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LINEAR DEPENDENCE OF POWERS OF LINEAR FORMS

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Abstract. The main goal of the paper is to examine the dimension of the vector space spanned by powers of linear forms. We also find a lower bound for the number of summands in the presentation of zero form as a sum of d-th powers of linear forms.

1. Introduction and terminology

B. Reznick, in his 2003 paper [4] introduced the ticket

$$T(F) = \{d \in \mathbb{N} : \{f_j^d\} \text{ is linearly dependent}\}$$

for any finite set of polynomials $F = \{f_j\}$. If f_j' is the homogenisation of the polynomial f_j, then $T(F) = T(F')$, where $F' = \{f_j'\}$. Thus, examination of tickets can be confined to forms. Observing interesting results in Reznick's paper one can ask about "degree" of linear dependence of the set of powers of forms within its ticket, i.e. $\dim \operatorname{span}\{f_j^d\}$ for $d \in T(F)$. The question seems to be difficult in the case of any sets of forms. The ticket of a set of linear forms is an initial segment of the set $\mathbb{N}$ of natural numbers (see [4, Lemma 2.2]) and the problem has a chance for at least partial solution in this case.

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In section 2 we examine the growth of the sequence $(\dim \operatorname{span}\{l_j^d\})_{d\in\mathbb{N}}$ for a finite set $\{l_j\}$ of linear forms. In section 3 we find a lower bound for the number of summands in the presentation of zero form as a sum of d-th powers of linear forms. This is connected with a conjecture formulated by A. Schinzel.

Throughout the paper the field K is of characteristic 0, however some obtained results hold true for fields with big enough (greater than members of considered tickets) positive characteristic. Let $L_K(n)$ be the set of linear forms over a field K. For any $A = \{l_1, \ldots, l_r\} \subset L_K(n)$ and $d \in \mathbb{N}$ let A^d denote the subset $\{l_1^d, \ldots, l_r^d\}$ of the vector space $F_K(n, d)$ of homogeneous forms in n variables of degree d over K and let $w_d(A) = \dim \operatorname{span}(A^d)$. We need one more invariant

$$s_d(A) = \max\{k : B \subset A, \#B = k \Longrightarrow B^d \text{ is linearly independent}\}.$$

Of course, by the definition $s_d(A) \leq w_d(A) \leq \#A$. Throughout the paper we consider distinct elements of $L_K(n)$ as projectively distinct, i.e. any $l_1, l_2 \in L_K(n), l_1 \neq l_2$, are supposed to be linearly independent. Thus, if $\#A \geq 2$, then $2 \leq s_1(A) \leq n$. Let us start with a simple lemma.

Lemma 1.1. *Assume* $l_1, \ldots, l_r \in L_K(n), \lambda_1, \ldots, \lambda_r \in K$ *and*

$$\sum_{j=1}^{r} \lambda_j l_j^d = 0. \tag{1.1}$$

Then for every $\alpha = [a_1, \ldots, a_n] \in K^n$, *we have*

$$\sum_{j=1}^{r} \lambda_j l_j(\alpha) l_j^{d-1} = 0. \tag{1.2}$$

Proof. It suffices to apply the differential operator $\frac{1}{d}\frac{\partial}{\partial l}$ to the equation (1.1), where $l = a_1X_1 + \ldots + a_nX_n$. $\square$

Corollary 1.2. *For any finite subset* A *of* $L_K(n)$ *the sequences* $(s_d(A))_{d\in\mathbb{N}}$ *and* $(w_d(A))_{d\in\mathbb{N}}$ *are nondecreasing.*

In the next section we will show that these sequences are increasing until the moment they stabilize.

2. Powers of linear forms and the dimensions

Let us start this section with the main theorem.

THEOREM 2.1. *Assume* $A \subset L_K(n)$, $d \geq 2$, $(l_1^{d-1}, \ldots, l_{w_{d-1}(A)}^{d-1})$ *is a basis of* A^{d-1} *and* $l_{i_1}, \ldots, l_{i_k} \in A$ *satisfy the following conditions:*
(a) $\{l_{i_1}, \ldots, l_{i_k}\} \cap \{l_1, \ldots, l_{w_{d-1}(A)}\} = \emptyset$,
(b) $l_{i_1}, \ldots, l_{i_k}$ *are linearly independent,*
(c) *at most* $s_{d-1}(A) - 1$ *of* $l_1, \ldots, l_{w_{d-1}(A)}$ *belong to* $\operatorname{span}(l_{i_1}, \ldots, l_{i_k})$.
Then

$$w_d(A) - w_{d-1}(A) \geq k.$$

PROOF. By Corollary 1.2, the sets $\{l_1^d, \ldots, l_{w_{d-1}(A)}^d\}$ and $\{l_{i_1}^d, \ldots, l_{i_k}^d\}$ are linearly independent. Thus, it suffices to show that

$$\operatorname{span}(l_{i_1}^d, \ldots, l_{i_k}^d) \cap \operatorname{span}(l_1^d, \ldots, l_{w_{d-1}(A)}^d) = \{\theta\}.$$

Suppose

$$b_1 l_{i_1}^d + \ldots + b_k l_{i_k}^d = c_1 l_1^d + \ldots + c_{w_{d-1}(A)} l_{w_{d-1}(A)}^d, \quad b_i, c_j \in K, b_1 \neq 0. \tag{2.1}$$

Let

$$l_{i_1}^{d-1} = \sum_{j=1}^{w_{d-1}(A)} a_j l_j^{d-1}. \tag{2.2}$$

By the definition of $s_{d-1}(A)$, at least $s_{d-1}(A)$ of $a_1, \ldots, a_{w_{d-1}(A)}$ must be different from zero. Thus by (c), there exists j_0 such that $a_{j_0} \neq 0$ and the set $\{l_{i_1}, \ldots, l_{i_k}, l_{j_0}\}$ is linearly independent. We can take $\alpha \in K^n$ such that

$$l_{i_1}(\alpha) \neq 0, \quad l_{i_2}(\alpha) = \ldots = l_{i_k}(\alpha) = l_{j_0}(\alpha) = 0. \tag{2.3}$$

If $\alpha = [a_1, \ldots, a_n]$ and $l = a_1 X_1 + \ldots + a_n X_n$, then after applying differential operator $\frac{1}{d}\frac{\partial}{\partial l}$ to both sides of the equation (2.1) we get

$$b_1 l_{i_1}(\alpha) l_{i_1}^{d-1} = \sum_{i=1, i \neq j_0}^{w_{d-1}(A)} c_i l_i(\alpha) l_i^{d-1},$$

which is in contradiction with (2.2). $\square$

Corollary 2.2. *Suppose* $A \subset L_K(n), \#A = r < \infty, d \geq 2$. *Then*

(a) $w_d(A) - w_{d-1}(A) \geq \min\{s_1(A) - 1, r - w_{d-1}(A)\}$*;*
(b) *if* $r - w_{d-1}(A) \geq s_1(A) - 1$, *then* $w_d(A) \geq (d-1)(s_1(A) - 1) + w_1(A)$;
(c) *if* $s_1(A) = n$ *and* $r = d(n-1) + 1$, *then* A^d *is linearly independent;*
(d) *the sequence* $(w_d(A))_{d\in\mathbb{N}})$ *is increasing untill the* d *for which* A^d *is linearly independent,* $w_d(A) = r$ *for* $d \geq r - 1$*;*
(e) *the sequence* $(s_d(A))_{d\in\mathbb{N}})$ *is increasing untill the* d *for which* A^d *is linearly independent,* $s_d(A) = r$ *for* $d \geq r - 1$*;*
(f) $s_d(A) \geq \min\{s_1(A) + d - 1, r\}$.

Proof. (a) Let $k = \min\{s_1(A)-1, r-w_{d-1}(A)\}$ and let $(l_1^{d-1}, \ldots, l_{w_{d-1}(A)}^{d-1})$ be a basis of A^{d-1}. Any linearly independent subset $\{l_{i_1}, \ldots, l_{i_k}\}$ of A which is disjoint with $\{l_1, \ldots, l_{w_{d-1}(A)}\}$ satisfies assumptions of Theorem 2.1. Thus, $w_d(A) - w_{d-1}(A) \geq k$.

(b) If $r - w_{d-1}(A) \geq s_1(A) - 1$, then $r - w_k(A) \geq s_1(A) - 1$ for every $k \leq d - 1$ and by (a)

$$w_d(A) = \sum_{k=2}^{d}(w_k(A) - w_{k-1}(A)) + w_1(A) \geq (d-1)(s_1(A) - 1) + w_1(A).$$

(c) If $s_1(A) = n$, then $w_1(A) = n$ and the statement results immediately from (b).

(d) It follows easily from (a).

(e) If $r > s_d(A) = s_{d-1}(A)$, then there exists $B \subset A, \#B = s_d(A)+1$ such that B^d is linearly dependent. Take $B_1 \subset B, \#B_1 = s_d(A)$. Then

$$\begin{aligned} s_d(A) + 1 > \dim \operatorname{span} B^d &\geq \dim \operatorname{span} B^{d-1} \\ &\geq \dim \operatorname{span} B_1^{d-1} = s_{d-1}(A) = s_d(A) \end{aligned}$$

and $w_d(B) = \dim \operatorname{span} B^d = \dim \operatorname{span} B^{d-1} = w_{d-1}(B)$. By (d) the set B^d is linearly independent and we obtained a contradiction.

(f) We perform an induction on d. For obvious reasons the inequality is also true for $d = 1$. Suppose $d \geq 2$ and $s_{d-1}(A) \geq \min\{s_1(A) + d - 2, r\}$. If $s_d(A) \geq r$, then $s_d(A) = r$ and we are done. Suppose $r > s_d(A)$ and $s_1(A) + d - 1 > s_d(A)$. Since the sequence $(s_d(A))_{d\in\mathbb{N}}$ is nondecreasing, we have

$$s_1(A) + d - 1 > s_d(A) \geq s_{d-1}(A) \geq s_1(A) + d - 2.$$

Thus, $r > s_d(A) = s_{d-1}(A)$ which by (e) is impossible. $\square$

Remark. The statement (c) in Corollary 2.2 was proved by A. Białynicki-Birula and A. Schinzel [1, Theorem 2] for fields of characteristic 0 or greater than d. Moreover, they showed an example that the statement is no longer true when $r = d(n-1)+2 \leq \#K+1$.

The lower bound for the difference of two consecutive elements of the sequence $(w_d(A))_{d\in\mathbb{N}}$ in Theorem 2.1 does not exceed the number of variables. It is not hard to show examples that this bound is strict. However, there are examples of sets of linear forms for which $w_d(A) - w_{d-1}(A)$ is much bigger. For example, if

$$A = \{a_1X_1 + \ldots + a_nX_n : a_1, \ldots, a_n \in \mathbb{N}\cup\{0\}, \sum_{i=1}^{n} a_i = d\}$$

then A^d is a basis for $F_{\mathbb{R}}(n,d)$ (see [3, Proposition 2.11]) and

$$w_d(A) - w_{d-1}(A) \geq \dim F_{\mathbb{R}}(n,d) - \dim F_{\mathbb{R}}(n,d-1) = \binom{n+d-2}{n-2}.$$

The statement (c) in 2.2 indicates that $w_d(A)$ strongly depends on the configuration of the linear forms in A, so it is natural to ask what can we say about $w_d(A)$ if we have information on linearly independent subsets of A. We conclude this section with citing two interesting results due to A. Chlebowicz and M. Wołowiec-Musiał [2] which shed light on this problem.

Lemma 2.3 ([2, Lemma 2.4]). *Let $A = \{l_1, \ldots, l_m\} \subset L_K(n)$. Suppose there exists a number k and subsets $A_{i1}, \ldots, A_{ik}$ of A for $i = 1, \ldots, m$ such that*
(a) $A = \bigcup_{j=1}^{k} A_{ij},\ i = 1, \ldots, m,$
(b) $l_i \notin \operatorname{span}(A_{ij} \setminus \{l_i\}),\ i = 1, \ldots, m,\ j = 1, \ldots, k.$
Then A^k is linearly independent.

The proof of this lemma based on very useful criterion due to P. Serret (see [3, Proposition 2.6]). Actually the original Serret's Theorem refers to the field $\mathbb{R}$, but one can check that its proof holds true for any field of characteristic 0. The following theorem is an easy consequence of the above lemma.

Theorem 2.4 ([2, Theorem 2.5]). *If $A \subset L_K(n)$ can be decomposed into a union of s subsets, which are disjoint and linearly independent, then A^{2s-1} is linearly independent.*

Remark. B. Reznick (see [4, Lemma 3.2]) proved inequality $\#T(A) < \#A - 1$ for any finite subset A of $L_K(n)$. If A is as in Lemma 2.3, then we have the better bound $\#T(A) < k$. If A is a disjoint union of s linearly independent subsets, then $\#T(A) < 2s - 2$.

Lemma 2.3 and Theorem 2.4 turn out to be very useful in constructing examples.

Example. Let $A = \{l_1, \ldots, l_n, l_1 + l_n, \ldots, l_{n-1} + l_n\} \subset L_K(n)$, where $l_1, \ldots, l_n$ are linearly independent. The sets

$$A_{i1} = \{l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n\}, \quad A_{i2} = \{l_1 + l_n, \ldots, l_{n-1} + l_n, l_i\}$$

satisfy conditions (a) and (b) from Lemma 2.3. Thus, A^2 is linearly independent and $w_1(A) = n, w_d(A) = 2n - 1$ for $d \geq 2$.

We may generalize this example. Let $B_0 = \{l_1, \ldots, l_n\} \subset L_K(n)$, where $l_1, \ldots, l_n$ are linearly independent, $B_j = \{l_j + l_i : j = 1, \ldots, n, j \neq i\}$ and $A = \bigcup_{j=0}^{n} B_j$. The sets $A_{i0} = B_0 \setminus \{l_i\}, A_{ij} = B_j \cup \{l_i\}$ for $j = 1, \ldots, n$, satisfy conditions (a) and (b) from Lemma 2.3. Thus, A^{n+1} is linearly independent and $w_1(A) = n, w_d(A) = n + n(n-1)$ for $d \geq n + 1$.

3. Schinzel's Conjecture

In this section we will show one more application of Theorem 2.4. A. Schinzel, during his lecture at 18th Czech and Slovak International Conference on Number Theory (2007, Smolenice, Slovakia), presented the following conjecture.

Conjecture. Let K be a field of characteristic 0 or greater than d. If $l_1, \ldots, l_r \in L_K(n)$, $\dim \operatorname{span}(l_1, \ldots, l_r) = n$ and $\sum_{i=1}^{r} l_i^d = 0$, but no proper subsum is 0, then $r \geq d(n-1) + 2$.

In [1] it was shown that the conjecture holds true for $n \leq 4$ and any d. We shall show that a weaker version of this conjecture holds true.

Lemma 3.1. *Let $l_1, \ldots, l_k, m_1, \ldots, m_s \in L_K(n)$ be pairwise projectively distinct. If*

$$l_1^d + \ldots + l_k^d = m_1^d + \ldots + m_s^d, \tag{3.1}$$

$\dim \operatorname{span}(l_1, \ldots, l_k) = n$ *and the set* $\{l_1^{d-1}, \ldots, l_k^{d-1}\}$ *is linearly independent, then* $\dim \operatorname{span}(m_1, \ldots, m_s) = n$.

Proof. Suppose that $\dim \operatorname{span}(m_1, \ldots, m_s) < n$. Then there exists $\alpha = [a_1, \ldots, a_n] \in K^n, \alpha \neq \theta$, such that $m_i(\alpha) = 0$ for $i = 1, \ldots, s$. Let us take $l = a_1 X_1 + \ldots + a_n X_n$ and apply the differential operator $\frac{1}{d}\frac{\partial}{\partial l}$ to both sides of equation (3.1). We have

$$\sum_{j=1}^{k} l_j(\alpha) l_j^{d-1} = \sum_{i=1}^{s} m_i(\alpha) m_i^{d-1} = 0.$$

Since $\dim \operatorname{span}(l_1, \ldots, l_k) = n$, there exists $j_0 \in \{1, \ldots, k\}$ such that $l_{j_0}(\alpha) \neq 0$. We get a contradiction with linear independence of $\{l_1^{d-1}, \ldots, l_k^{d-1}\}$. $\square$

Theorem 3.2. *If* $l_1, \ldots, l_r \in L_K(n)$, $\dim \operatorname{span}(l_1, \ldots, l_r) = n$ *and*

$$\sum_{i=1}^{r} l_i^d = 0,$$

but no proper subsum is 0, *then* $r \geq \frac{n(d+1)}{2}$, *if* d *is odd and* $r \geq \frac{n(d+2)}{2}$, *if* d *is even.*

Proof. First we show that for any $t \leq \frac{d}{2}+1$ we may find disjoint linearly independent subsets $A_1, \ldots A_t$ of $\{l_1, \ldots, l_r\}$, the number of elements of each is n, and such that (after reindexing elements of $\{l_1, \ldots, l_r\}$)

$$A_1 \cup \ldots \cup A_t = \{l_1, \ldots, l_{nt}\} \subset \{l_1, \ldots, l_r\}.$$

By the assumptions, we may find linearly independent subset $A_1 \subset \{l_1, \ldots, l_r\}$. Take the maximal t for which there exist $A_1, \ldots, A_t$ satisfying the above requirement. If $t \leq \frac{d}{2}+1$ and $t+1 > \frac{d}{2}+1$, then we are done. Thus, suppose $t+1 \leq \frac{d}{2}+1$. Then $2t-1 \leq d-1$ and by Theorem 2.4, the set $\{l_1^{d-1}, \ldots, l_{nt}^{d-1}\}$ is linearly independent. Let $\alpha = \sqrt[d]{-1}$. We have

$$l_1^d + \ldots + l_{nt}^d = (\alpha\, l_{nt+1})^d + \ldots + (\alpha\, l_r)^d$$

and by Lemma 3.1, applied over the field $K(\alpha)$, the set $\{l_{nt+1}, \ldots, l_r\}$ contains at least one more linearly independent subset with n elements. We obtained a contradiction with the choice of t.

Now, for even d we may take $t = \frac{d}{2}+1$, and then $r \geq nt = \frac{n(d+2)}{2}$. If d is odd, then $t = \frac{d-1}{2}+1$ is possible and $r \geq nt = \frac{n(d+1)}{2}$. $\square$

Remark. From the above Theorem it follows that Schinzel's conjecture is true for $d = 2$ and any n, since then $\frac{n(d+2)}{2} = d(n-1)+2$. Unfortunately, our lower bound for r is the worst the bigger d is. It seems that to prove Schinzel's conjecture one should use other tools. It would be interesting to check this conjecture for $d = 3$ and any n.

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