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EXISTENCE AND UNIQUENESS OF CLASSICAL SOLUTION TO DARBOUX PROBLEM TOGETHER WITH NONLOCAL CONDITIONS

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Abstract. The existence and uniqueness of a classical solution to a semilinear hyperbolic differential Darboux problem together with semilinear nonlocal conditions in a bounded domain are studied. The Banach fixed point theorem is applied.

1. Introduction

In this paper we prove a theorem on the existence and uniqueness of a classical solution to a semilinear hyperbolic differential Darboux problem together with semilinear nonlocal conditions in the domain $[0, a] \times [0, b]$, where $a > 0$ and $b > 0$.

The result obtained is a generalization of results given by Krzyżański in [5], by Chi, Poorkarimi, Wiener and Shah in [4] and by the author in [1] and [2].

In monograph [5], Krzyżański gives the existence and uniqueness of a classical solution to a semilinear Darboux problem, in the domain $[0, a] \times [0, a]$, together with the classical local conditions.

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Moreover, in publication [4], Chi, Poorkarimi, Wiener and Shah study the existence and uniqueness of classical solutions to semilinear Darboux problems, in the domains $[0,a]\times[0,b]$ and $[0,a]\times[0,\infty)$, together with the classical local conditions.

In publications [1] and [2], the author considers theorems on the existence and uniqueness of semilinear Darboux problems together with linear nonlocal conditions in two domains: $[0,a]\times[a,b]$ and $[0,\infty)\times[0,\infty)$.

The study of parabolic problems together with semilinear nonlocal conditions was initiated by Chabrowski in [3].

2. Preliminaries

Let $Q := [0,a]\times[0,b]$, where $a>0, b>0$, and let a_i $(i=1,\ldots,p)$, b_j $(j=1,\ldots,s)$ be given numbers such that

$$a_1 < a_2 < \ldots < a_p \le a,$$
$$b_1 < b_2 < \ldots < b_s \le b.$$

Moreover, let $Z := Q\times[-A,A]^3$, where $A>0$.

We mean by $C^1(Q,\mathbb{R})$ the set of all continuous functions $w\colon Q\to\mathbb{R}$ ($w=w(x,y)$) such that the derivatives w'_x and w'_y are continuous in Q. Moreover, we mean by $C^1(Q,[-A,A])$ the set of all continuous functions $w\colon Q\to[-A,A]$ such that the derivatives w'_x and w'_y are continuous in Q and satisfy the inequalities

$$\|w'_x\|\le A,\quad \|w'_y\|\le A, \tag{2.1}$$

where $\|\cdot\|$ is the norm of the uniform convergence in Q. In $C^1(Q,[-A,A])$ we use the following metric ρ:

$$\rho(w,\tilde{w}) = \|w-\tilde{w}\| + \|w'_x-\tilde{w}'_x\| + \|w'_y-\tilde{w}'_y\| \tag{2.2}$$

for $w,\tilde{w}\in C^1(Q,[-A,A])$. By $C^{1,2}(Q,[-A,A])$ we denote the class of all functions $w\in C^1(Q,[-A,A])$ such that the derivative w''_{xy} is continuous in Q.

In this paper we prove a theorem on the existence and uniqueness of a classical solution of the following Darboux problem together with semilinear nonlocal conditions:

$$u''_{xy}(x,y) = F(x,y,u(x,y),u'_x(x,y),u'_y(x,y)),\quad (x,y)\in Q, \tag{2.3}$$

$$u(x,0)+\sum_{i=1}^{p}h_i(x)H(u(x,b_i))=\phi(x),\quad x\in[0,a], \tag{2.4}$$

$$u(0,y)+\sum_{j=1}^{s}k_j(y)K(u(a_j,y))=\psi(y),\quad y\in[0,b], \tag{2.5}$$

where F, H, K, h_i $(i=1,\dots,p)$, k_j $(j=1,\dots,s)$, ϕ, and ψ are given functions satisfying some assumptions.

A function $u\in C^{1,2}(Q,[-A,A])$ is said to be a *classical solution* to problem (2.3)–(2.5) if u satisfies the differential equation (2.3) and the nonlocal conditions (2.4) and (2.5).

To find the classical solution of problem (2.3)–(2.5) we apply the Banach fixed point theorem.

Similarly as in paper [1], the theorem from this paper can be applied in the theory of elasticity with better effects than the analogous known theorem with classical local conditions.

3. Theorem on the existence and uniqueness

THEOREM 1. *Assume that:*

(i) $F\in C(Z,\mathbb{R})$ *and there is a constant* $L>0$ *such that*

$$\big|F(x,y,z,p,q)-F(x,y,\tilde z,\tilde p,\tilde q)\big|\le L(\big|z-\tilde z\big|+\big|p-\tilde p\big|+\big|q-\tilde q\big|) \tag{3.1}$$

for $(x,y,z,p,q),\ (x,y,\tilde z,\tilde p,\tilde q)\in Z$.

Moreover,

$$M:=\max_{(x,y,z,p,q)\in Z}\big|F(x,y,z,p,q)\big|; \tag{3.2}$$

(ii) $H\in C^1([-A,A],\mathbb{R}),\ K\in C^1([-A,A],\mathbb{R})$ *and there are constants* $L_i>0$ $(i=1,\dots,4)$ *such that*

$$\big|H(z)-H(\tilde z)\big|\le L_1\big|z-\tilde z\big|,\quad z,\tilde z\in[-A,A], \tag{3.3}$$

$$\big|K(z)-K(\tilde z)\big|\le L_2\big|z-\tilde z\big|,\quad z,\tilde z\in[-A,A], \tag{3.4}$$

$$\big|H'(z)-H'(\tilde z)\big|\le L_3\big|z-\tilde z\big|,\quad z,\tilde z\in[-A,A], \tag{3.5}$$

$$\big|K'(z)-K'(\tilde z)\big|\le L_4\big|z-\tilde z\big|,\quad z,\tilde z\in[-A,A]. \tag{3.6}$$

Moreover,

$$M_1 := \max\Big(\max_{z\in[-A,A]}\big|H(z)\big|,\ \max_{z\in[-A,A]}\big|H'(z)\big|\Big) \tag{3.7}$$

and

$$M_2 := \max\Big(\max_{z\in[-A,A]}\big|K(z)\big|,\ \max_{z\in[-A,A]}\big|K'(z)\big|\Big); \tag{3.8}$$

(iii) $\phi \in C^1([0,a],\mathbb{R})$, $\psi \in C^1([0,b],\mathbb{R})$, $\phi(0) = \psi(0)$, $h_i \in C^1([0,a],\mathbb{R})$, $h_i(0) = 0$ $(i = 1,\ldots,p)$, $k_j \in C^1([0,b],\mathbb{R})$, $k_j(0) = 0$ $(j = 1,\ldots,s)$. *Moreover,*

$$K_1 := \max\Big(\max_{x\in[0,a]}\big|\phi(x)\big|, \max_{x\in[0,a]}\big|\phi'(x)\big|\Big), \tag{3.9}$$

$$K_2 := \max\Big(\max_{y\in[0,b]}\big|\psi(y)\big|, \max_{y\in[0,a]}\big|\psi'(y)\big|\Big), \tag{3.10}$$

$$K_3 := \max_{i=1,\ldots,p}\Big(\max_{x\in[0,a]}\big|h_i(x)\big|,\ \max_{x\in[0,a]}\big|h_i'(x)\big|\Big), \tag{3.11}$$

$$K_4 := \max_{j=1,\ldots,s}\Big(\max_{y\in[0,b]}\big|k_j(y)\big|, \max_{y\in[0,b]}\big|k_j'(y)\big|\Big). \tag{3.12}$$

(iv) *The following inequalities are satisfied:*

$$\begin{aligned}(1+a)K_1 + 2K_2 + pK_3M_1(A+2)&\\ +sK_4M_2(A+2) + (a+b+ab)M &\le A,\end{aligned} \tag{3.13}$$

$$q < 1, \tag{3.14}$$

where $q := pK_3(2L_1+M_1+L_3A)+sK_4(2L_2+M_2+L_4A)+(a+b+ab)L$. *Then problem* (2.3)–(2.5) *has a unique classical solution.*

Proof. It is evident that if the function $u \in C^{1,2}(Q,[-A,A])$ satisfies problem (2.3)–(2.5) then it also satisfies the integral equation

$$\begin{aligned}u(x,y) = {}&\phi(x) - \phi(0) + \psi(y)\\ &-\sum_{i=1}^{p} h_i(x)H(u(x,b_i)) - \sum_{j=1}^{s} k_j(y)K(u(a_i,y))\\ &+\int_0^x\int_0^y F(\xi,\eta,u(\xi,\eta),u'_\xi(\xi,\eta),u'_\eta(\xi,\eta))d\xi d\eta.\end{aligned} \tag{3.15}$$

Conversely, if the function $u \in C^1(Q, [-A, A])$ and satisfies equation (3.15) then it has the continuous derivative $u''_{xy} = u''_{yx}$ in Q, satisfies equation (2.3) and, moreover, conditions (2.4)–(2.5). Therefore, we will seek the solution of equation (3.15). For this purpose introduce the operator T given by the following formula:

$$\begin{aligned}(Tw)(x,y) &:= \phi(x) - \phi(0) + \psi(y)\\ &\quad - \sum_{i=1}^{p} h_i(x) H(w(x,b_i)) - \sum_{j=1}^{s} k_j(y) K(w(a_j,y))\\ &\quad + \int_0^x \int_0^y F(\xi,\eta,w(\xi,\eta),w'_\xi(\xi,\eta),w'_\eta(\xi,\eta))\,d\xi d\eta\end{aligned} \tag{3.16}$$

for $w \in C^1(Q, [-A, A])$.

Since $\phi \in C^1([0,a],\mathbb{R})$, $\psi \in C^1([0,b],\mathbb{R})$, $h_i \in C^1([0,a],\mathbb{R})$ $(i = 1,\dots,p)$, $k_j \in C^1([0,b],\mathbb{R})$ $(j = 1,\dots,s)$, $H, K \in C^1([-A,A],\mathbb{R})$, and $F \in C(Z,\mathbb{R})$ then operator T maps $C^1(Q,[-A,A])$ into $C^1(Q,\mathbb{R})$. Now, we will show that operator T maps $C^1(Q,[-A,A])$ into $C^1(Q,[-A,A])$. To this end observe that by (3.16), (3.7)–(3.12) and (3.2),

$$\begin{aligned}\big|(Tw)(x,y)\big| &\le \big|\phi(x) - \phi(0)\big| + \big|\psi(y)\big|\\ &\quad + \sum_{i=1}^{p} \big|h_i(x)\big|\cdot\big|H(w(x,b_i))\big| + \sum_{j=1}^{s} \big|k_j(y)\big|\cdot\big|K(w(a_j,y))\big|\\ &\quad + \int_0^x \int_0^y \big|F(\xi,\eta,w(\xi,\eta),w'_\xi(\xi,\eta),w'_\eta(\xi,\eta))\big|\,d\xi d\eta\\ &\le aK_1 + K_2 + pK_3M_1 + sK_4M_2 + abM\end{aligned} \tag{3.17}$$

for $w \in C^1(Q, [-A, A])$,

$$\begin{aligned}\big|[(Tw)(x,y)]'_x\big| &\le \big|\phi'(x)\big| + \sum_{i=1}^{p} \big|h'_i(x)\big|\cdot\big|H(w(x,b_i))\big|\\ &\quad + \sum_{i=1}^{p} \big|h_i(x)\big|\cdot\big|H'(w(x,b_i))\big|\cdot\big|w'_x(x,b_i)\big|\\ &\quad + \int_0^y \big|F(x,\eta,w(x,\eta),w'_x(x,\eta),w'_\eta(x,\eta))\big|\,d\eta\\ &\le K_1 + pK_3M_1 + pK_3M_1A + bM\end{aligned} \tag{3.18}$$

for $w \in C^1(Q, [-A, A])$, and

$$
\begin{aligned}
\big|[(Tw)(x,y)]'_y\big| \le \big|\psi'(y)\big| &+ \sum_{j=1}^{s}\big|k'_j(y)\big|\cdot\big|K(w(a_j,y))\big| \\
&+ \sum_{j=1}^{s}\big|k_j(y)\big|\cdot\big|K'(w(a_j,y))\big|\cdot\big|w'_y(a_j,y)\big| \\
&+ \int_0^x \big|F(\xi,y,w(\xi,y),w'_\xi(\xi,y),w'_y(\xi,y))\big|d\xi \\
&\le K_2 + sK_4M_2 + sK_4M_2A + aM,\ w \in C^1(Q,[-A,A]).
\end{aligned}
\tag{3.19}
$$

Consequently, from (2.2), (3.17)–(3.19) and (3.13),

$$
\rho(Tw,0) = \|Tw\| + \|(Tw)'_x\| + \|(Tw)'_y\| \le A \quad \text{for } w \in C^1(Q,[-A,A]).
$$

Therefore,

$$
T\colon C^1(Q,[-A,A]) \to C^1(Q,[-A,A]). \tag{3.20}
$$

Now, we will show that

$$
\rho(Tw,T\tilde{w}) \le q\rho(w,\tilde{w}), \quad w,\tilde{w} \in C^1(Q,[-A,A]). \tag{3.21}
$$

For this purpose observe that, by (3.16),

$$
\begin{aligned}
(Tw)(x,y) - (T\tilde{w})(x,y) = &-\sum_{i=1}^{p} h_i(x)[H(w(x,b_i)) - H(\tilde{w}(x,b_i))] \\
&-\sum_{j=1}^{s} k_j(y)[K(w(a_j,y)) - K(\tilde{w}(a_j,y))] \\
&+\int_0^x\int_0^y [F(\xi,\eta,w(\xi,\eta),w'_\xi(\xi,\eta),w'_\eta(\xi,\eta)) \\
&\qquad - F(\xi,\eta,\tilde{w}(\xi,\eta),\tilde{w}'_\xi(\xi,\eta),\tilde{w}'_\eta(\xi,\eta))]d\xi d\eta,
\end{aligned}
$$

$w,\tilde{w} \in C^1(Q,[-A,A])$, and, therefore, from (3.11), (3.3), (3.12), (3.4), (3.1) and (2.2),

$$
\big|(Tw)(x,y) - (T\tilde{w})(x,y)\big| \le (pK_3L_1 + sK_4L_2 + abL)\rho(w,\tilde{w}), \tag{3.22}
$$

$w,\tilde{w} \in C^1(Q,[-A,A])$.

Moreover, observe that, by (3.16),

$$[(Tw)(x,y)]'_x - [(T\tilde{w})(x,y)]'_x$$
$$= \sum_{i=1}^{p} h'_i(x) \cdot [H(w(x,b_i)) - H(\tilde{w}(x,b_i))]$$
$$+ \sum_{i=1}^{p} h_i(x) \cdot [H'(w(x,b_i)) \cdot w'_x(x,b_i) - H'(\tilde{w}(x,b_i)) \cdot \tilde{w}'_x(x,b_i)] +$$
$$\int_0^y [F(x,\eta,w(x,\eta),w'_x(x,\eta),w'_\eta(x,\eta)) - F(x,\eta,\tilde{w}(x,\eta),\tilde{w}'_x(x,\eta),\tilde{w}'_\eta(x,\eta))]d\eta$$
$$= \sum_{i=1}^{p} h'_i(x) \cdot [H(w(x,b_i)) - H(\tilde{w}(x,b_i))]$$
$$+ \sum_{i=1}^{p} h_i(x) H'(w(x,b_i)) \cdot [w'_x(x,b_i) - \tilde{w}'_x(x,b_i)]$$
$$+ \sum_{i=1}^{p} h_i(x) [H'(w(x,b_i)) - H'(\tilde{w}(x,b_i))] \cdot \tilde{w}'_x(x,b_i) +$$
$$\int_0^y [F(x,\eta,w(x,\eta),w'_x(x,\eta),w'_\eta(x,\eta)) - F(x,\eta,\tilde{w}(x,\eta),\tilde{w}'_x(x,\eta),\tilde{w}'_\eta(x,\eta))]d\eta,$$

$w, \tilde{w} \in C^1(Q,[-A,A])$, and, therefore, from (3.11), (3.3), (3.7), (3.5), (3.1), and (2,2),

$$\text{(3.23)} \quad \big|[(Tw)(x,y)]_x - [(T\tilde{w})(x,y)]_x\big|$$
$$\le (pK_3L_1 + pK_3M_1 + pK_3L_3A + bL)\rho(w,\tilde{w}),$$

$w, \tilde{w} \in C^1(Q,[-A,A])$. Finally, observe that, by (3.16),

$$(Tw)(x,y)]'_y - [(T\tilde{w})(x,y)]'_y$$
$$= \sum_{j=1}^{s} k'_j(y) \cdot [K(w(a_j,y)) - K(\tilde{w}(a_j,y))]$$
$$+ \sum_{j=1}^{s} k_j(y) K'(w(a_j,y)) \cdot [w'_y(a_j,y) - \tilde{w}'_y(a_j,y)]$$
$$+ \sum_{j=1}^{s} k_j(y) \cdot [K'(w(a_j,y)) - K'(\tilde{w}(a_j,y))] \cdot \tilde{w}'_y(a_j,y) +$$
$$\int_0^x [F(\xi,y,w(\xi,y),w'_\xi(\xi,y),w'_y(\xi,y)) - F(\xi,y,\tilde{w}(\xi,y),\tilde{w}'_\xi(\xi,y),\tilde{w}'_y(\xi,y))]d\xi,$$

$w, \tilde{w} \in C^1(Q, [-A, A])$, and, therefore from (3.12), (3.4), (3.8), (3.6), (3.1) and (2.2),

$$(3.24) \quad \left|[Tw)(x,y)]'_y - [(T\tilde{w})(x,y)]'_y\right| \leq (sK_4L_2 + sK_4M_2 + sK_4L_4A + aL)\rho(w, \tilde{w}),$$

$w, \tilde{w} \in C^1(Q, [-A, A])$. Consequently, by (3.22)–(3.24), (2.2) and (3.14), inequality (3.21) is satisfied with $0 < q < 1$.

By (3.20) and (3.21) operator T satisfies all the assumptions of the Banach fixed point theorem. Therefore, in space $C^1(Q, [-A, A])$ there is the only one fixed point of T and this point is the classical solution of problem (2.3)–(2.5). So, the proof of Theorem 3.1 is complete. □

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References

[1] Byszewski L., *Theorem about the existence and uniqueness of continuous solution of nonlocal problem for nonlinear hyperbolic equation*, Appl. Anal. **40** (1991), 173–180.
[2] Byszewski L., *Existence and uniqueness of classical solutions to semilinear Darboux problems together with nonstandard conditions with integrals*, Comment. Math. Prace Mat. **43** (2003), 169–183.
[3] Chabrowski J., *On the non-local problem with a functional for parabolic equation,* Funkcial. Ekvac. **27** (1984), 101–123.
[4] Chi H., Poorkarimi H., Wiener J., Shah S.M., *On the exponential growth of solutions to nonlinear hyperbolic equations*, Internat. J. Math. Math. Sci. **12** (1989), 539–545.
[5] Krzyżański M., *Partial Differential Equations of Second Order*, Vol. II, PWN Polish Scientific Publishers, Warsaw, 1971.

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