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ON APPROXIMATE n-JORDAN HOMOMORPHISMS

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Abstract. The aim of this paper it to characterize n-Jordan homomorphisms and to investigate their connection with n-homomorphisms as well as homomorphisms. Furthermore, the following implication is also verified: if a function φ is additive and (for a fixed integer $n \geq 2$) the mapping $x \longmapsto \varphi(x^n) - \varphi(x)^n$ satisfies some mild regularity assumption, then the function φ is an n-homomorphism or it is a continuous additive function.

1. Introduction and preliminaries

The study of additive mappings from a ring into another ring which preserve squares was initiated by G. Ancochea in [1] in connection with problems arising in projective geometry. Later, these results were strengthened by (among others) Kaplansky [8] and Jacobson–Rickart [7].

In this paper we will present some characterization theorems concerning n-Jordan homomorphisms. Firstly, we will investigate the connection between n-Jordan homomorphisms and n-homomorphisms. Then, n-Jordan homomorphisms will be characterized among additive functions. Our main results will have the following form: if a function φ is additive and (for a fixed integer

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$n \geq 2$) the mapping $x \longmapsto \varphi(x^n) - \varphi(x)^n$ satisfies some mild regularity assumption, then the function φ is an n-homomorphism or it is a continuous additive function.

We remark that the topic of 'approximate homomorphisms' was investigated by several authors, see e.g. Badora [2], Šemrl [10, 11]. In the cited papers however the stability of homomorphisms was dealt with which is a different approach than ours.

In the remaining part of this section we will fix the notation and the terminology as well as the preliminaries that will be necessary in what follows.

Henceforth, $\mathbb{N}$ will denote the set of the positive integers.

Let R, R' be rings, the mapping $\varphi\colon R \to R'$ is called a *homomorphism* if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad (a, b \in R)$$

and

$$\varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in R)\,.$$

Furthermore, the function $\varphi\colon R \to R'$ is an *anti-homomorphism* if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad (a, b \in R)$$

and

$$\varphi(ab) = \varphi(b)\varphi(a) \quad (a, b \in R)\,.$$

Let $n \in \mathbb{N}, n \geq 2$ be fixed. The function $\varphi\colon R \to R'$ is called an *n-homomorphism* if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad (a, b \in R)$$

and

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n) \quad (a_1, \ldots, a_n \in R)\,.$$

The function $\varphi\colon R \to R'$ is called an n-Jordan homomorphism if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad (a, b \in R)$$

and

$$\varphi(a^n) = \varphi(a)^n \quad (a \in R)\,.$$

Finally, we remark that in case $n = 2$ we speak about homomorphisms and Jordan homomorphisms, respectively. The concept of n-homomorphisms was introduced in Hejazian et al. [5]. Furthermore, the notion of n-Jordan homomorphisms was dealt with firstly in Herstein [6]. From the above definitions immediately follows that every n-homomorphism is an n-Jordan homomorphism. The converse, however, does not hold in general.

Before reviewing the known results we recall some basic definitions from ring theory. Let $n \in \mathbb{N}$, we say that a ring R is *of characteristic larger than n* if $n!x = 0$ implies that $x = 0$.

The ring R is termed to be a *prime ring* if

$$a, b \in R \quad \text{and} \quad aRb = \{0\}$$

imply that either $a = 0$ or $b = 0$.

As we wrote above it was G. Ancochea who firstly dealt with the connection of Jordan homomorphisms and homomorphisms, see [1]. The results of G. Ancochea were generalized and extended in several ways, see for instance [7], [8], [13]. Later, in 1956 I.N. Herstein proved the following.

Theorem 1.1 (Herstein [6]). *If φ is a Jordan homomorphism of a ring R* ***onto*** *a prime ring R' of characteristic different from 2 and 3 then either φ is a homomorphism or an anti-homomorphism.*

In [6] not only Jordan homomorphisms but also n-Jordan mappings were considered. Concerning this the following statement was verified.

Theorem 1.2 (Herstein [6]). *Let φ be an n-Jordan homomorphism from a ring R* ***onto*** *a prime ring R' of characteristic larger than n. Suppose further that R has a unit element. Then $\varphi = \varepsilon\tau$ where τ is either a homomorphism or an anti-homomorphism and ε is an $(n-1)$st root of unity lying in the center of R'.*

At the end of the paper I.N. Herstein suggests: *'…One might conjecture that an appropriate variant of this theorem would hold even if R does not have a unit element.'* This problem was solved by M. Brešar, W. Martindale and R.C. Miers. In [3] they proved the following.

Theorem 1.3 (Brešar–Martindale–Miers [3]). *Let $n \geq 3$ and let φ be an n-Jordan homomorphism of the ring R* ***onto*** *the prime ring R'. Suppose further that the characteristic of R' is zero or larger than $2m(m+1)$ with $m = 4n - 8$. Then there exists $\varepsilon \in C'$ (the extended centroid of R') such that*

$\varepsilon^{n-1} = 1$ *and a homomorphism or an anti-homomorphism* $\tau\colon R \to R'C'$ *such that*

$$\varphi(x) = \varepsilon\tau(x) \quad (x \in R)\,.$$

On the score of the above theorems, we notice that the fact that the mapping in question is surjective and its range is a prime ring, is *essential.* However, there has been proved statements in which the surjectivity is not assumed. At the expense of this, we have to suppose more about the domain and also about the range. In 2009 M. Eshaghi Gordji proved that in case $n \in \{3, 4\}$ is fixed, A, B are commutative algebras, then every $\varphi\colon A \to B$ n-Jordan homomorphism is an n-homomorphism, see [4].

In this paper we would like to extend the results of Eshaghi Gordji [4] in several ways. Furthermore, we also would like to prove results similar to the above cited ones. To do this, we need to recall some definitions and statement, these can be found e.g. in Kuczma [9].

Let G, H be abelian groups, let $h \in G$ be arbitrary and consider a function $f\colon G \to H$. The *difference operator* Δ_h with the span h of the function f is defined by

$$\Delta_h f(x) = f(x+h) - f(x) \quad (x \in G)\,.$$

The iterates Δ_h^n of Δ_h, $n = 0, 1, \dots$ are defined by the recurrence

$$\Delta_h^0 f = f, \quad \Delta_h^{n+1} f = \Delta_h\left(\Delta_h^n f\right) \quad (n = 0, 1, \dots)\,.$$

Furthermore, the superposition of several difference operators will be denoted shortly

$$\Delta_{h_1 \dots h_n} f = \Delta_{h_1} \dots \Delta_{h_n} f,$$

where $n \in \mathbb{N}$ and $h_1, \dots, h_n \in G$.

Let $n \in \mathbb{N}$ and G, H be abelian groups. A function $F\colon G^n \to H$ is called *n-additive* if, for every $i \in \{\,1\,,\,2\,,\,\dots\,,\,n\,\}$ and for every $x_1, \dots, x_n, y_i \in G$,

$$\begin{aligned} &F\left(x_1, \dots, x_{i-1}, x_i + y_i, x_{i+1}, \dots, x_n\right) \\ &\quad = F\left(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n\right) + F\left(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n\right), \end{aligned}$$

i.e., F is additive in each of its variables $x_i \in G$, $i = 1, \dots, n$. For the sake of brevity we use the notation $G^0 = G$ and we call constant functions from G to H 0-additive functions. Let $F\colon G^n \to H$ be an arbitrary function. By the *diagonalization (or trace)* of F we understand the function $f\colon G \to H$ arising from F by putting all the variables (from G) equal:

$$f(x) = F(x, \dots, x) \quad (x \in G)\,.$$

It can be proved by induction that for any symmetric, n-additive function $F\colon G^n \to H$ the equality

$$\Delta_{y_1,\dots,y_k} f(x) = \begin{cases} n!F(y_1, \dots, y_n) & \text{for} \quad k = n, \\ 0 & \text{for} \quad k > n, \end{cases} \tag{1.1}$$

holds, whenever $x, y_1, \dots, y_n \in G$, where $f\colon G \to H$ denotes the trace of the function F. This means that a symmetric, n-additive function is uniquely determined by its trace.

The function $f\colon G \to H$ is called a *polynomial function* of degree at most n, where n is a nonnegative integer, if

$$\Delta_{y_1,\dots,y_{n+1}} f(x) = 0$$

is satisfied for all $x, y_1, \dots, y_{n+1} \in G$.

Theorem 1.4. *The function $p\colon G \to H$ is a polynomial at degree at most n if and only if there exist symmetric, k-additive functions $F_k\colon G^k \to H$, $k = 0, 1, \dots, n$ such that*

$$p(x) = \sum_{k=0}^{n} f_k(x) \quad (x \in G)\,,$$

where f_k denotes the trace of the function F_k, $k = 0, 1, \dots, n$. Furthermore, this expression for the function p is unique in the sense that the functions F_k different from identically zero are uniquely determined.

The following theorems will play a key role during the proof of our main result.

Theorem 1.5 (Székelyhidi [12]). *Let G be an abelian group and let X be a locally convex topological linear space. If a polynomial $p\colon G \to X$ is bounded on G, then it is constant.*

Theorem 1.6 (Székelyhidi [12]). *Let G be an abelian group which is generated by any neighborhood of the zero, and let X be a topological linear space, and $p\colon G \to X$ be a polynomial function. Then the following statements hold.*

(i) *If $p\colon G \to X$ is continuous at a point, then it is continuous on G.*

(ii) *Assume that G is locally compact and X is locally convex. If $p\colon G \to X$ is bounded on a measurable set of positive measure, then it is continuous.*
(iii) *Suppose that G is locally compact and X is locally convex and locally bounded. If $p\colon G \to X$ is measurable on a measurable set of positive measure, then it is continuous.*

2. Main results

The commutative case. We begin with the following generalization of Theorem 2.2 of [4].

THEOREM 2.1. *Let $n \in \mathbb{N}, n \geq 2$, R, R' be commutative rings such that $\operatorname{char}(R') > n$, and assume that the mapping $\varphi\colon R \to R'$ is an n-Jordan homomorphism. Then φ is an n-homomorphism. Moreover, if R is unitary then $\varphi(1) = \varphi(1)^n$ and the function ψ defined by*

$$\psi(x) = \varphi^{n-2}(1)\varphi(x) \quad (x \in R)$$

is a homomorphism between R and R'.

PROOF. With the aid of the function φ, let us define the function Φ on R^n by

$$\Phi(x_1, \ldots, x_n) = \varphi(x_1 \cdots x_n) - \varphi(x_1) \cdots \varphi(x_n) \quad (x_1, \ldots, x_n \in R)\,.$$

Due to the additivity of the function φ, the function Φ is a symmetric, n-additive function. Furthermore, its trace

$$\phi(x) = \Phi(x, \ldots, x) = \varphi(x^n) - \varphi(x)^n \quad (x \in R)$$

is identically zero on R, because of our assumptions. Therefore,

$$\Delta_{y_1,\ldots,y_n}\phi(x) = 0$$

also holds for all $x, y_1, \ldots, y_n \in R$. In view of formula (1.1), this yields that

$$n!\Phi(y_1, \ldots, y_n) = 0$$

for all $y_1, \dots, y_n \in R$. Due to the suppositions of the theorem, we obtain that the function Φ is identically zero on the set R^n. From this we get, however, that

$$\varphi(x_1 \cdots x_n) = \varphi(x_1) \cdots \varphi(x_n) \quad (x_1, \dots, x_n \in R)\,,$$

that is, the function φ is an n-homomorphism.

In the second part of the proof let us assume that the ring R is unitary with the unit element 1. If $n = 2$ then there is nothing to prove. If $n > 2$, then the previous equation with the substitution $x_3 = \ldots = x_n = 1$ implies that

$$\varphi(x_1 x_2) = \varphi(x_1)\varphi(x_2)\varphi(1)^{n-2}$$

is fulfilled for all $x_1, x_2 \in R$. Multiplying both the sides with $\varphi(1)^{n-2}$, we get that

$$\varphi(1)^{n-2}\varphi(x_1 x_2) = \varphi(1)^{n-2}\varphi(x_1)\varphi(1)^{n-2}\varphi(x_2)$$

holds for all $x_1, x_2 \in R$, that is, the function ψ defined by $\psi(x) = \varphi(1)^{n-2}\varphi(x)$ is a homomorphism between the rings R and R'. $\square$

A characterization of n-Jordan homomorphisms

THEOREM 2.2. *Let $n \in \mathbb{N}, n \geq 2$ R be a ring, R' be a locally convex algebra over the field $\mathbb{F}$ of characteristic zero, $\varphi\colon R \to R'$ be an additive function and assume that the mapping*

$$R \ni x \longmapsto \varphi(x^n) - \varphi(x)^n$$

is bounded on R. Then the function φ is an n-Jordan homomorphism.

PROOF. With the help of the function φ we define the mapping Φ on R^n through

$$\Phi(x_1, \dots, x_n) = \sum_{\sigma \in \mathcal{S}_n} \varphi(x_{\sigma(1)} \cdots x_{\sigma(n)}) - \varphi(x_{\sigma(1)}) \cdots \varphi(x_{\sigma(n)}) \quad (x_1, \dots, x_n \in R)\,,$$

where $\mathcal{S}_n$ denotes the symmetric group of $\{1,\ldots,n\}$. It is obvious that the function Φ is symmetric under all permutations of its variables. Furthermore, the additivity of φ yields that Φ is an n-additive mapping. Therefore, its trace

$$\phi(x)=\Phi(x,\ldots,x)=n!\,(\varphi(x^n)-\varphi(x)^n)\quad (x\in R)$$

is a polynomial function of degree at most n. On the other hand, from the suppositions of the theorem, the function ϕ is bounded on R. Thus, by Theorem 1.5,

$$\phi(x)=\text{const.}\quad (x\in R)\,.$$

Let us observe however that

$$\phi(0)=\Phi(0,\ldots,0)=n!\,(\varphi(0^n)-\varphi(0)^n)=0,$$

since $\varphi(0)=0$. Therefore, the function ϕ is identically zero on R, that is, for the additive function φ,

$$\varphi(x^n)=\varphi(x)^n$$

holds for any $x\in R$. This yields that φ is an n-Jordan mapping. □

Consequences of Theorem 2.1

THEOREM 2.3. *Let $n\in\mathbb{N}, n\geq 2$, $\mathbb{F}$ be a field of characteristic zero, R be a commutative topological ring and R' be a commutative topological algebra over the field $\mathbb{F}$. Furthermore, let us consider the additive function $\varphi\colon R\to R'$ and suppose that for the map ϕ defined on R by*

$$\phi(x)=\varphi(x^n)-\varphi(x)^n\quad (x\in R)$$

one of the following statements hold.

(i) *the function ϕ is continuous at a point*;
(ii) *assuming that R' is locally convex, the function ϕ is bounded on a nonvoid open set of B*;
(iii) *assuming that R is locally compact, R' is locally convex, the function ϕ is bounded on a measurable set of positive measure*;
(iv) *assuming that R is locally compact and R' is locally bounded and locally convex, the function ϕ is measurable on a measurable set of positive measure.*

Then and only then the function φ is a continuous function or it is an n-homomorphism.

Proof. Let us define the function Φ on R^n by

$$\Phi(x_1, \ldots, x_n) = \varphi(x_1 \cdots x_n) - \varphi(x_1) \cdots \varphi(x_n). \quad (x_1, \ldots x_n \in R)$$

By our assumptions the function Φ is a symmetric, n-additive function, and its diagonalization

$$\phi(x) = \Phi(x, \ldots, x) = \varphi(x^n) - \varphi(x)^n \quad (x \in R)$$

fulfills one of the suppositions (i), (ii), (iii), (iv). On the other hand, the function ϕ, being the trace of a symmetric, n-additive function, is a polynomial of degree at most n. On the other hand, due to the additivity of the function φ, the polynomial ϕ is in fact a continuous monomial function.

All in all, this means that there exists a continuous monomial function $\phi\colon R \to R'$ such that

$$\varphi(x^n) - \varphi(x)^n = \phi(x) \quad (x \in R)\,.$$

In case $\phi \equiv 0$, this means that φ is an n-homomorphism. Thus, in the remaining part of the proof, we can and we do suppose that the continuous monomial function ϕ is not identically zero. Due to the additivity, the function φ is either continuous (everywhere) ot it is nowhere continuous. Suppose that φ is nowhere continuous. Then the mapping $x \longmapsto \varphi(x^n) - \varphi(x)^n$ is a nowhere continuous monomial function. This contradicts to the fact the ϕ is continuous.

All in all, the additive function φ is continuous or it is an n-homomorphism. □

Obviously, it may occur that some rings admit n-homomorphisms that are continuous, too. Thus the disjunctions appearing in our results are not necessarily exclusive.

In case $R = R' = \mathbb{R}$ the situation is rather simple because the notions of n-homomorphisms as well as n-Jordan homomorphisms coincides with the notion of homomorphisms. Furthermore, in $\mathbb{R}$ if a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a homomorphism, then it is either identically zero or $\varphi = \mathrm{id}$. Concerning this case we can state the following.

Corollary 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$ be arbitrarily fixed and assume that for the additive function $\varphi\colon \mathbb{R} \to \mathbb{R}$ the mapping defined by*

$$\mathbb{R} \ni x \longmapsto \varphi(x^n) - \varphi(x)^n$$

fulfills at least one of (i), (ii), (iii) and (iv) appearing in Theorem 2.3. Then and only then

$$\varphi(x) = \varphi(1) \cdot x$$

is satisfied for any $x \in \mathbb{R}$.

However, in $\mathbb{C}$ the situation is completely different. Since the only *continuous* endomorphisms $\varphi \colon \mathbb{C} \to \mathbb{C}$ are

$$\varphi(x) = 0 \quad \text{or} \quad \varphi(x) = x \quad \text{or} \quad \varphi(x) = \overline{x} \quad (x \in \mathbb{C}),$$

where $\overline{x}$ denotes the complex conjugate of x.

These endomorphisms are referred to *trivial endomorphisms*. Of these the identically zero function is only an endomorphism, whereas the others are automorphisms.

In view of [9, Theorem 14.5.1], there exist *nontrivial automorphisms* of $\mathbb{C}$. Such functions behave rather pathologically. We just mention for example that if $\varphi \colon \mathbb{C} \to \mathbb{C}$ is a nontrivial automorphisms then the set $\varphi(\mathbb{R})$ is dense in $\mathbb{C}$.

Especially, our main result in this case reads as follows.

Corollary 2.2. *Let* $n \in \mathbb{N}$, $n \geq 2$ *be arbitrarily fixed and assume that for the additive function* $\varphi \colon \mathbb{C} \to \mathbb{C}$ *the mapping defined by*

$$\mathbb{C} \ni x \longmapsto \varphi(x^n) - \varphi(x)^n$$

fulfills at least one of (i), (ii), (iii) and (iv) appearing in Theorem 2.3. Then and only then the function φ *is a continuous additive function or it is an automorphism of* $\mathbb{C}$.

Surjective maps to prime algebras.

Theorem 2.4. *Let* $n \in \mathbb{N}, n \geq 2$, R *be a ring,* R' *be a locally convex algebra over the field* $\mathbb{F}$ *of characteristic zero,* $\varphi \colon R \to R'$ *be a surjective additive function, and assume that the mapping*

$$R \ni x \longmapsto \varphi(x^n) - \varphi(x)^n$$

is bounded on R. *Then the function* φ *is an* n*-Jordan homomorphism.*

Furthermore, in case R' is prime, there exists $\varepsilon \in C'$ (the extended centroid of R') such that $\varepsilon^{n-1} = 1$ and a homomorphism or an anti-homomorphism $\tau\colon R \to R'C'$ such that

$$\varphi(x) = \varepsilon\tau(x) \quad (x \in R).$$

Proof. In view of Theorem 2.2, we immediately obtain that φ is an n-Jordan mapping. Now, the last part of our statement immediately follows from Theorem 1.3. □

Finally, we end our paper with the *unitary version* of the previous theorem.

Corollary 2.3. *Let $n \in \mathbb{N}, n \geq 2$, R be a ring, R' be a locally convex algebra over the field $\mathbb{F}$ of characteristic zero, $\varphi\colon R \to R'$ be a surjective additive function, and assume that the mapping*

$$R \ni x \longmapsto \varphi(x^n) - \varphi(x)^n$$

is bounded on R. Then the function φ is an n-Jordan homomorphism.

Furthermore, in case R' is prime, R is unitary and φ is surjective then

$$\varphi = \varepsilon\tau$$

where τ is either a homomorphism or an anti-homomorphism and ε is an $(n-1)$st root of unity lying in the center of R'.

References

[1] Ancochea G., *Le théorème de von Staudt en géométrie projective quaternionienne*, J. Reine Angew. Math. **184** (1942), 193–198.

[2] Badora R., *On approximate ring homomorphisms*, J. Math. Anal. Appl. **276** (2002), no. 2, 589–597.

[3] Brešar M., Martindale 3rd W.S., Miers C.R., *Maps preserving n^{th} powers*, Comm. Algebra **26** (1998), no. 1, 117–138.

[4] Eshaghi Gordji M., *n-Jordan homomorphisms*, Bull. Aust. Math. Soc. **80** (2009), no. 1, 159–164.

[5] Hejazian Sh., Mirzavaziri M., Moslehian M.S., *n-homomorphisms*, Bull. Iranian Math. Soc. **31** (2005), no. 1, 13–23.

[6] Herstein I.N., *Jordan homomorphisms*, Trans. Amer. Math. Soc. **81** (1956), 331–341.

[7] Jacobson N., Rickart C.E., *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. **69** (1950), 479–502.

[8] Kaplansky I., *Semi-automorphisms of rings*, Duke Math. J. **14** (1947), 521–525.

[9] Kuczma M., *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Second edition, Birkhäuser Verlag, Basel, 2009.

[10] Šemrl M., *Nonlinear perturbations of homomorphisms on* $C(X)$, Quart. J. Math. Oxford Ser. (2) **50** (1999), no. 197, 87–109.

[11] Šemrl P., *Almost multiplicative functions and almost linear multiplicative functionals*, Aequationes Math. **63** (2002), no. 1–2, 180–192.

[12] Székelyhidi L., *Regularity properties of polynomials on groups*, Acta Math. Hungar. **45** (1985), no. 1–2, 15–19.

[13] Żelazko W., *A characterization of multiplicative linear functionals in complex Banach algebras*, Studia Math. **30** (1968), 83–85.

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