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Prace Naukowe Uniwersytetu Śląskiego nr 3240, Katowice

STRONG MAXIMUM PRINCIPLES FOR INFINITE IMPLICIT PARABOLIC SYSTEMS

Ludwik Byszewski

Abstract. In this paper we prove a theorem on strong maximum principles for infinite implicit systems of parabolic differential-functional inequalities together with nonlocal inequalities with functionals in $(n+1)$-dimensional sets more general than the cylindrical domain. Results obtained are generalizations of those from [1]–[8] and [10].

1. Introduction

In this paper we study the following infinite parabolic systems:

$$F_i(x,t,u^i(x,t),\ u^i_t(x,t),\ u^i_x(x,t),\ u^i_{xx}(x,t),u)$$
$$\geq F_i(x,t,v^i(x,t),v^i_t(x,t),\ v^i_x(x,t),v^i_{xx}(x,t),v) \quad (i=1,2,\dots)$$

together with the following nonlocal inequalities with functionals:

$$[u^j(x,t_0)-K^j]+\sum_{i\in I_*} h_i(x)[g_i(x,u^j\mid_{Z_i})-K^j]\leq 0$$

in $(n+1)$-dimensional space-time more general than the cylindrical domain.

We prove strong maximum principles for the above problems.

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Results obtained are generalizations of those from [1]–[8] and [10].

Infinite parabolic systems have physical application. For this purpose please see the publication [9] by Wrzosek.

2. Preliminaries

The notation, definitions and assumptions given in this section are applied throughout the paper.

We shall use the following notations:

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{N} = \{1, 2, \ldots\}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \quad (n \in \mathbb{N}).$$

By ℓ_∞ we denote the Banach space of real sequences $\xi = (\xi^1, \xi^2, \ldots)$ such that

$$\sup\{| \xi^j |: j = 1, 2, \ldots\} < \infty$$

and

$$\|\xi\|_{\ell^\infty} = \sup\{| \xi^j |: j = 1, 2, \ldots\}.$$

For $\xi = (\xi^1, \xi^2, \ldots)$, $\eta = (\eta^1, \eta^2, \ldots) \in \ell^\infty$ we write $\xi \leq \eta$ in the sense $\xi^i \leq \eta^i$ $(i \in \mathbb{N})$.

Let t_0 be a real number and let $T \in (0, \infty)$.

A set $D \subset \{(x, t) : x \in \mathbb{R}^n, t_0 < t \leq t_0 + T\}$ is called a *set of type* (P) if

(a) The projection of the interior of set D on the t-axis is the interval $(t_0, t_0 + T)$.

(b) For every $(\tilde{x}, \tilde{t}) \in D$ there exists a positive number $r = r(\tilde{x}, \tilde{t})$ such that

$$\{(x, t) : \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 + (t - \tilde{t}) < r, t < \tilde{t}\} \subset D.$$

(c) All the boundary points $(\tilde{x}, \tilde{t})$ of D for which there is a positive number $r = r(\tilde{x}, \tilde{t})$ such that

$$\{(x, t) : \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 + (t - \tilde{t}) < r, t \leq \tilde{t}\} \subset D$$

belong to D.

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} \operatorname{int}\{x \in \mathbb{R}^n : (x, t_0) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in \mathbb{R}^n : (x, t) \in D\} & \text{for } t \neq t_0, \end{cases}$$

$$\sigma_t = \begin{cases} \operatorname{int}[\bar{D} \cap (\mathbb{R}^n \times \{t_0\})] & \text{for } t = t_0, \\ D \cap (\mathbb{R}^n \times \{t\}) & \text{for } t \neq t_0. \end{cases}$$

Let $\tilde{D}$ be an arbitrary set such that

$$\bar{D} \subset \tilde{D} \subset \mathbb{R}^n \times (-\infty, t_0 + T].$$

We introduce the following sets:

$$\partial_p D := \tilde{D} \setminus D \quad \text{and} \quad \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{x}, \tilde{t}) \in D$, we denote by $S^-(\tilde{x}, \tilde{t})$ the set of points $(x, t) \in D$, that can be joined to $(\tilde{x}, \tilde{t})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (x, t) to $(\tilde{x}, \tilde{t})$.

Let $Z_\infty(\tilde{D})$ denote the space of mappings

$$w\colon \mathbb{N} \times \tilde{D} \ni (i, x, t) \to w^i(x, t) \in \mathbb{R},$$

where the functions

$$w^i\colon \tilde{D} \ni (x, t) \to w^i(x, t) \in \mathbb{R}$$

are continuous in $\bar{D}$ and

$$\sup\{\mid w^i(x, t) \mid : (x, t) \in \tilde{D},\ \ i \in \mathbb{N}\} < \infty.$$

For $w, \tilde{w} \in Z_\infty(\tilde{D})$ we write $w \leq \tilde{w}$ in the sense $w^i \leq \tilde{w}^i$ $(i \in \mathbb{N})$.

By $Z^{2,1}_\infty(\tilde{D})$ we denote the linear subspace of $Z_\infty(\tilde{D})$. A mapping w belongs to $Z^{2,1}_\infty(\tilde{D})$ if $w^i_t, w^i_x = (w^i_{x_1}, \ldots, w^i_{x_n})$, $w^i_{xx} = [w^i_{x_j x_k}]_{n\times n}$ $(i \in \mathbb{N})$ are continuous in D.

By $M_{n\times n}(\mathbb{R})$ we denote the space of real symmetric matrices $r = [r_{jk}]_{n\times n}$.

Let the mappings

$$\begin{aligned} F_i\colon D \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n}(\mathbb{R}) \times Z_\infty(\tilde{D}) \ni (x, t, z, p, q, r, w) \\ \longrightarrow F_i(x, t, z, p, q, r, w) \in \mathbb{R}\ \ (i \in \mathbb{N}) \end{aligned}$$

be given and let for an arbitrary function $w \in Z^{2,1}_\infty(\tilde{D})$

$$F_i[x,t,w] := F_i(x,t,w^i(x,t),w_t^i(x,t),w_x^i(x,t),w_{xx}^i(x,t),w), \qquad (x,t)\in D \quad (i\in\mathbb{N}).$$

For a given subset $E \subset D$ and a given mapping $w \in Z_\infty^{2,1}(\tilde{D})$, and a fixed index $i \in \mathbb{N}$ the function F_i is called *uniformly parabolic* with respect to w in E if there is a constant $\kappa > 0$ (depending on E) such that for any two matrices $r = [r_{jk}] \in M_{n\times n}(\mathbb{R}), \tilde{r} = [\tilde{r}_{jk}] \in M_{n\times n}(\mathbb{R})$ with $r \le \tilde{r}$ and for $(x,t) \in E$, we have

$$\begin{aligned} &F_i(x,t,w^i(x,t),w_t^i(x,t),w_x^i(x,t),\tilde{r},w) \\ &\qquad - F_i(x,t,w^i(x,t),w_t^i(x,t),w_x^i(x,t),r,w) \ge \kappa \sum_{j=1}^n (\tilde{r}_{jj} - r_{jj}). \end{aligned} \tag{2.1}$$

If (2.1) is satisfied for $\kappa = 0$ and $r = w_{xx}^i(x,t)$, where $(x,t) \in D$, and for $\tilde{r} = w_{xx}^i(x,t) + \hat{r}$, where $(x,t) \in E$ and $\hat{r} \ge 0$, then F_i is called *parabolic* with respect to w in E.

Two functions $u, v \in Z_\infty^{2,1}(\tilde{D})$ are called *solutions* of the system

$$F_i[x,t,u] \ge F_i[x,t,v] \quad (i\in\mathbb{N}) \tag{2.2}$$

in D, if they satisfy (2.2) for $(x,t) \in D$.

Assumption (L)**.** There are constants $L_i > 0$ $(i = 1,2)$ such that

$$F_i(x,t,z,p,q,r,w) - F_i(x,t,z,\tilde{p},q,r,w) \le L_1(\tilde{p} - p) \quad (i\in\mathbb{N})$$

for all $(x,t) \in D$, $z \in \mathbb{R}, p > \tilde{p}, q \in \mathbb{R}^n, r \in M_{n\times n}(\mathbb{R}), w \in Z_\infty(\tilde{D})$ and

$$\begin{aligned} &F_i(x,t,z,p,q,r,w) - F_i(x,t,\tilde{z},\tilde{p},\tilde{q},\tilde{r},\tilde{w}) \\ &\qquad \le L_2\Big(|z-\tilde{z}| + |p-\tilde{p}| + |x|\sum_{j=1}^n |q_j - \tilde{q}_j| \\ &\qquad\qquad + |x|^2 \sum_{j,k=1}^n |r_{jk} - \tilde{r}_{jk}| + [w-\tilde{w}]_{t,\infty}\Big) \quad (i\in\mathbb{N}) \end{aligned}$$

for all $(x,t) \in D, z, \tilde{z} \in \mathbb{R}, p, \tilde{p} \in \mathbb{R}, q, \tilde{q} \in \mathbb{R}^n, r, \tilde{r} \in M_{n\times n}(\mathbb{R}), w, \tilde{w} \in Z_\infty(\tilde{D})$.

For every set $A \subset \tilde{D}$ and for each function $w \in Z_\infty(\tilde{D})$ we apply the notation:

$$\max_{(x,t)\in A} w(x,t) := \Big(\max_{(x,t)\in A} w^1(x,t), \max_{(x,t)\in A} w^2(x,t), \dots\Big).$$

Let $I = \mathbb{N}$ or I is a finite set of natural numbers.

Let us define the set

$$S = \bigcup_{i\in I}(\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}),$$

where, in $I = \mathbb{N}$, the following conditions are satisfied:

(i) $t_0 < T_{2i-1} < T_{2i} \le t_0 + T$ for $i \in I$ and

$$T_{2i-1} \neq T_{2j-1}, \quad T_{2i} \neq T_{2j} \quad \text{for } i,j \in I, i \neq j;$$

(ii) $T_0 := \inf\{T_{2j-1} : i \in I_0\} > t_0$;

(iii) $S_t \supset S_{t_0}$ for every $t \in \bigcup_{i\in I}\Big[T_{2i-1}, T_{2i}\Big]$;

(iv) $S_t \supset S_{t_0}$ for every $t \in \Big[T_0, t_0 + T\Big]$,

and if I is a finite set of natural numbers, the conditions (i), (iii) are satisfied.

An unbounded set D of type (P) is called *a set of type* $(P_{S\Gamma})$, if:

(a) $S \neq \phi$,

(b) $\Gamma \cap \bar{\sigma}_{t_0} \neq \phi$.

Let S_* denote a non-empty subset of S. We define the following set:

$$I_* = \{i \in I : (\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}) \subset S_*\}.$$

An bounded set D of type (P) satisfying condition (a) of the definition of a set of type $(P_{S\Gamma})$ is called *a set of type* (P_{SB}).

Let

$$Z_i := S_{t_0} \times [T_{2i-1}, T_{2i}] \quad (i \in I_*).$$

Assumption (N)**.** The functions $g_i \colon S_{t_0} \times C(Z_i, \mathbb{R}) \to \mathbb{R}$ $(i \in I_*)$ and $h_i \colon S_{t_0} \to \mathbb{R}_-$ $(i \in I_*)$ satisfy the following conditions:

$$(N_1) \quad g_i(x, \xi\,|_{Z_i}) \le \max_{i\in[T_{2i-1},T_{2i}]} \xi(x,t) \quad \text{for } x \in S_{t_0},\ \xi \in C(\bar{D}, \mathbb{R}) \quad (i \in I_*)$$

and

$$(N_2) \qquad -1 \leq \sum_{i \in I_*} h_i(x) \leq 0 \quad \text{for } x \in S_{t_0}.$$

3. Strong maximum principles

THEOREM 3.1. *Assume that:*

(1) *D is a set of type $(P_{S\Gamma})$ or (P_{SB}).*

(2) *The functions F_i $(i \in \mathbb{N})$ satisfy Assumption (L) and the functions g_i, h_i $(i \in I_*)$ satisfy Assumption (N).*

(3) *$u \in Z^{2,1}_{\infty}(\tilde{D})$ and the maximum of function u of Γ is attained. Moreover, $K \in \ell^{\infty}, K \colon \mathbb{N} \times \tilde{D} \ni (i, x, t) \to K^i$ is given by*

$$K^i := \max_{(x,t) \in \Gamma} u^i(x, t) \quad (i \in \mathbb{N}). \tag{3.1}$$

(4) *The following inequalities hold*

$$[u^j(x, t_0) - K^j] + \sum_{i \in I_*} h_i(x)[g_i(x, u^j \mid_{Z_i}) - K^j] \leq 0 \tag{3.2}$$

for $x \in S_{t_0}$ $(j \in \mathbb{N})$, where the series $\sum\limits_{i \in I_*} h_i(x) g_i(x, u^j \mid_{Z_i})$ $(j \in \mathbb{N})$ are convergent for $x \in S_{t_0}$ if $\operatorname{card} I_* = \aleph_0$.

(5) *There exists a point $(x^*, t^*) \in \tilde{D}$ such that*

$$u(x^*, t^*) = \max_{(x,j) \in \tilde{D}} u(x, t).$$

Moreover

$$M^i := u^i(x^*, t^*) \quad (i \in \mathbb{N}) \tag{3.3}$$

and $M \in \ell^{\infty}$ is defined by

$$M \colon \mathbb{N} \times \tilde{D} \ni (i, x, t) \to M^i.$$

(6) *u and $v = M$ are solutions of system (2.2) in D.*

(7) *F_i $(i \in \mathbb{N})$ are parabolic with respect to u in D and uniformly parabolic with respect to M in any compact subset of D.*

Then

$$\max_{(x,t)\in\tilde{D}} u(x,t) = \max_{(x,t)\in\Gamma} u(x,t). \tag{3.4}$$

Moreover, if there is a point $(\tilde{x},\tilde{t}) \in D$ *such that*

$$u(\tilde{x},\tilde{t}) = \max_{(x,t)\in\tilde{D}} u(x,t)$$

then

$$u(x,t) = \max_{(\hat{x},\hat{t})\in\Gamma} u(\hat{x},\hat{t}) \quad \textit{for } (x,t) \in S^-(\tilde{x},\tilde{t}).$$

Proof. We shall prove Theorem 3.1 for a set of type $(P_{S\Gamma})$ only, since the proof of this theorem for a set of type (P_{SB}) is analogous.

We shall argue by contradiction. Suppose that

$$M \neq K.$$

Next, (3.1) and (3.3) imply inequalities

$$K^i \leq M^i \quad (i \in \mathbb{N}).$$

Consequently,

$$\text{There is } \ell \in \mathbb{N} \text{ such that } K^t < M^t. \tag{3.5}$$

Observe, from assumption (5), that:

$$\begin{gathered}\text{There is a point } (x^*,t^*) \in \tilde{D} \text{ such that}\\ u(x^*,t^*) = M := \max_{(x,t)\in\tilde{D}} u(x,t).\end{gathered} \tag{3.6}$$

By (3.6), assumption (3) and (3.5), we have

$$(x^*,t^*) \in \tilde{D} \setminus \Gamma = D \cup \sigma_{t_0}. \tag{3.7}$$

An argument analogous to the proof of Theorem 3.1 of [3] yields

$$(x^*,t^*) \notin D. \tag{3.8}$$

Conditions (3.7) and (3.8) give

$$(x^*, t^*) \in \sigma_{t_0}. \tag{3.9}$$

Consider now two possible cases:

$$\text{(I)} \quad \sum_{i \in I_*} h_i(x) = 0, \qquad \text{(II)} \quad -1 \le \sum_{i \in I_*} h_i(x) < 0.$$

In case (I) condition (3.9) leads, by (3.5), to a contradiction of (3.2) with (3.6). From this contradiction the proof of (3.4) is complete in case (I).

In case (II), by the definition of sets I and I_*, we must consider the following cases:

(A) I_* is a finite set, i.e. (without loss of generality), there is a number $P \in \mathbb{N}$ such that $I_* = \{1, \dots, P\}$.

(B) card $I_* = \aleph_0$.

First we shall consider case (A). And so, since $u^i \in C(\bar{D}, \mathbb{R})$, it follows that for every $j \in I_*$ there is $\tilde{T}_i^j \in [T_{2i-1}, T_{2i}]$ such that

$$u^j(x^*, \tilde{T}_i^j) = \max_{t \in [T_{2i-1}, T_{2i}]} u^j(x^*, t). \tag{3.10}$$

Consequently, by (3.2), Assumption (N_1), (3.10) and the inequality

$$u(x^*, t) < u(x^*, t_0) \quad \text{for } \ t \in \bigcup_{i=1}^{P} \Big[T_{2i-1}, T_{2i}\Big]$$

(being a consequence of (3.6), (3.9) and of (a)(i), (a)(iii) in the definition of a set of type $(P_{S\Gamma})$), there is

$$\begin{aligned}
0 &\ge [u^j(x^*, t_0) - K^j] + \sum_{i=1}^{P} h_i(x^*)[g^i(x^*, u^j\mid_{Z_i}) - K^j] \\
&\ge [u^j(x^*, t_0) - K^j] + \sum_{i=1}^{P} h_i(x^*)[u^j(x^*, \tilde{T}_i^j) - K^j] \\
&\ge [u^j(x^*, t_0) - K^j] + \sum_{i=1}^{P} h_i(x^*)[u^j(x^*, t_0) - K^j] \\
&= [u^j(x^*, t_0) - K^j] \cdot [1 + \sum_{i=1}^{P} h_i(x^*)] \quad (j \in \mathbb{N}).
\end{aligned}$$

Hence

$$u(x^*,t_0) \le K \ \text{ if } \ 1+\sum_{i\in I_*} h_i(x^*) > 0. \tag{3.11}$$

Then, from (3.5) and (3.9), we obtain a contradiction of (3.11) with (3.6). Assume now that

$$\sum_{i\in I_*} h_i(x^*) = -1. \tag{3.12}$$

Observe that for every $j \in \mathbb{N}$ there is a number $\ell_j \in \{1, ..., P\}$ such that

$$u^j(x^*, \tilde{T}^j_{l_j}) = \max_{i=1,\dots,P} u^j(x^*, \tilde{T}^j_i). \tag{3.13}$$

Consequently, by (3.12), (3.13), (3.10), by Assumption (N_1), and by (3.2), we obtain

$$\begin{aligned} u^i(x^*,t_0) - u^j(x^*,\tilde{T}^j_{l_j}) &= [u^j(x^*,t_0) - K^j] - [u^j(x^*,\tilde{T}^j_{l_j}) - K^j] \\ &= [u^j(x^*,t_0) - K^j] + \sum_{i=1}^{P} h_i(x^*)[u^j(x^*,\tilde{T}^j_{l_j}) - K^j] \\ &\le [u^j(x^*,t_0) - K^j] + \sum_{i=1}^{P} h_i(x^*)[u^j(x^*,\tilde{T}^j_{i}) - K^j] \\ &\le [u^j(x^*,t_0) - K^j] + \sum_{i=1}^{P} h_i(x^*)[g_i(x^*,u^j\mid_{Z_i}) - K^j] \\ &\le 0 \quad (j \in \mathbb{N}). \end{aligned}$$

Hence

$$u^i(x^*,t_0) \le u^j(x^*,\tilde{T}^j_{l_j}) \ (j\in\mathbb{N}) \quad \text{if} \ \sum_{i=1}^{P} h_i(x^*) = -1. \tag{3.14}$$

Since, by (a)(i) of the definition of a set of type $(P_{S\Gamma})$, $\tilde{T}^j_{l_j} > t_0$ $(j \in \mathbb{N})$, from (3.9), we get that condition (3.14) contradicts (3.6). This completes the proof of (3.4) in case (A).

It remains to investigate case (B). Analogously as in the proof of (3.4) in case (A), by (3.2), Assumption (N_1), (3.10) and the inequality

$$u(x^*,t) < u(x^*,t_0) \quad \text{for } t \in \bigcup_{i\in I_*} [T_{2i-1}, T_{2i}]$$

(being a consequence of (3.6), (3.9), and of the (a)(i), (a)(iii) of the definition of a set of type $(P_{S\Gamma})$), there is

$$\begin{aligned} 0 &\geq [u^j(x^*,t_0) - K^j] + \sum_{i\in I_*} h_i(x^*)[g_i(x^*, u^j\mid_{Z_i}) - K^j] \\ &\geq [u^j(x^*,t_0) - K^j] + \sum_{i\in I_*} h_i(x^*)[u^j(x^*, \tilde{T}_i^j) - K^j] \\ &\geq [u^j(x^*,t_0) - K^j] + \sum_{i\in I_*} h_i(x^*)[u^j(x^*,t_0) - K^j] \\ &= [u^j(x^*,t_0) - K^j] \cdot [1 + \sum_{i\in I_*} h_i(x^*)] \quad (j \in \mathbb{N}). \end{aligned}$$

Hence

$$u(x^*,t_0) \leq K \quad \text{if } 1 + \sum_{i\in I_*} h_i(x^*) > 0. \tag{3.15}$$

Then, from (3.5) and (3.9), we obtain a contradiction of (3.15) with (3.6). Assume now that

$$\sum_{i\in I_*} h_i(x^*) = -1. \tag{3.16}$$

Let

$$\tilde{T}_*^j = \inf_{i\in I_*} \tilde{T}_i^j \quad (j \in \mathbb{N}). \tag{3.17}$$

Since $u^i \in C(\bar{D})(i \in \mathbb{N})$ and since (by (3.9) and by (a)(iv), (a)(ii) of the definition of a set of type $(P_{S\Gamma})$) $x^* \in S_t$ for every $t \in [T_0, t_0 + T]$ if $I = \mathbb{N}$, from (3.17), it follows that for every $j \in \mathbb{N}$ there is a number $\hat{t}_j \in [\tilde{T}_*^j, t_0 + T]$ such that

$$u^j(x^*, \hat{t}_j) = \max_{t\in[\tilde{T}_*^j, t_0+T]} u^j(x^*, t). \tag{3.18}$$

Consequently, by (3.16), (3.18), (3.10), by Assumption (N_1) and Assumption (4), there follows

$$\begin{aligned} u^j(x^*,t_0) - u^j(x^*,\hat{t}_j) &= [u^j(x^*,t_0) - K^j] - [u^j(x^*,\hat{t}_j) - K^j] \\ &= [u^j(x^*,t_0) - K^j] + \sum_{i\in I_*} h_i(x^*)[u^j(x^*,\hat{t}_j) - K^j] \\ &\le [u^j(x^*,t_0) - K^j] + \sum_{i\in I_*} h_i(x^*)[u^j(x^*,\tilde{T}_i^j) - K^j] \\ &\le [u^j(x^*,t_0) - K^j] + \sum_{i\in I_*} h_i(x^*)[g_i(x^*,u^j\mid_{Z_i}) - K^j] \\ &\le 0 \quad (j\in\mathbb{N}). \end{aligned}$$

Hence

$$u^j(x^*,t_0) \le u^j(x^*,\hat{t}_j) \;\; (j\in\mathbb{N}) \quad \text{if} \sum_{i\in I_*} h_i(x^*) = -1. \tag{3.19}$$

Since, by (a)(ii) of the definition of a set of type $(P_{S\Gamma}), \hat{t}_j > t_0$ $(j\in\mathbb{N})$, from (3.9), we get that condition (3.19), contradicts condition (3.6). This completes the proof of equality (3.4).

The second part of Theorem 3.1 is a consequence of (3.4) and of Theorem 4.1 from [2]. Therefore, the proof of Theorem 3.1 is complete. □

Remark 1. It is easy to see that the functionals

$$g_i\colon S_{t_0}\times C(Z_i,\mathbb{R}) \to \mathbb{R} \quad (i\in I_*)$$

given by the formulae

$$\begin{aligned} g_i(x,w^j\mid_{Z_i}) = \frac{1}{T_{2i}-T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} w^j(x,\tau)d\tau, \quad x\in S_{t_0}, \\ w^j\in C(\bar{D},\mathbb{R}) \quad (i\in I_*,\ j\in\mathbb{N}) \end{aligned}$$

satisfy Assumption (N_1).

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Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Cracow
Poland
e-mail: lbyszews@pk.edu.pl