Annales Mathematicae Silesianae **28** (2014), 59–86
Prace Naukowe Uniwersytetu Śląskiego nr 3240, Katowice

EXISTENCE OF GENERALIZED, POSITIVE AND PERIODIC SOLUTIONS FOR SOME DIFFERENTIAL EQUATIONS OF ORDER II

Jan Ligęza

Abstract. We study the existence of positive periodic solutions of the equations

$$y''(x) - P'(x)y(x) + \mu Q'(x)f(x,y(x)) = 0,$$
$$y''(x) + P'(x)y(x) = \mu Q'(x)f(x,y(x)),$$

where $\mu > 0$, P and Q are real nondecreasing functions, P' and Q' are 1-periodic distributions, f is a continuous function and 1-periodic in the first variable. The Krasnosielski fixed point theorem on cone is used.

1. Introduction

Positive solutions of various boundary value problem for ordinary differential equations have been considered by several authors (see for instance [1], [4], [15], [18], [19]). Many papers on the generalized ordinary differential equations have appeared too (for instance [5], [8], [10], [11], [14], [16], [17]). The paper deals with existence of positive periodic solutions of nonlinear differential equations of the form:

Received: 4.06.2013. Revised: 15.01.2014.
(2010) Mathematics Subject Classification: 34K05, 34K30, 47G20, 45J05, 46F10.
Key words and phrases: positive, periodic solutions, cone, distributions, Krasnosielski fixed point theorem, Green function.

$$y''(x) - P'(x)y(x) + \mu Q'(x)f(x, y(x)) = 0, \tag{1.1}$$

$$y''(x) + P'(x)y(x) = \mu Q'(x)f(x, y(x)), \tag{1.2}$$

where $\mu > 0$, P and Q are real, nondecreasing functions, P' and Q' are 1-periodic distribution. The derivative is understood in the distributional sense. The solutions of equations (1.1) and (1.2) are considered in the class of all distributions for which the first derivatives (in the distribution sense) are functions of locally of bounded variation on the interval $(-\infty, \infty)$. This class will be denoted by V^1. The class of all functions of locally of bounded variation on the interval $(-\infty, \infty)$ will be denoted by V. The product $P'y$ we mean in the following way

$$P'y = \Big(\int_{x_0}^{x} y(s)dP(s)\Big)',$$

where the integral is understood in the sense of Riemann–Stieltjes, $y \in C$ and $P \in V$ (C denotes the space of all continuous functions $y\colon \mathbb{R} \to \mathbb{R}$).

By a solution of equation (1.1) or (1.2) we mean every function $y \in V^1$, which satisfies the equation (1.1) or (1.2) in the distributional sense.

2. Notation and lemmas

We denote $I = [0, 1] \times [0, 1]$ and $I_0 = (0, 1) \times (0, 1)$.

By a delta sequence we mean a sequence of real, $C^\infty(\mathbb{R})$, nonnegative, scalar functions $\{\delta_n(x)\}$ satisfying:

(a) $\int_{-\infty}^{\infty} \delta_n(x)dx = 1$,

(b) $\delta_n(x) = 0$ for $|x| \geq \alpha_n$, where $\{\alpha_n\}$ is a sequence of positive numbers which $\alpha_n \to 0$,

(c) $\delta_n(x) = \delta_n(-x)$ for $x \in \mathbb{R}$ (see [3], p. 75).

We say that a distribution g in $\mathbb{R}$ is 1-periodic, if

$$g(x + 1) = g(x) \quad \text{(see [17], p. 229).}$$

Now we assume two hypotheses:
Hypothesis H_1**.** The functions P and Q have the following properties: $P \in V, Q \in V, P' \geq 0, Q' \geq 0, P'$ and Q' are 1-periodic distributions.
Hypothesis H_2**.** Assumptions H_1 are fulfilled, $P' \neq 0$ and $Q' \neq 0$.

Lemma 2.1. *If hypothesis H_1 is satisfied and $\{\delta_n(x)\}$ is a delta sequence, then*

$$\lim_{n\to\infty}(P * \delta_n)(x_0) = \frac{P(x_0^+) + P(x_0^-)}{2} = P^*(x_0),$$

where $x_0 \in (-\infty, \infty)$, $P(x_0^+)$ $(P(x_0^-))$ denotes the left-hand (the right-hand) side limits of P at the point x_0 (the asterisk $*$ denotes the convolution of functions P and δ_n).

Proof. Let

$$g(x) = P(x_0^+)H(x - x_0) + P(x_0^-)H(x_0 - x)$$

and let

$$P(x) = (P(x) - g(x)) + g(x),$$

where

$$H(x - x_0) = \begin{cases} 1, & \text{if } x \geq x_0, \\ 0, & \text{if } x < x_0. \end{cases}$$

Then

$$P_n(x_0) = ((P - g) * \delta_n)(x_0) + g_n(x_0),$$

where

$$g_n(x_0) = (g * \delta_n)(x_0).$$

Evidently

$$\lim_{n\to\infty}((P - g) * \delta_n)(x_0) = 0$$

and

$$\lim_{n\to\infty} P_n(x_0) = \lim_{n\to\infty} g_n(x_0) = \lim_{n\to\infty} \int_{-\alpha_n}^{0} P(x_0^+)H(-t)\delta_n(t)dt$$

$$+ \int_{0}^{\alpha_n} P(x_0^-)H(t)\delta_n(t)dt = \frac{P(x_0^+)}{2} + \frac{P(x_0^-)}{2} = P^*(x_0). \quad \square$$

REMARK 2.2. Now we define the value of the distribution at the point in the Łojasiewicz sense (see [13]). If G is a distribution defined on the interval $(x_0 - \alpha, x_0 + \alpha) \subset \mathbb{R}$ and if the limit

$$\lim_{\varepsilon\to 0} G\left[\frac{1}{|\varepsilon|}\varphi\left(\frac{x - x_0}{\varepsilon}\right)\right]$$

exists, for each $\varphi \in \mathcal{D}$, it is a constant distribution C ($\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support). The constant distribution C is said to be the value of the distribution G at the point x_0 and is denoted by $G(x_0)$ (see [13]). So

$$G(x_0)[\varphi] = \lim_{\varepsilon\to 0} G\left[\frac{1}{|\varepsilon|}\varphi\left(\frac{x - x_0}{\varepsilon}\right)\right] = C \int_{-\infty}^{\infty} \varphi(x)dx.$$

LEMMA 2.3. *If g is an 1-periodic distribution and if $G' = g$, then there exists the value of the distribution $G(x+1) - G(x)$ at the point zero (see [3], p. 50).*

Now we introduce the definite integral of a distribution g (defined on the interval $(a - \varepsilon, b + \varepsilon), \varepsilon > 0$). Namely, we put

$$\int_{a}^{b} g(x)dx = (G(x+b) - G(x+a))(0),$$

provided that the value of the distribution $G(x+b) - G(x+a)$ at the point 0 exists and $G' = g$ (see [3], p. 47, [13]).

Lemma 2.4. *If $P \in V$ and P' is 1-periodic distribution, then*

$$\begin{aligned}\int_0^1 P'(x)dx &= P(1^+) - P(0^+) = P(1^-) - P(0^-)\\ &= \frac{P(1^+) + P(1^-)}{2} - \frac{P(0^+) + P(0^-)}{2}\\ &= P^*(1) - P^*(0) = P_n(x+1) - P_n(x) = P_n(1) - P_n(0),\end{aligned}$$

where

$$P_n = P * \delta_n.$$

Proof. Since

$$(P(x+1) - P(x))' = P'(x+1) - P'(x) = 0,$$

therefore

$$P(x+1) - P(x) \equiv C \quad (C \text{ denotes a constant distribution}).$$

Hence

$$\begin{aligned}P_n(x+1) - P_n(x) &= P_n(1) - P_n(0) = C\\ &= P(1^+) - P(0^+) = P(1^-) - P(0^-)\end{aligned}$$

and (by Lemma 2.1)

$$\lim_{n\to\infty} P_n(1) - P_n(0) = P^*(1) - P^*(0) = C = \int_0^1 P'(x)dx. \qquad \square$$

Lemma 2.5. *Let hypothesis H_2 be satisfied. Then the equation*

$$y''(x) - P'(x)y(x) = 0 \tag{2.1}$$

has only the trivial 1-periodic solution of the class V^1.

Proof. If $y \in V^1$ and $y \not\equiv 0$ is an 1-periodic solution of equation (2.1), then

$$y''(x)y(x) - P'(x)y^2(x) = 0.$$

Hence

$$\int_0^1 y(x)y''(x)dx - \int_0^1 P'(x)y^2(x)dx = 0.$$

On the other hand

$$\begin{aligned}
&\int_0^1 y(x)y''(x)dx - \int_0^1 P'(x)y^2(x)dx \\
&\qquad = \int_0^1 (y(x)y'(x))' - {y'}^2(x)dx - \int_0^1 P'(x)y^2(x)dx \\
&\qquad = y^*(1){y'}^*(1) - y^*(0){y'}^*(0) - \int_0^1 {y'}^2(x)dx - \int_0^1 P'(x)y^2(x)dx \\
&\qquad = -\int_0^1 {y'}^2(x)dx - \int_0^1 P'(x)y^2(x)dx = 0.
\end{aligned}$$

The last equality gives

$$y'(x) = 0$$

and

$$y(x) = C,$$

where C is a constant. If $C \neq 0$, then we obtain contradiction (by hypothesis H_2). □

Now we give three hypothesis.

Hypothesis H_3**.** Assumptions H_2 are satisfied and

$$0 < \int_0^1 P'(x)dx < 16.$$

Hypothesis H_4**.** Assumptions H_2 are fulfilled and

$$0 < \int_0^1 P'(x)dx < 4.$$

Hypothesis H_5**.** 1^o The function $f\colon \mathbb{R}^2 \to \mathbb{R}_0^+$ is continuous ($\mathbb{R}_0^+ = [0, \infty)$).
2^o $f(x+1, v) = f(x, v)$ for all $(x, v) \in \mathbb{R}^2$.

LEMMA 2.6. *Let* $P \in V, P_n = P * \delta_n$ *and*

$$\int_a^b |P_n'(x)|dx < \frac{4}{b-a}.$$

Then the problem

$$y''(x) + P_n'(x)y(x) = 0, \quad y(a) = 0, y(b) = 0$$

has only the trivial solution (see [7], p. 408, Corollary 5.1).

LEMMA 2.7. *Let hypothesis* H_3 *be satisfied. Then the equation*

$$y''(x) + P'(x)y(x) = 0$$

has only the trivial, 1*-periodic solution of the class* V^1 *(see [11]).*

LEMMA 2.8. *Let* $a, x_0, x_1 \in \mathbb{R}$. *We assume that* $P \in V$ *and*

$$P_n(x) = (P * \delta_n)(x).$$

Then

(a) *the problem*

$$y''(x) + P'(x)y(x) = 0, \quad y(a) = x_0, y'^*(a) = x_1$$

has exactly one solution y *of the class* V^1 *(see [10]),*

(b) $y = \lim_{n\to\infty} y_n$ *(almost uniformly)*

$$y'^{*}(a) = \lim_{n\to\infty} {y'}_n(a),$$

where y_n is the solution of the problem

$$y''(x) + {P'}_n(x)y(x) = 0, \quad y_n(a) = x_0, {y'}_n(a) = x_1 \quad \textit{(see [10])},$$

(c) *the sequences $\{y_n(x)\}$ and $\{{y'}_n(x)\}$ are locally equibounded on $\mathbb{R}$,*

(d) $y(x) = y_0 + x_1(x - x_0) - \int_{x_0}^{x} (x - s)y(s)dP^*(s)$ *(see [2], p. 341–342, Theorem 11.2.1),*

(e) $\left(\int_{x_0}^{x} y(s)dP^*(s)\right)' = \left(\int_{x_0}^{x} y(s)d\,\widetilde{P}(s)\right)'$,

where $\widetilde{P} \in V, \widetilde{P}(s) = \widetilde{P}^*(s)$ for every point of continuity of functions $\widetilde{P}$ and $\widetilde{P}^*$ and the derivative is understood in the distributional sense. (The last equality follows from [16], p. 38, Lemma 4.23.)

Lemma 2.9. *Suppose that all assumptions of Lemma 2.5 are fulfilled and let ${P'}_n(x) = (P * \delta_n)'(x)$. Then*

(i) *the problem*

$$y''(x) - {P'}_n(x)y(x) = 0, \quad y(0) = y(1), y'(0) = y'(1) \qquad (2.3)_n$$

has only the trivial 1-periodic solution for $n \in \mathbb{N}$.

(ii) *the Green function $G_{1n}(x,s)$ of problem (2.3)$_n$ is negative for all $(x,s) \in I$ and $n \in \mathbb{N}$,*

(iii) *there exist constants $\overline{\gamma}_1$ and $\overline{M}_1$ such that*

$$0 < \overline{\gamma}_1 \le |G_{1n}(x,s)| \le \overline{M}_1 < \infty$$

for $n \in \mathbb{N}$ and $(x,s) \in I$,

(iv) *there exist constants d_1 and M_1 such that*

$$d_1|G_{1n}(x,s)| \ge |G_{1n}(s,s)|$$

for $n \in \mathbb{N}$ and $(x,s) \in I$, $\left(d_1 \ge \frac{\overline{M}_1}{\overline{\gamma}_1}\right)$ and

$$|G_{1n}(s,s)| \ge M_1|G_{1n}(x,s)|$$

for $n \in \mathbb{N}$ and $(x,s) \in I$, $\left(M_1 \in \left(0, \frac{\overline{\gamma}_1}{\overline{M}_1}\right)\right)$.

PROOF. The proof of property (i) follows from Lemma 2.5. Now we will examine property (ii). Let

$$(2.4)_n \qquad G_{1n}(x,s) = \begin{cases} a_{1n}(s)\varphi_{1n}(x) + a_{2n}(s)\psi_{1n}(x), & \text{if } 0 \le x \le s \le 1, \\ b_{1n}(s)\varphi_{1n}(x) + b_{2n}(s)\psi_{1n}(x), & \text{if } 0 \le s \le x \le 1, \end{cases}$$

where φ_{1n}, and ψ_{1n} are solutions of the problems

$$(2.5)_n \qquad {\varphi''}_{1n}(x) = {P'}_n(x)\varphi_{1n}(x), \quad \varphi_{1n}(0) = 1, \quad {\varphi'}_{1n}(0) = 0,$$

$$(2.6)_n \qquad {\psi''}_{1n}(x) = {P'}_n(x)\psi_{1n}(x), \quad \psi_{1n}(0) = 0, \quad {\psi'}_{1n}(0) = 1,$$

and $a_{1n}, a_{2n}, b_{1n}, b_{2n}$ satisfy the following system of equations

$$(2.7)_n \begin{cases} a_{1n}(s)\varphi_{1n}(s) - b_{1n}(s)\varphi_{1n}(s) + a_{2n}(s)\psi_{1n}(s) - b_{2n}(s)\psi_{1n}(s) = 0, \\ -a_{1n}(s){\varphi'}_{1n}(s) + b_{1n}(s){\varphi'}_{1n}(s) - a_{2n}(s){\psi'}_{1n}(s) + b_{2n}(s){\psi'}_{1n}(s) = 1, \\ a_{1n}(s) - b_{1n}(s)\varphi_{1n}(1) - b_{2n}(s)\psi_{1n}(1) = 0, \\ -b_{1n}(s){\varphi'}_{1n}(1) + a_{2n}(s) - b_{2n}(s){\psi'}_{1n}(1) = 0. \end{cases}$$

Let

$$(2.8)_n \qquad W^o_{1n} = \begin{vmatrix} \varphi_{1n}(0) - \varphi_{1n}(1) & \psi_{1n}(0) - \psi_{1n}(1) \\ {\varphi'}_{1n}(0) - {\varphi'}_{1n}(1) & {\psi'}_{1n}(0) - {\psi'}_{1n}(1) \end{vmatrix}$$

and let

$$(2.9)_n \qquad W_{1n} = \begin{vmatrix} \varphi_{1n}(s) & -\varphi_{1n}(s) & \psi_{1n}(s) & -\psi_{1n}(s) \\ -{\varphi'}_{1n}(s) & {\varphi'}_{1n}(s) & -{\psi'}_{1n}(s) & {\psi'}_{1n}(s) \\ 1 & -\varphi_{1n}(1) & 0 & -\psi_{1n}(1) \\ 0 & -{\varphi'}_{1n}(1) & 1 & -{\psi'}_{1n}(1) \end{vmatrix}.$$

Let us assume that

$$y_n(x) = c_{1n}\varphi_{1n}(x) + \psi_{1n}(x)$$

is a solution of equation $(2.3)_n$. Then, by (i) we have

$$(2.10)_n \qquad W^o_{1n} = 2 - \varphi_{1n}(1) - {\psi'}_{1n}(1) \neq 0 \qquad \text{for } n \in \mathbb{N}$$

and

$$(2.11)_n \qquad W_{1n} = W^o_{1n} \neq 0.$$

The relations $(2.7)_n$–$(2.11)_n$ guarantee the existence of the Green functions $G_{1n}(x,s)$ of problem $(2.3)_n$. It is not difficult to prove that $G_{1n}(x,s) < 0$ for $n \in \mathbb{N}$ and $(x,s) \in I$ (see [18]).

We now show (iii). First we prove that

$$\inf_{n\in\mathbb{N}} |W_{1n}| = m > 0. \tag{2.12}$$

If $m = 0$ then there exists a subsequence $\{W_{1n_\nu}\}$ such that

$$\lim_{\nu\to\infty} W_{1n_\nu} = 0.$$

Without loss of a generality we can assume that

$$\lim_{n\to\infty} W_{1n} = 0.$$

From Helly's theorem (see [12], p. 29, Theorem 1.6.10) it follows that there exist subsequences $\{\varphi_{1nk}^{(i)}\}$ and $\{\psi_{nk}^{(i)}\}$ of sequences $\{\varphi_{1n}^{(i)}\}$ and $\{\psi_{1n}^{(i)}\}$ convergent to functions $\varphi_1^{(i)} \in V$ and $\psi_1^{(i)} \in V$ for $i = 0, 1$; respectively. Besides

$$\lim_{k\to\infty} \varphi_{1nk}(x) = \varphi_1(x), \quad \lim_{k\to\infty} \psi_{nk}(x) = \psi_1(x)$$

almost uniformly on $(-\infty, \infty)$. Thus

$$\lim_{k\to\infty} W_{1nk}^o = \lim_{k\to\infty} (2 - \varphi_{1nk}(1) - {\psi'}_{1nk}(1)) = 0 = W_1^o \tag{2.13}$$

and

$$\varphi''_1(x) = P'(x)\varphi_1(x), \quad \varphi_1(0) = 1, \quad {\varphi'}^*_1(0) = 0, \tag{2.14}$$

$$\psi''_1(x) = P'(x)\psi_1(x), \quad \psi_1(0) = 0, \quad {\psi'}^*_1(0) = 1 \quad \text{(see [10])}. \tag{2.15}$$

On the other hand the function

$$y = c_1\varphi_1 + c_2\psi_1 \qquad (c_1, c_2 \text{ denote constants})$$

is also a solution of the equation

$$y''(x) = P'(x)y(x) \tag{2.16}$$

and

$$y(0) = c_1 = y(1) = c_1\varphi_1(1) + c_2\psi_1(1) \tag{2.17}$$

and

$$y'^{*}(0) = c_2 = y'^{*}(1) = c_1 {\varphi'}^{*}_1(1) + c_2 {\psi'}^{*}_1(1). \tag{2.18}$$

By (2.16)–(2.18) we have

$$\begin{vmatrix} 1-\varphi_1(1) & -\psi_1(1) \\ -{\varphi'}^{*}_1(1) & 1-{\psi'}^{*}_1(1) \end{vmatrix} = W_1^o = 0.$$

Hence, there exists a non trivial, 1-periodic solution of equation (2.16) (of the class V^1), i.e. (2.12) holds.

Existence of a constant $\overline{M}_1$ follows from Lemma 2.8 and from $(2.5)_n$–(2.12). We will show that there exists a constant $\overline{\gamma}_1$ such that

$$\inf_{n\in\mathbb{N}} \inf_{(x,s)\in I} |G_{1n}(x,s)| = \overline{\gamma}_1 > 0. \tag{2.19}$$

If $\overline{\gamma}_1 = 0$ then there exists a subsequence $\{G_{1n\nu}(x_\nu, s_\nu)\}$ of sequence $\{G_{1n}(x,s)\}$ such that

$$\lim_{n\to\infty} \inf_{(x,s)\in I} G_{1n\nu}(x,s) = \lim_{\mu\to\infty} G_{1n\nu}(x_\nu, s_\nu) = 0,$$

where $(x_\nu, s_\nu) \in I$.

Without loss of a generality we can assume that

$$\lim_{n\to\infty} G_{1n}(x_n, s_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} G_{1n}(x,s) = G_1(x,s) \tag{2.20}$$

uniformly for $(x,s) \in I$ and

$$\lim_{n\to\infty} \varphi_{1n}(x) = \varphi_1(x), \qquad \lim_{n\to\infty} {\varphi'}^{*}_{1n}(x) = {\varphi'}^{*}_1(x), \tag{2.21}$$

$$\lim_{n\to\infty} \psi_{1n}(x) = \psi_1(x), \qquad \lim_{n\to\infty} {\psi'}^{*}_{1n}(x) = {\psi'}^{*}_1(x), \tag{2.22}$$

$$\lim_{n\to\infty} a_{1nk}(s) = a_1(s), \qquad \lim_{n\to\infty} b_{1n}(s) = b_1(s), \tag{2.23}$$

uniformly on $[0,1]$,

$$\lim_{n\to\infty} a_{2n}(s) = a_2(s), \qquad \lim_{n\to\infty} b_{2n}(s) = b_2(s) \tag{2.24}$$

uniformly on $[0,1]$. Then there exists a point $(x_0, s_0) \in I$ such that

$$G_1(x_0, s_0) = 0. \tag{2.25}$$

Without loss of a generality we can assume that $(x_0, s_0) \in I_0$. Let

$$\widetilde{y_1}(x) = \begin{cases} G_1(x, s_0), & \text{if } x \in [s_0, 1] \\ G_1(x-1, s_0), & \text{if } x \in [1, s_0+1] \quad \text{(see [18])}. \end{cases}$$

Then $\widetilde{y_1}(x_0) = 0$ and $\widetilde{y_1}(x)$ is a solution of the equation

$$\widetilde{y_1}''(x) - P'(x)\,\widetilde{y_1}(x) = 0 \quad \text{for } x \in (s_0, s_0+1)$$

i.e.

$$\widetilde{y_1}''(x) - \left(P(x)\,\widetilde{y_1}(x)\right)' + P(x)\widetilde{y_1}'(x) = 0.$$

Let

$$z_1 = \widetilde{y_1}' - P\,\widetilde{y_1}\,.$$

Then we get the following system of equations

$$\begin{cases} \widetilde{y_1}'(x) = P(x)\widetilde{y_1}(x) + z_1(x), \\ z_1{}'(x) = -P^2(x)\widetilde{y_1}(x) - P(x)z_1(x) \quad \text{(see [14])}. \end{cases}$$

If $\widetilde{y_1}(x_0) = 0$ then $\widetilde{y_1}'(x_0) = z_1(x_0)$. So $\widetilde{y_1}'$ is a continuous function at the point x_0. The inequality $G_1(x, s) \le 0$ (for all $(x, s) \in I$) implies $\widetilde{y_1}'(x_0) = 0$. By the uniqueness of the solution of the Cauchy problem (Lemma 2.8) we get

$$\widetilde{y_1}(x) = 0 \quad \text{for } x \in (s_0, s_0+1).$$

Let $\overline{y}_{1n}(x)$ be a solution of the problem

$$\begin{cases} \overline{y}_{1n}''(x) - P_n{}'(x)\overline{y}_{1n}(x) = 0, \\ \overline{y}_{1n}(x_0) = \widetilde{y_{1n}}(x_0), \quad \overline{y}_{1n}'(x_0) = \widetilde{y_{1n}}'(x_0), \end{cases}$$

where

$$\widetilde{y_{1n}}(x) = \begin{cases} G_{1n}(x, s_0), & \text{if } x \in [s_0, 1], \\ G_{1n}(x-1, s_0), & \text{if } x \in [1, s_0+1]. \end{cases}$$

Let

$$z_{1n}(x) = \overline{y}'_{1n}(x) - P_n(x)\overline{y}_{1n}(x).$$

Then

$$\lim_{n\to\infty} z_{1n}(x_0) = \lim_{n\to\infty} \overline{y}_{1n}(x_0) = 0.$$

Consequently,

$$\lim_{n\to\infty} z_{1n}(x) = 0 = \lim_{n\to\infty} \overline{y}_{1n}(x) = \lim_{n\to\infty} \overline{y'}_{1n}(x) \quad \text{for } x \in (-\infty, \infty).$$

This gives

$$\lim_{n\to\infty} (\overline{y}_{1n}{}'(s_0) - \overline{y}_{1n}{}'(s_0+1)) = 0 = \lim_{n\to\infty} [b_{1n}(s_0)\varphi_{1n}{}'(s_0)$$
$$+ b_{2n}(s_0)\psi_{1n}{}'(s_0) - a_{1n}(s_0)\varphi_{1n}{}'(s_0) - a_{2n}(s_0)\psi_{1n}{}'(s_0)] = 1,$$

which is impossible. Thus (iii) holds. The property (iv) is evident. □

Lemma 2.10. *If P' satisfies H_4 and $P_n{}'(x) = (P * \delta_n)'(x)$, then*

(j) *the problem*

$$y''(x) + P_n{}'(x)y(x) = 0, \quad y(0) = y(1), \quad y'(0) = y'(1) \tag{2.26_n}$$

has only the trivial 1-periodic solution for $n \in \mathbb{N}$;

(jj) *the Green function $G_{2n}(x,s)$ of problem $(2.26)_n$ is positive for all $(x,s) \in I$ and $n \in \mathbb{N}$;*

(jjj) *there exist constants $\overline{\gamma}_2$ and $\overline{M}_2$ such that*

$$0 < \overline{\gamma}_2 \le G_{2n}(x,s) \le \overline{M}_2 < \infty$$

for $n \in \mathbb{N}$ and $(x,s) \in I$;

(jv) *there exist constants d_2 and M_2 such that*

$$d_2 G_{2n}(x,s) \ge G_{2n}(s,s) \quad \textit{for } n \in \mathbb{N}$$

and $(x,s) \in I$, $\left(d_2 \ge \frac{\overline{M}_2}{\overline{\gamma}_2}\right)$ and

$$G_{2n}(s,s) \ge M_2 G_{2n}(x,s) \quad \textit{for } n \in \mathbb{N},$$

$(x,s) \in I$, $\left(M_2 \in \left(0, \frac{\overline{M}_2}{\overline{\gamma}_2}\right)\right)$.

Proof. The proof of property (j) follows from Lemma 2.7 and [9]. The proof of property (jj) is similar to that of property (ii). Let

$$(2.27)_n \quad G_{2n}(x,s)=\begin{cases}\overline{a}_{1n}(s)\varphi_{2n}(x)+\overline{a}_{2n}(s)\psi_{2n}(x), & \text{if } 0\le x\le s\le 1,\\ \overline{b}_{1n}(s)\varphi_{2n}(x)+\overline{b}_{2n}(s)\psi_{2n}(x), & \text{if } 0\le s\le x\le 1,\end{cases}$$

where φ_{2n} and ψ_{2n} are solutions of the problems

$$(2.28)_n \qquad {\varphi_{2n}}''(x)+{P_n}'(x)\varphi_{2n}(x)=0,\quad \varphi_{2n}(0)=1,\quad {\varphi_{2n}}'(0)=0,$$

$$(2.29)_n \qquad {\psi_{2n}}''(x)+{P_n}'(x)\psi_{2n}(x)=0,\quad \psi_{2n}(0)=0,\quad {\psi_{2n}}'(0)=1,$$

and $\overline{a}_{1n},\overline{a}_{2n},\overline{b}_{1n},\overline{b}_{2n}$ satisfy the system of equations

$$(2.30)_n \begin{cases}\overline{a}_{1n}(s)\varphi_{2n}(s)-\overline{b}_{1n}(s)\varphi_{2n}(s)+\overline{a}_{2n}(s)\psi_{2n}(s)-\overline{b}_{2n}(s)\psi_{2n}(s)=0,\\ -\overline{a}_{1n}(s){\varphi_{2n}}'(s)+\overline{b}_{1n}(s){\varphi_{2n}}'(s)-\overline{a}_{2n}(s){\psi_{2n}}'(s)+\overline{b}_{2n}(s){\psi_{2n}}'(s)=1,\\ -\overline{a}_{1n}(s)-\overline{b}_{1n}(s)\varphi_{2n}(1)-\overline{b}_{2n}(s)\psi_{2n}(1)=0,\\ -\overline{b}_{1n}(s){\varphi_{2n}}'(1)+\overline{a}_{2n}(s)-\overline{b}_{2n}(s){\psi_{2n}}'(1)=0.\end{cases}$$

Let us put

$$(2.31)_n \qquad W_{2n}^0=\begin{vmatrix}\varphi_{2n}(0)-\varphi_{2n}(1) & \psi_{2n}(0)-\psi_{2n}(1)\\ {\varphi'}_{2n}(0)-{\varphi'}_{2n}(1) & {\psi'}_{2n}(0)-{\psi'}_{2n}(1)\end{vmatrix}$$

and

$$(2.32)_n \qquad W_{2n}=\begin{vmatrix}\varphi_{2n}(s) & -\varphi_{2n}(s) & \psi_{2n}(s) & -\psi_{2n}(s)\\ -\varphi_{2n}(s) & {\varphi_{2n}}'(s) & -{\psi_{2n}}'(s) & {\psi_{2n}}'(s)\\ 1 & {\varphi_{2n}}'(1) & 0 & \psi_{2n}(1)\\ 0 & -{\varphi_{2n}}'(1) & 1 & {\psi_{2n}}'(1)\end{vmatrix}.$$

Then

$$(2.33)_n \qquad W_{2n}^{\circ}=2-\varphi_{2n}(1)-{\psi_{2n}}'(1)=W_{2n}\neq 0$$

for $n\in\mathbb{N}$.

The relations $(2.30)_n$–$(2.33)_n$ imply the existence of the Green function $G_{2n}(x,s)$ of problem $(2.26)_n$ for $n\in\mathbb{N}$. It is not difficult to prove that $G_{2n}(x,s)>0$ for $n\in\mathbb{N}$ and $(x,s)\in I$ (see [18]). The proof of (jjj) is similar to that of (iii). The property (jv) is evident. □

3. Positive periodic solution

In this section we present results on the existence of positive, 1-periodic solutions of equations (1.1) and (1.2). Existence in this paper will be established using Krasnosielski fixed point theorem in a cone which we state here for the convenience of the reader. First, we shall give definition of a cone (see [6], p. 1–2).

A nonempty subset K of a real Banach space E is called a cone if K is closed, convex and

1^o $\alpha x \in K$ for all $x \in K$ and $\alpha \geq 0$,
2^o $x, -x \in K$ implies $x = 0$.

Theorem 3.1 ([6], p. 94, Theorem 2.3.4). *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded and open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and let $A\colon K \cap (\overline{\Omega}_2 | \Omega_1) \to K$ be continuous and completely continuous. In addition suppose either*

$$\|Au\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \geq \|u\|$$

for $u \in K \cap \partial\Omega_2$ or

$$\|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \leq \|u\|$$

for $u \in K \cap \partial\Omega_2$ hold.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3.2. *Let hypotheses H_2 and H_5 be satisfied. Suppose that there exists a continuous nondecreasing function*

$$\psi\colon [0, \infty) \to [0, \infty) \quad \text{such that } \psi(u) > 0 \quad \text{for } u > 0$$

and

$$|f(x, v)| \leq \psi(v) \quad \text{for } (x, v) \in (-\infty, \infty) \times [0, \infty) \tag{3.1}$$

and there exists $r > 0$ such that

$$r \geq \psi(r) \cdot \mu m_1, \tag{3.2}$$

where

$$m_1 \geq \sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} \int_0^1 {Q_n}'(s)|G_{1n}(x,s)|ds, \tag{3.3}$$

$${Q_n}'(x) = (Q * \delta_n)'(x) \quad \textit{and} \quad G_{1n}(x,s)$$

is the Green function defined by $(2.4)_n$. *Assume, additionally that*

$$f(x,v) \geq \tau(x)g(v) \quad \textit{for } x \in \mathbb{R} \quad \textit{and} \quad v \in \mathbb{R}_0^+, \tag{3.4}$$

where $\tau\colon (-\infty,\infty) \to [0,\infty)$ *is continuous,* 1*-periodic and* $g\colon [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing and*

$$g(u) > 0 \quad \textit{for } u > 0.$$

Suppose that there exists $R > 0$ *such that* $R > r$ *and*

$$R \leq \mu \int_0^1 \tau(s){Q_n}'(s) \left| G_{1n}\left(\frac{1}{2}, s\right) \right| g\left(\frac{M_1 R}{d_1}\right) ds \tag{3.5}$$

for $n \in \mathbb{N}$, *where* d_1 *and* M_1 *are defined by relation* (iv).

Then (1.1) *has a positive,* 1*-periodic solution of the class* V^1.

Proof. To show (1.1) has a positive 1-periodic solution we will look at

$$y(x) = -\mu \int_0^1 G_{1n}(x,s){Q_n}'(s)f(s,y(s))ds. \tag{3.6$_n$}$$

We will show that there exists a solution y_n to $(3.6)_n$ for $n \in \mathbb{N}$ with

$$y_n(x) \geq \frac{M_1 R}{d_1} \quad \text{for } x \in [0,1].$$

Let $E = (P_1(\mathbb{R}), \|\cdot\|)$, where $P_1(\mathbb{R})$ denotes the space of all continuous, real, 1-periodic functions y on $\mathbb{R}$ with the norm

$$\|y\| = \max_{x\in[0,1]} |y(x)|.$$

Let

$$K_1 = \{u \in P_1(\mathbb{R}) : \min_{x\in[0,1]} d_1 u(x) \geq M_1 |u\|\},$$

where d_1 and M_1 are defined by (iv). Obviously K_1 is a cone on E. Let

$$\Omega_1 = \{u \in P_1(\mathbb{R}) : \|u\| < r\} \tag{3.7}$$

and

$$\Omega_2 = \{u \in P_1(\mathbb{R}) : \|u\| < R\}. \tag{3.8}$$

Now let $A_{1n}\colon K_1 \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P_1(\mathbb{R})$ be defined by $(A_{1n})(\varphi) = y_{n\varphi}$, where $\varphi \in P_1(\mathbb{R})$ and $y_{n\varphi}$ is the unique 1-periodic solution of the equation

$$y''(x) - {P_n}'(x) y(x) = -\mu {Q_n}'(x) f(x, \varphi(x)), \tag{3.9$_n$}$$

where

$${P_n}'(x) = (P * \delta_n)'(x), \quad {Q_n}'(x) = (Q * \delta_n)'(x).$$

First we show $A_{1n}\colon K_1 \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K_1$ for $n \in \mathbb{N}$. If $\varphi \in K_1 \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $x \in [0,1]$, then we have

$$(A_{1n}(\varphi)(x) = -\mu \int_0^1 G_{1n}(x,s) {Q_n}'(s) f(s, \varphi(s)) ds. \tag{3.10$_n$}$$

We have

$$\begin{aligned} d_1(A_{1n})(\varphi)(x) &\geq \mu d_1 \int_0^1 -G_{1n}(x,s) {Q_n}'(s) f(s,\varphi(s)) ds \\ &\geq \mu d_1 \int_0^x |G_{1n}(x,s)| {Q_n}'(s) f(s,\varphi(s)) ds \\ &\quad + \mu d_1 \int_x^1 |G_{1n}(x,s)| {Q_n}'(s) f(s,\varphi(s)) ds. \end{aligned}$$

The property (iv) implies

$$d_1(A_{1n})(\varphi)(x) \geq \mu \int_0^1 |G_{1n}(s,s)| Q_n{}'(s) f(s,\varphi(s))ds$$
$$\geq \mu M_1 \int_0^1 |G(\overline{x},s)| Q_n{}'(s) f(s,\varphi(s))ds \geq M_1 \|A_{1n}\varphi\|,$$

where $\overline{x} \in [0,1]$. Hence

$$d_1(A_{1n}\varphi)(x) \geq M_1 \|A_{1n}(\varphi)\|. \tag{3.11}$$

Consequently $A_{1n}\varphi \in K_1$ for $n \in \mathbb{N}$. So

$$A_{1n} \colon K_1 \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K_1$$

for $n \in \mathbb{N}$.

We now show

$$\|A_{1n}(\varphi)\| \leq \|\varphi\| \quad \text{for } \varphi \in K_1 \cap \partial\Omega_1 \tag{3.12}$$

and $n \in \mathbb{N}$. To see this let $\varphi \in K_1 \cap \partial\Omega_1$. Then $\|\varphi\| = r$ and

$$\varphi(x) \geq \frac{M_1 r}{d_1} \quad \text{for } x \in \mathbb{R}.$$

From (3.2)–(3.3) we have

$$(A_{1n}\varphi)(x) \leq \mu\psi(r)m_1 \leq r \leq \|\varphi\|.$$

So (3.12) holds.

Next we show

$$\|A_{1n}\varphi\| \geq \|\varphi\| \quad \text{for } \varphi \in K_1 \cap \partial\Omega_2 \tag{3.13}$$

and $n \in \mathbb{N}$. To see it let $\varphi \in K_1 \cap \partial\Omega_2$. Then $\|\varphi\| = R$ and

$$d_1\varphi(x) \geq RM_1 \quad \text{for } x \in \mathbb{R}.$$

The relations (3.4)–(3.5) yield

$$\|A_{1n}(\varphi)\| \geq A_{1n}(\varphi)\left(\frac{1}{2}\right) \geq \mu\int_0^1 \left|G_{1n}\left(\frac{1}{2},s\right)\right| {Q_n}'(s)f(s,\varphi(s))ds$$
$$\geq \mu\int_0^1 \tau(s)G_{1n}\left(\frac{1}{2},s\right){Q_n}'(s)g\left(\frac{RM_1}{d_1}\right)ds \geq R$$

for $n \in \mathbb{N}$. Hence we have (3.13).

Next we show A_{1n} is continuous and completely continuous. The continuity of A_{1n} follows from the continuity of G_{1n}, Q_n' and f. Let $\Omega \subset P_!(\mathbb{R})$ be bounded i.e. $\|u\| \leq R_1$ for each $u \in \Omega$. Then if $\varphi \in \Omega$ we have

$$(A_{1n}\varphi)'(x) = -\mu\int_0^x [b_{1n}(s){\varphi'}_{1n}(x) + b_{2n}(s){\psi'}_{1n}(x)]{Q'}_n(s)f(s,\varphi(s))]ds$$
$$-\mu\int_x^1 [a_{1n}(s){\varphi'}_{1n}(x) + a_{2n}(s){\psi'}_{1n}(x)]{Q'}_n(x)f(s,\varphi(s))]ds,$$

so (by Lemmas 2.8–2.9)

$$|A_{1n}\varphi)'(x)| \leq \mu\psi(R_1)K_0(Q^*(1) - Q^*(0)) < \infty, \tag{3.14}$$

where $n \in \mathbb{N}$ and

$$K_0 = \sup_{n\in\mathbb{N}}\sup_{x\in[0,1]}[|\varphi_{1n}'(x)| + |\psi_{1n}'(x)|]\cdot$$
$$\sup_{n\in\mathbb{N}}\sup_{s\in[0,1]}[|a_{1n}(s)| + |a_{2n}(s)| + |b_{1n}(s)| + |b_{2n}(s)|] < \infty.$$

The boundedness of $A_{1n}(\Omega)$ is immediate from $(3.10)_n$ and Lemmas 2.8–2.9, whereas $A_{1n}(\Omega)$ is equicontinuous on $[0,1]$, because of (3.14). Consequently the Arzela theorem implies A_{1n} is completely continuous. This together with Theorem 3.1 implies A_{1n} has a fixed point $y_n \in K_1 \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e.

$$r \leq \|y_n\| \leq R \quad \text{and} \quad y_n(x) \geq \frac{M_1 r}{d_1} \tag{3.15}$$

for $n \in \mathbb{N}$ and $x \in (-\infty,\infty)$.

Now we will prove that there exists a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ uniformly convergent to an 1-periodic function y. The relations (3.15) imply that the sequence $\{y_n\}$ is equibounded. By (3.14) we conclude that $\{y_n\}$ is a family of equicontinuous functions on the interval $[0,1]$. From the Arzela theorem it follows that there exists a subsequence $\{y_{nk}\}$ of $\{y_n\}$ uniformly convergent to a 1-periodic continuous function y. By (3.15) we get

$$r \le \|y\| \le R \quad \text{and} \quad y(x) \ge \frac{M_1 r}{d_1}. \tag{3.16}$$

We will prove $y \in V^1$. In fact, by

$$y_{nk}''(x) - P_{nk}'(x) y_{nk}(x) = -\mu Q_{nk}'(x) f(x, y_{nk}(x)) \tag{3.17}$$

and

$$y_{nk}''(x) - \Big(\int_0^x y_{nk}(s)\,dP_{nk}(s)\Big)' = -\mu\Big(\int_0^x f(s, y_{nk}(s))\,dQ_k(s)\Big)' \tag{3.18}$$

and Helly's theorem (see [12], p. 29, Theorem 1.6.10), we have

$$\lim_{k\to\infty} \int_0^x y_{nk}(s)\,dP_{nk}(s)ds = \int_0^x y(s)\,dP^*(s)$$

and

$$\lim_{k\to\infty} \int_0^x f(s, y_{nk}(s))\,dQ_{nk}(s) = \int_0^x f(s, y(s))\,dQ^*(s),$$

so

$$\begin{aligned} \lim_{k\to\infty} y_{nk}''(x) = y''(x) &= \Big(\int_0^x y(s)\,dP^*(s)\Big)' - \mu\Big(\int_0^x f(s, y(s))\,dQ^*(s)\Big)' \\ &= P'(x)y(x) - \mu Q'(x) f(x, y(x)) \end{aligned}$$

and $y \in V^1$. This completes the proof of Theorem 3.2. $\square$

Theorem 3.3. *Let hypotheses H_2, H_4 and H_5 be satisfied. Suppose that a function f has properties* (3.1), (3.4) *and there exists $r > 0$ such that*

$$r \geq \psi(r)\mu m_2, \tag{3.19}$$

where

$$m_2 \geq \sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} \int_0^1 {Q_n}'(s) G_{2n}(x,s)ds,$$

${Q_n}'(x) = (Q * \delta_n)'(x)$ and $G_{2n}(x,s)$ is the Green function defined by $(2.27)_n$. Assume, additionally that there exists $R > 0$ such that $R > r$ and

$$R \leq \mu \int_0^1 \tau(s) G_{2n}\left(\frac{1}{2}, s\right) {Q_n}'(s) g\left(\frac{M_2 R}{d_2}\right) ds \quad \text{for } n \in \mathbb{N}, \tag{3.20}$$

where d_2 and M_2 are defined by relations (jv).

Then (1.2) *has a positive,* 1*-periodic solution of the class V^1.*

Proof. The proof is similar to the proof of Theorem 3.2. Let Ω_1 and Ω_2 be as in Theorem 3.2. Let

$$K_2 = \{u \in P_1(\mathbb{R}) : \min_{x\in[0,1]} d_2 u(x) \geq M_2 \|u\|\}.$$

Then K_2 is a cone of E. Now, let $\varphi \in P_1(\mathbb{R})$ and let $y_{n\varphi}$ be the unique, 1-periodic solution of the equation

$$y''(x) + {P_n}'(x) y(x) = \mu {Q_n}'(x) f(x, \varphi(x)). \tag{3.21$_n$}$$

Let $A_{2n} \colon K_1 \cap (\overline{\Omega}_2 | \Omega_1) \to E$ be defined by $(A_{2n})(\varphi) = y_{n\varphi}$. Then

$$(A_{2n}\varphi)(x) = \mu \int_0^1 G_{2n}(x,s) {Q_n}'(s) f(s, \varphi(s)) ds. \tag{3.22$_n$}$$

It is not difficult to prove that $A_{2n} \colon K_2 \cap (\overline{\Omega}_2 | \Omega_1) \to K_2$, A_{2n} is continuous and completely continuous. Similar arguments as in Theorem 3.2 guarantee that

$$\|A_{2n}\varphi\| \leq \|\varphi\| \quad \text{for } \varphi \in K_2 \cap \partial\Omega_1$$

and

$$\|A_{2n}\varphi\| \geq \|\varphi\| \quad \text{for } \varphi \in K_2 \cap \partial\Omega_2.$$

Theorem 3.1 implies A_{2n} has a fixed point $y_n \in K_2 \cap (\overline{\Omega}_2 | \Omega_1)$ i.e. $y_n(x) \geq \frac{M_2 r}{d_2}$ and $r \leq \|y_n\| \leq R$ for $n \in \mathbb{N}$. Arzela's and Helly's theorems imply that there exists a subsequence $\{y_{nk}\}$ of the sequence $\{y_n\}$ uniformly convergent to a 1-periodic, positive function y of the class V^1 and y is a solution of problem (1.2). The proof of Theorem 3.3 is finished. □

Example 3.4. Consider the following equation

$$y''(x) + \left(\sum_{k=-\infty}^{\infty} \delta(x+k) \right) y(x) = \left(\sum_{k=-\infty}^{\infty} \delta(x+k) \right) y^2(x), \tag{3.23}$$

where δ denotes the delta Dirac distribution. We have

$$P'(x) = Q'(x) = \sum_{k=-\infty}^{\infty} \delta(x+k).$$

Evidently $P' \geq 0, Q' \geq 0, P' \neq 0, Q' \neq 0, P'$ and Q' are 1-periodic distribution. The distribution P' and Q' are derivatives of the function $E(x)$, where symbol $E(a)$ denotes the greatest integer not exceeding a. Without loss of a generality we can assume that

$$P(x) = Q(x) = E(x).$$

It is not difficult to verify that $E(x+1) - E(x) = 1$ and

$$0 < \int_0^1 P'(x)dx = 1 < 4.$$

Thus the equation

$$y'' + P'(x)y(x) = 0$$

has only the trivial, 1-periodic solution of the class V^1. Let $G_{2n}(x,s)$ be defined by $(2.27)_n$ and let $\lim_{n\to\infty} G_{2n}(x,s) = G_2(x,s)$ (uniformly on I). We will prove that

$$G_2(x,s) = \begin{cases} x(s-1)+1, & \text{if } 0 \le x \le s \le 1, \\ s(x-1)+1, & \text{if } 0 \le s \le x \le 1. \end{cases} \tag{3.24}$$

To see this let $\varphi(x)$ and $\psi(x)$ be solutions of the following problems

$$\begin{cases} \varphi''(x) + P'(x)\varphi(x) = 0, \\ \varphi(0) = 1,\ \varphi'^*(0) = 0, \end{cases}$$

$$\begin{cases} \psi''(x) + P'(x)\psi(x) = 0, \\ \psi(0) = 0,\ \psi'^*(0) = 1. \end{cases}$$

Then

$$\begin{aligned} &\varphi(x) = -xH(x) - \frac{1}{2}(x-1)H(x-1) + \frac{1}{2}x + 1 \quad \text{for } x \in (-1,2), \\ &\psi(x) = -(x-1)H(x-1) + x \quad \text{for } x \in (-1,2), \\ &\varphi(1) = \frac{1}{2}, \quad \varphi'^*(1) = -\frac{3}{4}, \quad \psi(1) = 1, \quad \psi'^*(1) = \frac{1}{2}, \end{aligned}$$

where H denotes the Heaviside function.

Now let

$$G_2(x,s) = \begin{cases} a_1(s)\varphi(x) + a_2(s)\psi(x), & \text{if } 0 \le x \le s \le 1, \\ b_1(s)\varphi(x) + b_2(s)\psi(x), & \text{if } 0 \le s \le x \le 1. \end{cases}$$

Then functions a_1, a_2, b_1, b_2 satisfy the system of equations (similar to that of $(2.30)_n$)

$$\begin{cases} a_1 - \frac{1}{2}b_1 - b_2 = 0, \\ -\frac{1}{2}b_1 + b_2 + \frac{1}{2}a_1 - a_2 = 1, \\ a_1\left(-\frac{1}{2}s + 1\right) + a_2 s + \left(\frac{1}{2}s - 1\right)b_1 - b_2 s = 0, \\ a_2 + \frac{3}{4}b_1 - \frac{1}{2}b_2 = 0. \end{cases}$$

Consequently $a_1 = 1, a_2 = s - \frac{1}{2}, b_1 = 1 - s$ and $b_2 = \frac{1}{2} + \frac{1}{2}s$, so (3.24) holds.

It is not difficult to verify that

$$\sup_{(x,s)\in I} G_2(x,s) = 1, \qquad \inf_{(x,s)\in I} G_2(x,s) = \frac{3}{4},$$

$$G_2(x,0) = G_2(x,1) = 1 = G_2(0,s) = G_2(1,s).$$

Let us take

$$\overline{\gamma}_2 = \frac{1}{2}, \quad \overline{M}_2 = \frac{10}{9}, \quad d_2 = \frac{20}{9}, \quad M_2 = \frac{1}{3},$$

$$\tau(x) = 1, \quad f(x,v) = g(v) = \psi(v) = v^2,$$

$$\mu = 1, \quad m_2 = 3, \quad r = \frac{1}{3} \quad \text{and} \quad R = 40.$$

Then the inequalities (3.5)–(3.6) are satisfied for sufficiently large n.

Theorem 3.3 implies the existence of positive and 1-periodic solution of equation (3.23).

Next we show $y = 1$ is the unique 1-periodic and positive solution of equation (3.23) (of the class V^1). To see it, let $\overline{y}$ be an 1-periodic solution of equation (3.23). Then

$$\overline{y}''(x) + \sum_{k=-\infty}^{\infty} c\delta(x+k) = \sum_{k=-\infty}^{\infty} c^2\delta(x+k),$$

where $c = \overline{y}(0) = \overline{y}(1)$. So

$$y'(x) = (c^2 - c)E(x) + c_1$$

and

$$\overline{y}(x) = (c^2 - c)\,\widetilde{E}(x) + c_1 x + c_2,$$

where c_1, c_2 denote constants and $(\widetilde{E}(x))' = E(x)$.

Without loss of generality we can assume that

$$\overline{y}(x) = (-c + c^2)xH(x) + (-c + c^2)(x-1)H(x-1) + c_1 x + c_2 \tag{3.25}$$

for $x \in (-1,2)$. By (3.25), we have

$$\overline{y}(0) = c_2 = \overline{y}(1) = (-c + c^2) + c_1 + c_2.$$

Consequently

$$c_1 = -c^2 + c \tag{3.26}$$

and

$$y'(0^+) = (-c + c^2) + c_1 = \overline{y}'(1^+) = (-c + c^2) + (-c + c^2) + c_1. \tag{3.27}$$

The relations (3.26)–(3.27) yield $c = 0$ or $c = 1$. Thus $y = 1$ is the unique, positive, 1-periodic solution of equation (3.23).

Remark 3.5. It is not difficult to prove that $y = 1$ is the unique, 1-periodic, positive solution of the class V^1 of the equation

$$y''(x) - \Big(\sum_{k=-\infty}^{\infty} \delta(x+k)\Big) y(x) + \Big(\sum_{k=-\infty}^{\infty} \delta(x+k)\Big) y^2(x) = 0. \tag{3.28}$$

Theorem 3.6. *Let hypothesis H_2 and H_5 be satisfied. Suppose that there exist $r > 0$ and $R > 0$ such that $r < R$ and for $x \in [0,1]$*

$$\begin{aligned} f(x,v) &\le \frac{1}{\overline{M}_1 q_1 \mu} v, \quad \textit{if } 0 \le v \le r,\\ f(x,v) &\ge \frac{d_1}{\mu \overline{\gamma}_1 q_1 M_1} v, \quad \textit{if } R \le v < \infty, \end{aligned} \tag{3.29}$$

where $q_1 = \int\limits_0^1 Q'(x)dx$ and constants $\overline{M}_1, M_1, \overline{\gamma}_1$ have properties (iii)–(iv).

Then (1.1) *has a positive,* 1*-periodic solution of the class V^1.*

Proof. Let Ω_1, Ω_2 and K_1 be as in Theorem 3.2. Let $\varphi \in P_1(\mathbb{R})$ and let $y_{n\varphi}$ be the unique solution, 1-periodic of equation (3.9) n and let $(A_{1n})(\varphi) = y_{n\varphi}$. Then

$$A_{1n} \colon K_1 \cap (\overline{\Omega}_2 | \Omega_1) \to K_1 \quad \text{for } n \in \mathbb{N},$$

A_{1n} is continuous and completely continuous.

For $\varphi \in K_1 \cap \partial\Omega_1$ and $n \in \mathbb{N}$, we have (by (3.29))

$$\begin{aligned}\|A_{1n}(\varphi)\| &\le \mu\overline{M}_1 \int_0^1 Q_n{}'(s) f(s,\varphi(s))ds \\ &\le \mu\overline{M}_1 \frac{1}{\overline{M}_1 q_1 \mu} \int_0^1 Q'(s)\varphi(s)ds \\ &\le \frac{1}{q_1}\int_0^1 Q_n{}'(s)ds\|\varphi\| = \|\varphi\|.\end{aligned}$$

If $\varphi \in K_1 \cap \partial\Omega_2$, then by (3.29) and (iii) we obtain

$$\begin{aligned}\|A_{1n}\varphi\| &\ge \mu\overline{\gamma}_1 \int_0^1 Q_n{}'(s) f(s,\varphi(s))ds \\ &\ge \mu\overline{\gamma}_1 \frac{d_1}{\mu\overline{\gamma}_1 q_1 M_1} \int_0^1 Q_n{}'(s)\varphi(s)ds \\ &\ge \frac{d_1}{q_1 M_1}\int_0^1 Q_n{}'(s)\frac{M_1\|\varphi\|}{d_1}ds = \|\varphi\|\end{aligned}$$

for $n \in \mathbb{N}$. Theorem 3.1 implies A_{1n} has a fixed point $y_n \in K_1 \cap (\overline{\Omega}_2|\Omega_1)$, i.e.

$$r \le \|y_n\| \le R \quad \text{and} \quad y_n(x) \ge \frac{M_1 r}{d_1} \quad \text{for } n \in \mathbb{N}.$$

It is not difficult to prove that there exists a subsequence $\{y_{nk}\}$ of the sequence $\{y_n\}$ uniformly convergent to an 1-periodic and positive function $y \in V^1$ and y is a solution of (1.1), which completes the proof of Theorem 3.6. □

REMARK 3.7. If

$$\lim_{v\to 0^+} \frac{f(x,v)}{v} = 0 \quad \text{and} \quad \lim_{v\to\infty} \frac{f(x,v)}{v} = \infty$$

uniformly on $x \in [0,1]$, then condition (3.29) will be satisfied for r sufficiently small and for $R > 0$ sufficiently large.

Corollary 3.8. *Let hypotheses H_2 and H_5 be satisfied, suppose that there exist $r > 0$ and $R > 0$ such that $r < R$ and for $x \in [0,1]$*

$$
\begin{aligned}
f(x,v) &\geq \frac{d_1}{\mu\overline{\gamma}_1 q_1 M_1} v, \quad \text{if } 0 \leq v \leq r,\\
f(x,v) &\leq \frac{1}{\overline{M}_1 q_1 \mu} v, \quad \text{if } R \leq v < \infty.
\end{aligned} \tag{3.30}
$$

Then (1.1) *has a positive,* 1*-periodic solution of the class* V^1.

The proof is analogous to that of Theorem 3.6 and uses the second part of Theorem 3.1.

Remark 3.9. If

$$\lim_{v\to 0^+} \frac{f(x,v)}{v} = \infty \quad \text{and} \quad \lim_{v\to\infty} \frac{f(x,v)}{v} = 0$$

uniformly on $x \in [0,1]$, then conditions (3.30) will be satisfied for $r > 0$ sufficiently small and for $R > 0$ sufficiently large.

Theorem 3.10. *Let hypotheses H_2, H_4 and H_5 be satisfied. We assume that there exist $r > 0$ and $R > 0$ such that $r < R$ and for $x \in [0,1]$*

$$
\begin{aligned}
f(x,v) &\leq \frac{1}{\mu\overline{M}_2 q_1} v, \quad \text{if } 0 \leq v \leq r,\\
f(x,v) &\geq \frac{d_2}{\mu\overline{\gamma}_2 q_1 M_2} v, \quad \text{if } R \leq v < \infty,
\end{aligned} \tag{3.31}
$$

where $\overline{M}_2, M_2, \overline{\gamma}_2$ have properties (jjj)–(jv). *Then* (1.2) *has a positive,* 1*-periodic solution of the class* V^1.

The proof is analogous to that of Theorem 3.6.

Theorem 3.11. *Let hypotheses H_2, H_4 and H_5 be satisfied. Suppose that there exist $r > 0$ and $R > 0$ such that $r < R$ and for $x \in [0,1]$*

$$
\begin{aligned}
f(x,v) &\geq \frac{d_2}{\mu\overline{\gamma}_2 q_1 M_2} v, \quad \text{if } 0 \leq v \leq r,\\
f(x,v) &\leq \frac{1}{\overline{M}_2 q_1 \mu} v, \quad \text{if } R \leq v < \infty.
\end{aligned}
$$

Then (1.2) *has a positive,* 1*-periodic solution of the class* V^1.

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Institute of Mathematics
Silesian University
Bankowa 14
40–007 Katowice
Poland
e-mail: ligeza@math.us.edu.pl