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MIXED TYPE OF ADDITIVE AND QUINTIC FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate the general solution and Hyers–Ulam–Rassias stability of a new mixed type of additive and quintic functional equation of the form

$$f(3x+y) - 5f(2x+y) + f(2x-y) + 10f(x+y) - 5f(x-y) = 10f(y) + 4f(2x) - 8f(x)$$

in the set of real numbers.

1. Introduction

In 1940, Ulam [17] raised the following question. Under what conditions does there exist an additive mapping near an approximately addition mapping? The case of approximately additive functions was solved by Hyers [7] under the assumption that for $\epsilon > 0$ and $f\colon E_1 \to E_2$ with

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

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for all $x, y \in E_1$, then there exist a unique additive mapping $T\colon E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in E_1$, where E_1 is a normed space and E_2 is a Banach space.

In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Th.M. Rassias [15]. He proved that for a mapping $f\colon E_1 \to E_2$ for which $f(tx)$ is continuous in $t \in \mathbb{R}$ and for each fixed $x \in E_1$, there exist constant $\varepsilon > 0$ and $p \in [0, 1)$ with

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E_1$, then there exist a unique linear mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p$$

for all $x \in E_1$. A number of mathematicians were attracted by the result of Th.M. Rassias (see also [1], [3], [6], [11], [8], [9], [10] and [16]). The stability concept that was introduced and investigated by Rassias is called the Hyers–Ulam–Rassias stability.

In 1982–1989, J.M. Rassias [13, 14] replaced the sum appeared in right hand side of the equation (1.1) by the product of powers of norms.

In [18], Xu et al. obtained the general solution and investigated the Ulam stability problem for the following quintic functional equation

$$\begin{aligned} f(x+3y) - 5f(x+2y) + 10f(x+y) - 10f(x) + 5f(x-y) - f(x-2y) \\ = 120f(y) \end{aligned}$$

in quasi-β-normed spaces via fixed point method. This method which is different from the "*direct method*", initiated by Hyers in [7], had been applied by Cădariu and Radu for the first time. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [5] and for the quadratic functional equation [4].

In [12], Park et al. introduced the following new form of quintic functional equations

$$\begin{aligned} f(3x+y) - 5f(2x+y) + f(2x-y) + 10f(x+y) - 5f(x-y) \\ = 10f(y) + f(3x) - 3f(2x) - 27f(x). \end{aligned} \tag{1.2}$$

They applied the fixed point method to establish the Hyers–Ulam stability of the orthogonally quintic functional equation (1.2) in Banach spaces and in non-Archimedean Banach spaces (see also [2]).

In this paper, we prove the general solution and Hyers–Ulam–Rassias stability of the new mixed additive and quintic functional equation of the form

$$\begin{aligned} f(3x+y)-5f(2x+y)+f(2x-y)+10f(x+y)-5f(x-y) \\ =10f(y)+4f(2x)-8f(x) \end{aligned} \tag{1.3}$$

in real numbers. It is easily verified that the function $f(x)=\alpha x^5+\beta x$ is a solution of the functional equation (1.3).

2. Main Results

Throughout this paper, we denote the set of real number by $\mathbb{R}$. Before proceeding the proof of main results in this section, we shall need the following lemma.

Lemma 2.1. *The only nonzero solution* $f\colon \mathbb{R}\to\mathbb{R}$ *such that* $f(x)=g(x)+h(x)$ *admitting a finite limit of the quotient* $\frac{g(x)}{x}$ *and* $\frac{h(x)}{x^5}$ *at zero, of the equation* (1.3) *is of the form* $g(x)=ax$ *and* $h(x)=bx^5$ *for all* $x\in\mathbb{R}$.

Proof. Putting $y=0$ in (1.3), we have

$$f(3x)-8f(2x)+13f(x)=0 \tag{2.1}$$

for all $x\in\mathbb{R}$. Replacing (x,y) by (x,x) in (1.3) and using (2.1), we get

$$f(4x)-34f(2x)+64f(x)=0 \tag{2.2}$$

for all $x\in\mathbb{R}$. Setting $g(x)=f(2x)-32f(x)$ in (2.2), we obtain

$$g(2x)=2g(x) \tag{2.3}$$

for all $x\in\mathbb{R}$. The relation (2.3) implies that

$$\frac{g(x)}{x}=\lim_{n\to\infty}\frac{g\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}}=a$$

for some $a\in\mathbb{R}$. Note that a cannot be zero, otherwise, we will have $g=0$. Now, setting $h(x)=f(2x)-2f(x)$ in (2.2), we get

$$h(2x)=32h(x) \tag{2.4}$$

for all $x \in \mathbb{R}$. It follows from (2.4) that

$$\frac{h(x)}{x^5} = \lim_{n\to\infty} \frac{h\left(\frac{x^5}{2^n}\right)}{\frac{x}{2^n}} = b$$

for some $b \in \mathbb{R}$, as claimed. Clearly, b cannot vanishes. This completes the proof. □

From now on, we use the abbreviation for the given mapping $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned}\mathcal{D}_q f(x,y) = {} & f(3x+y) - 5f(2x+y) + f(2x-y) + 10f(x+y)\\ & - 5f(x-y) - 10f(y) - 4f(2x) + 8f(x).\end{aligned}$$

THEOREM 2.2. *Let $\phi\colon \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be a mapping such that*

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\phi(2^j x, 2^j y) < \infty \tag{2.5}$$

for all $x, y \in \mathbb{R}$ in which $x \in \{x, 0\}$ and $y \in \{x, -x\}$. Suppose that a mapping $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ satisfies the inequality

$$|\mathcal{D}_q f(x,y)| \le \phi(x,y) \tag{2.6}$$

for all $x, y \in \mathbb{R}$. Then the limit

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n}\{f(2^{n+1}x) - 32f(2^n x)\} \tag{2.7}$$

exists for all $x \in \mathbb{R}$ and the mapping $A\colon \mathbb{R} \to \mathbb{R}$ is a unique additive mapping satisfying

$$|f(2x) - 32f(x) - A(x)| \le \frac{1}{2}\sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \tag{2.8}$$

for all $x \in \mathbb{R}$ where

$$\psi(x) = \phi(x,x) + \frac{25}{2}\phi(0,x) + 5\phi(x,-x). \tag{2.9}$$

Proof. Replacing (x,y) by $(0,x)$ in (2.6), we get

$$|f(x)+f(-x)| \leq \frac{1}{4}\phi(0,x) \tag{2.10}$$

for all $x \in \mathbb{R}$. Letting $x=y$ in (2.6), we have

$$|f(4x)-5f(3x)+6f(2x)-f(x)| \leq \phi(x,x) \tag{2.11}$$

for all $x \in \mathbb{R}$. Substituting (x,y) by $(x,-x)$ in (2.6), we obtain

$$|-8f(2x)+f(3x)+3f(x)-10f(-x)| \leq \phi(x,-x) \tag{2.12}$$

for all $x \in \mathbb{R}$. It follows from (2.10)–(2.12) that

$$|f(4x)-34f(2x)+64f(x)| \leq \frac{25}{2}\phi(0,x)+\phi(x,x)+5\phi(x,-x) \tag{2.13}$$

for all $x \in \mathbb{R}$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a mapping defined by $g(x) := f(2x)-32f(x)$ and let

$$\psi(x) = \frac{25}{2}\phi(0,x)+\phi(x,x)+5\phi(x,-x) \tag{2.14}$$

for all $x \in \mathbb{R}$. In other words, (2.13) means

$$|g(2x)-2g(x)| \leq \psi(x) \tag{2.15}$$

for all $x \in \mathbb{R}$. By (2.5), we see that

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\psi(2^j x) < \infty \tag{2.16}$$

for all $x \in \mathbb{R}$. Interchanging x into $2^n x$ in (2.15) and dividing both sides of (2.15) by 2^{n+1}, we get

$$\left|\frac{1}{2^{n+1}}g(2^{n+1}x)-\frac{1}{2^n}g(2^n x)\right| \leq \frac{1}{2^{n+1}}\psi(2^n x) \tag{2.17}$$

for all $x \in \mathbb{R}$ and all non-negative integers n. We have

$$\begin{aligned} \left|\frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^m} g(2^m x)\right| &\leq \sum_{j=m}^{n} \left|\frac{1}{2^{j+1}} g(2^{j+1}x) - \frac{1}{2^j} g(2^j x)\right| \\ &\leq \frac{1}{2} \sum_{j=m}^{n} \frac{1}{2^j} \psi(2^j x) \end{aligned} \tag{2.18}$$

for all $x \in \mathbb{R}$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (2.16) and (2.18) that the sequence $\{\frac{1}{2^n} g(2^n x)\}$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in \mathbb{R}$. Thus, the sequence $\{\frac{1}{2^n} g(2^n x)\}$ is convergent to the mapping $A \colon \mathbb{R} \to \mathbb{R}$. Indeed,

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} g(2^n x) \tag{2.19}$$

for all $x \in \mathbb{R}$. Letting $m = 0$ and allowing $n \to \infty$ in (2.18), we get

$$|g(x) - A(x)| \leq \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{2^j} \psi(2^j x) \tag{2.20}$$

for all $x \in \mathbb{R}$. Using (2.14) in (2.20), we arrive at the result (2.8). It follows from (2.5), (2.6) and (2.7) that

$$|D_q A(x, y)| = \lim_{n\to\infty} \frac{1}{2^n} |D_q A(2^n x, 2^n y)| \leq \lim_{n\to\infty} \frac{1}{2^n} \phi(2^n x, 2^n y) \tag{2.21}$$

for all $x, y \in \mathbb{R}$. Therefore the mapping A satisfies (1.3). By Lemma 2.1, we see that the mapping A is additive. To prove the uniqueness of $\mathcal{A}$, let $T \colon \mathbb{R} \to \mathbb{R}$ be another additive mapping satisfying (2.8). We have

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{j=0}^{\infty} \frac{1}{2^j} \phi\left(2^{n+j}x, 2^{n+j}y\right) = \lim_{n\to\infty} \sum_{j=n+1}^{\infty} \frac{1}{2^j} \phi\left(2^j x, 2^j y\right) = 0$$

for all $x, y \in \mathbb{R}$ for which $x \in \{x, 0\}$ and $y \in \{x, -x\}$. It follows the above relation and (2.8) that

$$\begin{aligned} |A(x) - T(x)| &= \lim_{n\to\infty} \frac{1}{2^n} |g(2^n x) - T(2^n x)| \\ &\leq \frac{1}{2^p} \lim_{n\to\infty} \frac{1}{2^n} \psi(2^n x) = 0 \end{aligned}$$

for all $x \in \mathbb{R}$. So $A = T$. Hence the theorem is proved. □

The upcoming result is a different form of Theorem 2.2.

THEOREM 2.3. *Let* $\phi\colon \mathbb{R}\times\mathbb{R}\to[0,\infty)$ *be a mapping such that*

$$\sum_{j=0}^{\infty} 2^j \phi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}\right) < \infty \tag{2.22}$$

for all $x, y \in \mathbb{R}$ *in which* $x \in \{x, 0\}$ *and* $y \in \{x, -x\}$. *Suppose that a mapping* $f\colon \mathbb{R}\to\mathbb{R}$ *with* $f(0) = 0$ *satisfies the inequality*

$$|\mathcal{D}_q f(x,y)| \le \phi(x,y) \tag{2.23}$$

for all $x, y \in \mathbb{R}$. *Then the limit*

$$A(x) = \lim_{n\to\infty} 2^n \left\{ f\left(2\frac{x}{2^n}\right) - 32 f\left(\frac{x}{2^n}\right)\right\} \tag{2.24}$$

exists for all $x \in \mathbb{R}$ *and the mapping* $A\colon \mathbb{R}\to\mathbb{R}$ *is a unique additive mapping satisfying*

$$|f(2x) - 32f(x) - A(x)| \le \sum_{n=0}^{\infty} 2^n \psi\left(\frac{x}{2^{n+1}}\right) \tag{2.25}$$

for all $x \in \mathbb{R}$ *where* $\psi(x)$ *is defined in* (2.9).

PROOF. Similar to the proof of Theorem 2.2, we have

$$|g(2x) - 2g(x)| \le \psi(x) \tag{2.26}$$

for all $x \in \mathbb{R}$, in which $g\colon \mathbb{R}\to\mathbb{R}$ is a mapping defined by $g(x) := f(2x) - 32f(x)$ and $\psi(x)$ is defined in (2.9). It follows (2.22) that

$$\sum_{j=0}^{\infty} 2^j \psi\left(\frac{x}{2^{j+1}}\right) < \infty \tag{2.27}$$

for all $x \in \mathbb{R}$. Replacing x by $\frac{x}{2^{n+1}}$ in (2.26) and multiply both sides of (2.26) by 2^n, we get

$$\left|2^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 2^n g\left(\frac{x}{2^n}\right)\right| \le 2^n \psi\left(\frac{x}{2^{n+1}}\right) \tag{2.28}$$

for all $x \in \mathbb{R}$ and all non-negative integers n. We have

$$\begin{aligned} \left|2^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 2^m g\left(\frac{x}{2^m}\right)\right| &\leq \sum_{j=m}^{n} \left|2^{j+1} g\left(\frac{x}{2^{j+1}}\right) - 2^j g\left(\frac{x}{2^j}\right)\right| \\ &\leq \sum_{j=m}^{n} 2^j \psi\left(\frac{x}{2^{j+1}}\right) \end{aligned} \tag{2.29}$$

for all $x \in \mathbb{R}$ and all non-negative integers n and m with $n \geq m$. It follows from (2.27) and (2.29) that for each $x \in \mathbb{R}$ the sequence $\{2^n g\left(\frac{x}{2^n}\right)\}$ is a Cauchy. Hence, the mentioned sequence converges for all $x \in \mathbb{R}$. So one can define the mapping $A\colon \mathbb{R} \to \mathbb{R}$ by

$$A(x) := \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) \tag{2.30}$$

for all $x \in X$. Putting $m = 0$ and tending n to infinity in (2.29), we obtain

$$|g(x) - A(x)| \leq \sum_{j=0}^{\infty} 2^j \psi\left(\frac{x}{2^{j+1}}\right)$$

for all $x \in \mathbb{R}$. Therefore, the inequality (2.25) hold. The rest of the proof is same as the proof of Theorem 2.2. □

The following corollaries are the direct consequences of Theorems 2.2 and 2.3 concerning the stability of (1.3).

Corollary 2.4. *Let λ be a nonnegative real number with $\lambda \neq 1$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfies the functional equation*

$$|\mathcal{D}_q f(x,y)| \leq \varepsilon(|x|^\lambda + |y|^\lambda)$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. If $g\colon \mathbb{R} \to \mathbb{R}$ is a mapping defined by $g(x) := f(2x) - 32f(x)$, then there exists a unique additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$|g(x) - A(x)| \leq \begin{cases} 49\frac{\varepsilon}{2(2-2^\lambda)}|x|^\lambda, & \lambda < 1, \\ 49\frac{2^\lambda \varepsilon}{2(2^\lambda-2)}|x|^\lambda, & \lambda > 1, \end{cases} \tag{2.31}$$

for all $x \in \mathbb{R}$.

PROOF. In Theorem 2.2 and Theorem 2.3, take $\phi(x,y) = \varepsilon(|x|^\lambda + |y|^\lambda)$ for all $x, y \in \mathbb{R}$. By equation (2.8) and (2.25), we obtain the desired result. □

COROLLARY 2.5. *Let r and s be nonnegative real numbers with $\lambda := r+s \neq 1$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfies the functional equation*

$$|\mathcal{D}_q f(x,y)| \leq \varepsilon\, |x|^r\, |y|^s$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. If $g\colon \mathbb{R} \to \mathbb{R}$ is a mapping defined by $g(x) := f(2x) - 32f(x)$, then there exists a unique additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$|g(x) - A(x)| \leq \begin{cases} \frac{6\varepsilon}{2-2^\lambda}|x|^\lambda, & \lambda < 1, \\ \frac{6\varepsilon 2^\lambda}{2^\lambda - 2}|x|^\lambda, & \lambda > 1, \end{cases} \tag{2.32}$$

for all $x \in \mathbb{R}$.

PROOF. The result follows from the equations (2.8) and (2.25) by defining $\phi(x,y) = \varepsilon\, |x|^r\, |y|^s$. □

COROLLARY 2.6. *Let r and s be nonnegative real numbers with $\lambda := r+s \neq 1$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the functional equation*

$$|\mathcal{D}_q f(x,y)| \leq \varepsilon \left(|x|^\lambda + |y|^\lambda + |x|^r\, |y|^s\right)$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. If $g\colon \mathbb{R} \to \mathbb{R}$ is a mapping defined by $g(x) := f(2x) - 32f(x)$, then there exists a unique additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$|g(x) - A(x)| \leq \begin{cases} \frac{61\varepsilon}{2(2-2^\lambda)}|x|^\lambda, & \lambda < 1 \\ \frac{61\varepsilon 2^\lambda}{2(2^\lambda - 2)}|x|^\lambda, & \lambda > 1, \end{cases} \tag{2.33}$$

for all $x \in \mathbb{R}$.

PROOF. Putting $\phi(x,y) = \varepsilon(|x|^\lambda + |y|^\lambda + |x|^r\, |y|^s)$ and employing Theorem 2.2 and Theorem 2.3, we get the result. □

We have the following theorem which is analogous to Theorem 2.2. The proof is similar but we bring some parts.

Theorem 2.7. *Let* $\phi\colon \mathbb{R}\times\mathbb{R}\to[0,\infty)$ *be a mapping such that*

$$\sum_{j=0}^{\infty}\frac{1}{32^j}\phi(2^jx,2^jy)<\infty \tag{2.34}$$

for all $x,y\in\mathbb{R}$ *and all* $x\in\{x,0\}$ *and* $y\in\{x,-x\}$. *Suppose that a mapping* $f\colon\mathbb{R}\to\mathbb{R}$ *with* $f(0)=0$ *satisfies the inequality*

$$|\mathcal{D}_qf(x,y)|\le\phi(x,y) \tag{2.35}$$

for all $x,y\in\mathbb{R}$. *Then the limit*

$$\lim_{n\to\infty}\frac{1}{32^n}\{f(2^{n+1}x)-2f(2^nx)\} \tag{2.36}$$

exists for all $x\in\mathbb{R}$ *and the mapping* $Q\colon\mathbb{R}\to\mathbb{R}$ *is a unique quintic mapping satisfying*

$$\|f(2x)-2f(x)-Q(x)\|_Y\le\frac{1}{32}\sum_{n=0}^{\infty}\frac{\psi(2^nx)}{32^n} \tag{2.37}$$

for all $x\in\mathbb{R}$ *where* $\psi(2^nx)$ *is defined in* (2.9).

Proof. Similar to the proof of Theorem 2.2, one can show that

$$|h(2x)-32h(x)|\le\psi(x) \tag{2.38}$$

for all $x\in\mathbb{R}$, where $h(x)=f(2x)-2f(x)$ and $\psi(x)$ is defined in (2.9). The rest of the proof is the same as the proof of Theorem 2.2. □

Theorem 2.8. *Let* $\phi\colon \mathbb{R}\times\mathbb{R}\to[0,\infty)$ *be a mapping such that*

$$\sum_{j=0}^{\infty}32^j\phi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}\right)<\infty \tag{2.39}$$

for all $x,y\in\mathbb{R}$ *and all* $x\in\{x,0\}$ *and* $y\in\{x,-x\}$. *Suppose that a mapping* $f\colon\mathbb{R}\to\mathbb{R}$ *with* $f(0)=0$ *satisfies the inequality*

$$|\mathcal{D}_qf(x,y)|\le\phi(x,y) \tag{2.40}$$

for all $x, y \in \mathbb{R}$. Then the limit

$$\lim_{n\to\infty} 32^n \left\{f\left(2\frac{x}{2^n}\right) - 2f\left(\frac{x}{2^n}\right)\right\} \tag{2.41}$$

exists for all $x \in \mathbb{R}$ and the mapping $Q\colon \mathbb{R} \to \mathbb{R}$ is a unique quintic mapping satisfying

$$|f(2x) - 2f(x) - Q(x)| \le \sum_{n=0}^{\infty} 32^n \psi\left(\frac{x}{2^{n+1}}\right) \tag{2.42}$$

for all $x \in \mathbb{R}$ where $\psi(x)$ is defined in (2.9).

Proof. The proof is the same as the proof of Theorem 2.3 with $h(x) = f(2x) - 2f(x)$. □

The following corollaries are the direct consequences of Theorems 2.7 and 2.8 concerning the stability of (1.3). Since the proofs are similar to the previous corollaries, we omit them.

Corollary 2.9. *Let λ be a nonnegative real number with $\lambda \neq 5$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the functional equation*

$$|\mathcal{D}_q f(x,y)| \le \varepsilon(|x|^\lambda + |y|^\lambda)$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. If $h\colon \mathbb{R} \to \mathbb{R}$ is a mapping defined by $h(x) := f(2x) - 2f(x)$, then there exists a unique quintic function $Q\colon \mathbb{R} \to \mathbb{R}$ such that

$$|h(x) - Q(x)| \le \begin{cases} 392\frac{\varepsilon}{32-2^\lambda}|x|^\lambda, & \lambda < 5, \\ 49\frac{2^\lambda \varepsilon}{2(2^\lambda-32)}|x|^\lambda, & \lambda > 5, \end{cases} \tag{2.43}$$

for all $x \in \mathbb{R}$.

Corollary 2.10. *Let r and s be nonnegative real numbers with $\lambda := r + s \neq 5$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the functional equation*

$$|\mathcal{D}_q f(x,y)| \le \varepsilon\left(|x|^r\,|y|^s\right)$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. If $h\colon \mathbb{R} \to \mathbb{R}$ is a mapping defined by $h(x) := f(2x) - 32f(x)$, then there exists a unique additive function $Q\colon \mathbb{R} \to \mathbb{R}$ such that

$$|h(x)-Q(x)| \leq \begin{cases} \frac{96\varepsilon}{32-2^{\lambda}}|x|^{\lambda}, & \lambda<5, \\ \frac{6\varepsilon 2^{\lambda}}{2^{\lambda}-32}|x|^{\lambda}, & \lambda>5, \end{cases} \tag{2.44}$$

for all $x \in \mathbb{R}$.

Corollary 2.11. *Let* r *and* s *be nonnegative real numbers with* $\lambda := r+s \neq 5$ *and let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a function satisfying the functional equation*

$$|\mathcal{D}_q f(x,y)| \leq \varepsilon(|x|^{\lambda}+|y|^{\lambda}+|x|^{r}|y|^{s})$$

for some $\varepsilon>0$ *and for all* $x,y \in \mathbb{R}$. *If* $h\colon \mathbb{R} \to \mathbb{R}$ *is a mapping defined by* $h(x) := f(2x)-2f(x)$, *then there exists a unique additive function* $Q\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$|h(x)-Q(x)| \leq \begin{cases} \frac{488\varepsilon}{32-2^{\lambda}}|x|^{\lambda}, & \lambda<5, \\ \frac{61\varepsilon 2^{\lambda}}{2(2^{\lambda}-32)}|x|^{\lambda}, & \lambda>5, \end{cases} \tag{2.45}$$

for all $x \in \mathbb{R}$.

The upcoming theorems show that the equation (1.3) is stable under some mild conditions.

Theorem 2.12. *Let* $\phi\colon \mathbb{R}\times\mathbb{R} \to [0,\infty)$ *be a mapping such that*

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\phi(2^j x, 2^j y) < \infty \tag{2.46}$$

for all $x,y \in \mathbb{R}$ *in which* $x \in \{x,0\}$ *and* $y \in \{x,-x\}$. *Suppose that a mapping* $f\colon \mathbb{R} \to \mathbb{R}$ *with* $f(0)=0$ *satisfies the inequality*

$$|\mathcal{D}_q f(x,y)| \leq \phi(x,y) \tag{2.47}$$

for all $x,y \in \mathbb{R}$. *Then there exist a unique additive mapping* $A\colon \mathbb{R} \to \mathbb{R}$ *and a unique quintic mapping* $Q\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$|f(x)-A(x)-Q(x)| \leq \frac{1}{30}\left[\frac{1}{2}\sum_{n=0}^{\infty}\frac{\psi(2^n x)}{2^n}+\frac{1}{32}\sum_{n=0}^{\infty}\frac{\psi(2^n x)}{32^n}\right] \tag{2.48}$$

for all $x \in \mathbb{R}$ *where* $\psi(x)$ *is defined in* (2.9).

PROOF. By Theorems 2.2 and 2.7, there exist an additive mapping $A_0 : \mathbb{R} \to \mathbb{R}$ and a quintic mapping $Q_0 \colon \mathbb{R} \to \mathbb{R}$ such that

$$|f(2x) - 32f(x) - A_0(x)| \leq \frac{1}{2} \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n}$$

and

$$|f(2x) - 2f(x) - Q_0(x)| \leq \frac{1}{32} \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{32^n}$$

for all $x \in \mathbb{R}$. Therefore it follows from the above inequalities that

$$\left| f(x) - (-\frac{1}{30} A_0(x)) - \frac{1}{30} Q_0(x) \right| \leq \frac{1}{30} \left[\frac{1}{2} \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} + \frac{1}{32} \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{32^n} \right]$$

for all $x \in \mathbb{R}$. So we obtain (2.48) by letting $A(x) = -\frac{1}{30} A_0(x)$ and $Q(x) = \frac{1}{30} Q_0(x)$ for all $x \in \mathbb{R}$. To prove the uniqueness of A and Q, let $A_1, C_1 \colon \mathbb{R} \to \mathbb{R}$ be another additive and quintic mappings satisfying (2.48). Put $A' = A - A_1$ and $Q' = Q - Q_1$. Hence,

$$|A'(x) + Q'(x)| \leq |f(x) - A(x) - Q(x)| + |f(x) - A_1(x) - Q_1(x)| \tag{2.49}$$

$$\leq \frac{1}{15} \left[\frac{1}{2} \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} + \frac{1}{32} \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{32^n} \right] \tag{2.50}$$

for all $x \in \mathbb{R}$. Since $\lim_{n\to\infty} \frac{1}{32^n} \phi(2^n x, 2^n y) = 0$, we have

$$\lim_{n\to\infty} \frac{1}{32^n} |A'(2^n x) + Q'(2^n x)| = 0$$

for all $x \in \mathbb{R}$. Thus $Q' = 0$. Now it follows from (2.48) that $A' = 0$. □

The next theorem is an alternative result of Theorem 2.12.

THEOREM 2.13. *Let $\phi \colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be a mapping such that*

$$\sum_{j=0}^{\infty} 2^j \phi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}} \right) < \infty \tag{2.51}$$

for all $x, y \in \mathbb{R}$ in which $x \in \{x, 0\}$ and $y \in \{x, -x\}$. Suppose that a mapping $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ satisfies the inequality

$$|\mathcal{D}_q f(x,y)| \leq \phi(x,y) \tag{2.52}$$

for all $x, y \in \mathbb{R}$. Then there exist a unique additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ and a unique quintic mapping $Q\colon \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x) - Q(x)| \leq \frac{1}{30}\left[\sum_{n=0}^{\infty} 2^n \psi\left(\frac{x}{2^{n+1}}\right) + \sum_{n=0}^{\infty} 32^n \psi\left(\frac{x}{2^{n+1}}\right)\right] \tag{2.53}$$

for all $x \in \mathbb{R}$ where $\psi(x)$ is defined in (2.9).

Proof. The proof is similar to the proof of Theorem 2.12 and the result follows from Theorems 2.3 and 2.8. □

In the next corollaries, by using Theorems 2.12 and 2.13, we show that the equation (1.3) can be stable when $|\mathcal{D}_q f(x,y)|$ is controlled by the sum and product of powers of absolute values. Due to similarity of the proofs with the previous corollaries, we present them without proof.

Corollary 2.14. *Let λ be a nonnegative real number with $\lambda \neq 1, 5$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the functional equation*

$$|\mathcal{D}_q f(x,y)| \leq \varepsilon(|x|^{\lambda} + |y|^{\lambda})$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. Then there exist a unique additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ and a unique quintic mapping $Q\colon \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x) - Q(x)| \leq \begin{cases} \frac{1}{30}\left[49\frac{\varepsilon}{2(2-2^{\lambda})} + 392\frac{\varepsilon}{32-2^{\lambda}}\right]|x|^{\lambda}, & \lambda < 1, \\ \frac{1}{30}\left[49\frac{2^{\lambda}\varepsilon}{2(2^{\lambda}-2)} + 392\frac{\varepsilon}{32-2^{\lambda}}\right]|x|^{\lambda}, & 1 < \lambda < 5, \\ \frac{1}{30}\left[49\frac{2^{\lambda}\varepsilon}{2(2^{\lambda}-2)} + 49\frac{2^{\lambda}\varepsilon}{2(2^{\lambda}-32)}\right]|x|^{\lambda}, & \lambda > 5, \end{cases}$$

for all $x \in \mathbb{R}$.

Corollary 2.15. *Let r and s be positive numbers with $\lambda := r + s \neq 1, 5$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the functional equation*

$$|\mathcal{D}_q f(x,y)| \leq \varepsilon(|x|^r\,|y|^s)$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. Then there exist a unique additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ and a unique quintic mapping $Q\colon \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x) - Q(x)| \le \begin{cases} \frac{1}{30}\left[\frac{61\varepsilon}{2(2-2^\lambda)} + \frac{96\varepsilon}{32-2^\lambda}\right]|x|^\lambda, & \lambda < 1, \\ \frac{1}{30}\left[\frac{6\varepsilon 2^\lambda}{2^\lambda-2} + \frac{96\varepsilon}{32-2^\lambda}\right]|x|^\lambda, & 1 < \lambda < 5, \\ \frac{1}{30}\left[\frac{6\varepsilon 2^\lambda}{2^\lambda-2} + \frac{6\varepsilon 2^\lambda}{2^\lambda-32}\right]|x|^\lambda, & \lambda > 5, \end{cases}$$

for all $x \in \mathbb{R}$.

COROLLARY 2.16. *Let r and s be positive numbers with $\lambda := r + s \ne 1, 5$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the functional equation*

$$|\mathcal{D}_q f(x, y)| \le \varepsilon(|x|^\lambda + |y|^\lambda + |x|^r |y|^s)$$

for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. Then there exist a unique additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ and a unique quintic mapping $Q\colon \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x) - Q(x)| \le \begin{cases} \frac{1}{30}\left[\frac{6\varepsilon}{2-2^\lambda} + \frac{488\varepsilon}{32-2^\lambda}\right]|x|^\lambda, & \lambda < 1, \\ \frac{1}{30}\left[\frac{61\varepsilon 2^\lambda}{2(2^\lambda-2)} + \frac{488\varepsilon}{32-2^\lambda}\right]|x|^\lambda, & 1 < \lambda < 5, \\ \frac{1}{30}\left[\frac{61\varepsilon 2^\lambda}{2(2^\lambda-2)} + \frac{61\varepsilon 2^\lambda}{2(2^\lambda-32)}\right]|x|^\lambda, & \lambda > 5, \end{cases}$$

for all $x \in \mathbb{R}$.

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