\varphi\colon [0,1] \to [0,1]$ of equation (1) satisfying

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(1) = 1. \tag{2}$$

We will show that equation (1) has many non-linear monotonic and continuous solutions $\varphi\colon [0,1] \to \mathbb{R}$ as well as many increasing and continuous solutions $\varphi\colon [0,1] \to [0,1]$ satisfying (2).

Reference

[1] Matkowski J., *Remark on BV-solutions of a functional equation connected with invariant measures*, Aequationes Math. **29** (1985), no. 2–3, 210–213.

Andrzej Olbryś: *On a separation theorem for generalized (s,t)-convex and (s,t)-concave maps*

Let D be a convex subset of a real linear space, $s,t \in (0,1)$ be fixed numbers and let $f\colon D \to \mathbb{R}$, $\omega\colon D \times D \times [0,1] \to \mathbb{R}$ be given maps. The function f is said to be an (ω,s,t)-*convex* if the inequality

$$f(sx + (1-s)y) \le tf(x) + (1-t)f(y) + \omega(x,y,s)$$

holds for all $x,y \in D$. If the above inequality is satisfied for $s = t = \alpha$, $\alpha \in [0,1]$, then f is said to be an ω-*convex*. If the inequality

$$tf(x) + (1-t)f(y) + \omega(x,y,s) \le f(sx + (1-s)y)$$

is satisfied for all $x,y \in D$, then f is said to be an (ω,s,t)-*concave*. We say that f is an (ω,s,t)-*affine* if it is at the same time (ω,s,t)-*convex* and (ω,s,t)-*concave*.

In our talk we give the necessary and sufficient conditions on the map ω under which for given two maps $f,g\colon D \to \mathbb{R}$, where f and $-g$ are (ω,s,t)-*convex*, and

$$g(x) \le f(x), \quad x \in D,$$

there exists an (ω,s,t)-*affine* map $h\colon D \to \mathbb{R}$ such that

$$g(x) \le h(x) \le f(x), \quad x \in D.$$

Zsolt Páles: *Computation of the best constant in Hardy type inequalities for quasi-arithmetic means* (Joint work with Paweł Pasteczka)

If A_f denotes the quasi-arithmetic mean generated by the strictly monotone continuous function $f\colon \mathbb{R}_+ \to \mathbb{R}$, then $A_f\colon \bigcup_{n\in\mathbb{N}} \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be a *Hardy mean* if, for all summable sequences (x_n) with positive terms, it satisfies the inequality

$$A_f(x_1) + A_f(x_1, x_2) + \cdots + A_f(x_1, \ldots, x_n) + \cdots \leq C(x_1 + x_2 + \cdots + x_n + \cdots)$$

for some finite positive constant C. The sharpest value of the constant C is called the *Hardy constant* of the mean A_f and will be denoted by C_f. In view of the classical inequalities by Hardy, Carleman and Knopp, if A_f equals the Hölder (or power) mean of parameter p, then it is a Hardy mean if and only if $p < 1$ and the exact form of the Hardy constant is well-known. In a much broader class of quasi-arithmetic means the authors in [1] established the following formula for C_f:

$$C_f = \sup_{x>0} \liminf_{n\to\infty} \frac{n}{x} A_f\Big(\frac{x}{1}, \frac{x}{2} \ldots, \frac{x}{n}\Big).$$

The aim is to provide a more efficient (i.e., easier to compute) formula for C_f instead of the above one assuming twice continuous differentiability of the function f.

Reference

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Bella Popovics: *On separation by strongly h-convex functions*

The sufficient and necessary conditions of the existence of h-convex separator for pairs of real valued functions are known from the paper [2]. Separations via strongly h-convex functions can be found in [3]. Although the main results of these papers assume the multiplicativity of h, this assumption turns out to be relaxed [1]. The aim of this talk is to give a common generalization of the above-mentioned results, that is, to present a separation theorem involving strongly h-convex functions, without assuming the multiplicativity of h.

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Barbara Przebieracz: *Remarks on Farah's theorem, I*

We present two theorems in the spirit of Theorems of I. Farah from [1] concerning approximate (in a sense) group homomorphisms.

Reference

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Éva Székelyné Radácsi: *Characterization of convexity with respect to smooth Chebyshev systems* (Joint work with Zsolt Páles)

Popoviciu's well-known theorem states that a real function f is n-convex if and only if it is $(n-1)$ times continuously differentiable and its $(n-1)$st-order derivative $f^{(n-1)}$ is convex. In 1966, Karlin and Studden managed to generalize the regularity part of this result by proving that if a function f is convex with respest to an n-dimensional sufficiently smooth extended Chebyshev system then it is $(n-2)$-times continuously differentiable. On the other hand, for the convexity part of Popoviciu's theorem, Bessenyei and Páles in 2003 showed that a function f is convex with respect to a 2-dimensional Chebyshev system $(\omega_1;\omega_2)$ if and only if the function $\frac{f}{\omega_1}\circ(\frac{\omega_2}{\omega_1})^{-1}$ is convex in the classical sense.

Motivated by the above two results, for the n-dimensional Chebyshev system setting, it has been an open problem to characterize convexity with respect to the Chebyshev system in terms of classical convexity notions. The aim of this talk is to construct, in terms of the members of the given n-dimensional Chebyshev system, an $(n-2)$nd order linear differential operator L, such that the convexity of f with respect to the n-dimensional Chebyshev system is equivalent to the convexity of Lf in the classical sense.

Maciej Sablik: *Characterizing polynomial functions*

Let G and H be commutative groups. Then $SA^i(G;H)$ denotes the group of all i-additive, symmetric mappings from G^i into H for $i\geq 2$, while $SA^0(G;H)$ denotes the family of constant functions from G to H and $SA^1(G;H)=\mathrm{Hom}(G;H)$. We also denote by $\mathcal{I}$ the subset of $\mathrm{Hom}(G;G)\times\mathrm{Hom}(G;G)$ containing all pairs (α,β) for which $\mathrm{Ran}(\alpha)\subset\mathrm{Ran}(\beta)$. Furthermore, we adopt a convention that a sum over empty set of indices equals 0. We present the following result.

Lemma. *Fix N, $M \in \mathbb{N} \cup \{0\}$, and let $I_{p,r}$, $0 \le p + r \le M$ be finite subsets of $\mathcal{I}$. Suppose further that H is uniquely divisible by $N!$ and let functions $\varphi_i \colon G \to SA^i(G; H)$, $i \in \{0, \dots, N\}$ and $\psi_{p,n-p,(\alpha,\beta)} \colon G \to SA^i(G; H)$, $(\alpha, \beta) \in I_{p,n-p}$, $0 \le p \le n, n \in \{0, \dots, M\}$ satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i)$$

$$= \sum_{n=0}^{M} \sum_{(\alpha,\beta)\in I_{p,n-p}} \psi_{p,n-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^p, y^{n-p})$$

for every $x, y \in G$. Then φ_N is a polynomial function.

The above statement is a generalization of earlier results from [1]–[4]. We also present examples of applications.

References

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Ekaterina Shulman: *On some generalizations of the Wilson equation*

We discuss the functional equation

$$(1)\quad \sum_{i=1}^{m} f_i(b_i x + c_i y) = \sum_{k=1}^{n} u_k(y) v_k(x) + \sum_{s=1}^{N} P_s(x) w_s(y) e^{\langle x, \varphi_s(y)\rangle}, \quad x, y \in \mathbb{R}^d,$$

where $f_i, v_k, u_k, w_s, \varphi_s$ are scalar functions defined on $\mathbb{R}^d$, P_s are polynomials $(1 \le i \le m,\ 1 \le k \le n,\ 1 \le s \le N)$ and $b_i, c_i \in GL(d, \mathbb{R})$ are the given matrices.

We prove [1] that *if the functions v_k are continuous, the matrices b_i, c_i and $b_i^{-1} c_i - b_j^{-1} c_j$ (for $i \ne j$) are invertible, then all continuous solutions f_i of (1) are exponential polynomials.*

The main technical tool is the analysis of finite-dimensional translation-invariant subspaces in function spaces.

The equation (1) can be formulated also for distributions, in this case we regard both sides of the equation as elements of $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)'$. It is shown that in such a setting the class of solutions remains the same.

The result can be extended from $\mathbb{R}^d$ to an arbitrary finitely generated abelian group, with the change of matrices b_i, c_i by automorphisms.

Reference

[1] Almira J.M., Shulman E.V., *On certain generalizations of the Levi-Civita and Wilson functional equations.* Preprint.

Patrícia Szokol: *Restricted skew-morphisms on matrix algebras* (Joint work with Gregor Dolinar, Bojan Kuzma and Gergő Nagy)

In this presentation, skew-morphisms, which are extensively studied in graph theory, are considered in the setting of matrix algebras. Let $n \geq 2$ be an integer and let M_n be the algebra of n-by-n matrices over a field $\mathbb{F}$. We say, that a transformation $\phi\colon M_n \to M_n$ is a restricted skew-morphism, if it satisfies the following property: there exists a *power function* $\kappa\colon M_n \to \{0, 1, 2, \ldots\}$ such that

$$\phi(AB) = \phi(A)\phi^{\kappa(A)}(B); \quad \forall\, A, B \in M_n,$$

where as usual $\phi^0 = \mathrm{id}$, the identity mapping, and $\phi^k(x) = \phi(\phi^{k-1}(x))$. Different properties of skew-morphisms are obtained and their classification in some specific cases is given.

Tomasz Szostok: *On some method of proving inequalities for convex functions*

We present a method of proving functional inequalities which is connected with the properties of the Stieltjes integral. First we show that this method sheds a new light on some well known inequalities and then we obtain some new inequalities.

2. Problems and Remarks

1. Remark (*Remark to the talk of Professor Ger*) In the following note we describe a new approach to prove the differentiability of delta convex functions. We start with a trivial observation.

Lemma. *If $(S,+)$ is a semigroup and $a,b\colon S\to\mathbb{R}$ are subadditive functions such that $a+b$ is additive, then a and b are also additive.*

Proof. From the decomposition $a=(a+b)+(-b)$ it follows that a is the sum of an additive and a superadditive function, therefore a is superadditive, henceforth additive. Similarly, b is also additive. □

Given an open set $D\subseteq\mathbb{R}^n$ and a function $f\colon D\to\mathbb{R}$, we say that f *is directionally differentiable at $p\in D$ in the direction $h\in\mathbb{R}^n$*, if the limit, called the directional derivative of f at p,

$$f'(p,h):=\lim_{t\to 0+}\frac{f(p+th)-f(p)}{t}$$

exists. We say that f *is Gâteaux-differentiable at* p, if the map $h\mapsto f'(p,h)$ is linear on $\mathbb{R}^n$. The following result is well-known from the theory of convex functions.

Lemma. *Let $D\subseteq\mathbb{R}^n$ be an open convex subset and $f\colon D\to\mathbb{R}$ be a convex function. Then, for every $p\in D$ and $h\in\mathbb{R}^n$, the directional derivative $f'(p,h)$ exists and the map $h\mapsto f'(p,h)$ is sublinear (i.e., subadditive and positively homogeneous) on $\mathbb{R}^n$.*

Theorem. *Let $D\subseteq\mathbb{R}^n$ be an open convex subset and $f,g\colon D\to\mathbb{R}$ be convex functions. If $f+g$ is Gâteaux-differentiable at $p\in D$, then f and g are also Gâteaux-differentiable at p.*

Proof. Assume that $f+g$ is Gâteaux-differentiable at p. Then $h\mapsto (f+g)'(p,h)=f'(p,h)+g'(p,h)$ is linear, i.e., additive and homogeneous. Using the first lemma, it follows that $h\mapsto f'(p,h)$ and $h\mapsto g'(p,h)$ are also additive functions. This property and also their positive homogeneity implies that they are also linear. Hence f and g are Gâteaux-differentiable at p. □

Theorem. *Let $D\subseteq\mathbb{R}^n$ be an open convex subset and let $f\colon D\to\mathbb{R}$ be a delta Jensen convex functions with control function $\varphi\colon D\to\mathbb{R}$, that is, assume that*

$$\left|\frac{f(x)+f(y)}{2}-f\Big(\frac{x+y}{2}\Big)\right|\le\frac{\varphi(x)+\varphi(y)}{2}-\varphi\Big(\frac{x+y}{2}\Big),\qquad x,y\in D.$$

If φ is Gâteaux-differentiable at $p\in D$, and f is bounded on a nonempty open subset of D, then f is also Gâteaux-differentiable at p.

Proof. By the delta Jensen convexity, it follows that $\varphi+f$ and $\varphi-f$ are Jensen convex. Since φ itself is Jensen convex and Gâteaux-differentiable at p, hence, by the Bernstein–Doetsch Theorem it is convex on D. This implies that φ is continuous on D. Thus $\varphi+f$ and $\varphi-f$ are bounded from above on the open set where f was assumed to be bounded. Therefore, the Bernstein–Doetsch Theorem yields that $\varphi+f$ and $\varphi-f$ are convex functions. On the other hand, $(\varphi+f)+(\varphi-f)=2\varphi$ is Gâteaux-differentiable at p, which, by the previous theorem implies that $\varphi+f$ and $\varphi-f$ are also Gâteaux-differentiable at p. Thus f possesses this property, too. □

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