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Report of Meeting

The Thirteenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Zakopane (Poland), January 30 – February 2, 2013

The Thirteenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held in Zakopane, Poland, from January 30 to February 2, 2013. It was organized by the Institute of Mathematics of the Silesian University from Katowice.

30 participants came from the University of Debrecen (Hungary) and the Silesian University of Katowice (Poland) at 15 from each of both cities.

Professor Roman Ger opened the Seminar and welcomed the participants to Zakopane.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iteration theory, equations on algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity. Interesting discussions were generated by the talks.

There was also a profitable Problem Session.

The social program included a festive dinner and an excursion to the top of Kasprowy Wierch by cable car.

The closing address was given by Professor Gyula Maksa. His invitation to hold the Fourteenth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in February 2014 in Hungary was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in section 1, problems and remarks in section 2, and the list of participants in the final section.

1. Abstracts of talks

Roman Badora: *The stability of the Cauchy equation in lattices* (Joint work with Barbara Przebieracz and Tomasz Kochanek)

Let $(X, \vee, \wedge)$ be a lattice and let $(Y, \vee, \wedge)$ be a boundedly complete lattice. The stability problem concerning the following Cauchy's type functional equation

$$f(x \vee y) = f(x) \vee f(y),$$

where $f\colon X \to Y$, is investigated.

Karol Baron: *On additive involutions and Hamel bases*

Assume X is a linear space over the field $\mathbb{Q}$ of rationals with $\dim X \geq 3$, let H_0 be a basis of X and fix an $h_0 \in H_0$.

Inspired by the foot–note on p. 325 of [1] (on p. 294 of the original edition) we show that the additive function $a\colon X \to X$ defined by

$$a(h_0) = h_0 \quad \text{and} \quad a(h) = -h \quad \text{for } h \in H_0 \setminus \{h_0\}$$

is involutory,

$$a(x) + x \in \mathbb{Q}h_0 \qquad \text{for } x \in X$$

and for every linearly independent set $H \subset X$ with $\mathrm{card} H \geq 3$ we have

$$a(H) \setminus H \neq \emptyset.$$

Reference

[1] Kuczma M., *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Second Edition, ed. by Gilányi A., Birkhäuser Verlag, Basel, 2009.

Mihály Bessenyei: *Solving functional equations via finite substitutions II* (Joint work with Csaba G. Kézi)

In this talk, we study single variable functional equations that involve one unknown function and a finite set of known functions that form a group under the operation of composition. The main theorems give sufficient conditions for the existence and uniqueness of a (local) solution and also stability-type result for the solution. In the proofs, beside the standard methods of classical analysis, some group theoretical tools play a key role.

ZOLTÁN BOROS: *Abstract subdifferentials and the Bernstein–Doetsch theorem*

The following abstract version of the celebrated Bernstein–Doetsch theorem [1] is established.

THEOREM. *Let K and L denote fields fulfilling $K \subset L \subset \mathbb{R}$, X be a non-trivial linear space over L, and D be a non-empty, L-convex and L-algebraically open subset of X. Suppose that $f\colon D \to \mathbb{R}$ is K-convex (i.e., the inequality*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

holds for every $x, y \in D$, $\lambda \in [0,1] \cap K$) and locally bounded above along L-lines at a point $x_0 \in D$ (in the sense that, for every $u \in X$, there exist $\delta_u > 0$ and a real number M_u such that

$$f(x_0 + \nu u) \le M_u$$

for all $\nu \in]-\delta_u, \delta_u[\cap L$). Then f is L-convex (i.e., f satisfies (1) *for every $x, y \in D$, $\lambda \in [0,1] \cap L$).*

Similar assumptions can be found in former or recent papers ([3], [4]). However, this statement can be easily proved with the aid of the existence of some abstract subgradients, which characterizes these convexity properties according to our joint results with Zsolt Páles [2].

REFERENCES

[1] Bernstein F., Doetsch G., *Zur Theorie der konvexen Funktionen*, Math. Ann. **76** (1915), 514–526.

[2] Boros Z., Páles Zs., *$\mathbb{Q}$-subdifferential of Jensen-convex functions*, J. Math. Anal. Appl. **321** (2006), 99–113.

[3] Kominek Z., Kuczma M., *Theorems of Bernstein–Doetsch, Piccard and Mehdi and semilinear topology*, Arch. Math. (Basel) **52** (1989), 595–602.

[4] Mureńko A., *A generalization of Bernstein–Doetsch theorem*, Demonstratio Math. **45**(2012), 35–38.

PÁL BURAI: *Some results concerning generalized quasi-arithmetic means* (Joint work with Justyna Jarczyk)

We present some results concerning means which are generated by a measure and a strictly monotone continuous function.

Włodzimierz Fechner: *A general functional inequality*

Let I be a nonvoid open interval, $k \in \mathbb{N}$ and let $c \in \mathbb{R}^+ \cup \{+\infty\}$ be arbitrarily fixed and denote $U = (0, c)$. Further, assume that we are given some mappings $\alpha_0, \alpha_1, \ldots, \alpha_k \colon U \to \mathbb{R}_0^+$ and $f \colon I \to \mathbb{R}$ is an unknown function. We are interested in the following functional inequality:

$$f(x + (k+1)h) \leq \sum_{i=0}^{k} \alpha_i(h) f(x + ih) \tag{1}$$

which is assumed to be satisfied for all $x \in I$, $h \in U$ such that $x + (k+1)h \in I$.

Inequality (1) provides a joint generalization of several older problems which have been studied in [1–6].

References

[1] Alsina C., Garcia Roig J.L., *On some inequalities characterizing the exponential function*, Arch. Math., Brno **26** (1990), no. 2–3, 67–71.

[2] Alsina C., Ger R., *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), no. 4, 373–380.

[3] Fechner W., *Some inequalities connected with the exponential function*, Arch. Math., Brno **44** (2008), no. 3, 217–222.

[4] Fechner W., *On some functional inequalities related to the logarithmic mean*, Acta Math. Hung. **128** (2010), no. 1–2, 36–45.

[5] Fechner W., *Functional inequalities and equivalences of some estimates*, in: *Inequalities and Applications 2010*, dedicated to the Memory of Wolfgang Walter, Hajdúszoboszló, Hungary International Series of Numerical Mathematics 161, Birkhäuser, Basel, 2012, pp. 231–240.

[6] Fechner W., Ger R., *Some stability results for equations and inequalities connected with the exponential functions*, in: *Functional Equations and Difference Inequalities and Ulam Stability Notions* (F.U.N.), Mathematics Research Developments, Nova Science Publishers Inc., New York, 2010, pp. 37–46.

Attila Gilányi: *On strongly Wright-convex functions of higher order* (Joint work with Nelson Merentes, Kazimierz Nikodem and Zsolt Páles)

Motivated by the results on strongly convex functions proved by Roman Ger and Kazimierz Nikodem in [1], we investigate strongly Wright-convex functions of higher order and we prove decomposition and characterization theorems for them.

Reference

[1] Ger R., Nikodem K., *Strongly convex functions of higher order*, Nonlinear Anal. **74** (2011), 661–665.

Tamás Glavosits: *On the Davison functional equation*

The general solution of the following equation (so named Davison functional equation)

$$f(xy) + f(x + y) = f(xy + x) + f(y)$$

was given by Roland Girgensohn and Károly Lajkó in their paper: *A functional equation of Davison and its generalization*, Aequationes Mathematicae **60** (2000), 219–22.

In my talk I will show a new method to give the general solution of the Pexider version of the above Davison equation.

Eszter Gselmann: *On some classes of partial difference equations*

In [1], J. A. Baker initiated the systematic investigation of some partial difference equations. The main purpose of my talk is the investigation of partial difference equations of the form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{m} \gamma_i(x_1, \ldots, x_n) f\left(x_1 + \rho_{1,i}h, \ldots, x_n + \rho_{i,n}h\right),$$

where $m \in \mathbb{N}$, $\rho_{j,i} \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are fixed as well as the functions γ_i, $i = 1, \ldots, m$.

Firstly, we present how such type of equations can be classified into elliptic, parabolic and hyperbolic subclasses, respectively. After that, we show solution methods in each of these classes. Here we will speak in details about the discrete version of the following partial differential equations: Laplace's equation, Poisson equation, inhomogeneous biharmonic equation, convective heat equation with a source, inhomogeneous Klein–Gordon equation.

Reference

[1] Baker, J.A., *An analogue of the wave equation and certain related functional equations*, Canad. Math. Bull. **12** (1969), 837–846.

Gábor Horváth: *Solving functional equations via finite substitutions I* (Joint work with Mihály Bessenyei and Csaba G. Kézi)

Motivated by some mathematical competition exercises, we consider functional equations involving one unknown function and a finite set of known functions forming a group under composition. We show how algebra can be utilized for solving such functional equations.

Tomasz Kochanek: *Compact perturbations of operator semigroups*

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators mapping $\mathcal{H}$ into itself. Suppose $(Q(t))_{t\geqslant 0} \subset \mathcal{B}(\mathcal{H})$ satisfies the semigroup condition modulo $\mathcal{K}(\mathcal{H})$, the ideal of compact operators, *i.e.*

$$(*) \qquad Q(s+t) - Q(s)Q(t) \in \mathcal{K}(\mathcal{H}) \quad \text{for all } s,t \geqslant 0.$$

We deal with the lifting problem, that is, our aim is to give sufficient conditions upon $(Q(t))_{t\geqslant 0}$ under which there exists a ($\mathcal{C}_0$-, *i.e.* SOT-continuous) operator semigroup $(T(t))_{t\geqslant 0}$ such that $Q(t) - T(t) \in \mathcal{K}(\mathcal{H})$. This task is not evident because $\mathcal{K}(\mathcal{H})$ is not even linearly complemented in $\mathcal{B}(\mathcal{H})$ whenever $\dim(\mathcal{H}) = \infty$ and therefore there is no lifting universal for all sequences $(Q(t))_{t\geqslant 0}$ as above.

Let $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the Calkin algebra (identified, via the GNS construction, with an algebra of operators on a Hilbert space) and let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ be the canonical projection. We will prove that if $(Q(t))_{t\geqslant 0}$ is a sequence of normal operators on $\mathcal{H}$ such that $(\pi Q(t))_{t\geqslant 0}$ is a $\mathcal{C}_0$-semigroup (so, we assume $(*)$ plus the continuity condition), and the spectrum of at least one of $Q(t)$'s lies inside a simply connected domain not containing zero, then there exists a $\mathcal{C}_0$-semigroup $(T(t))_{t\geqslant 0}$ of normal operators on $\mathcal{H}$ such that $\pi Q(t) = \pi T(t)$ for every $t \geqslant 0$. In other words, every such sequence $(Q(t))_{t\geqslant 0}$ is necessarily a compact perturbation of a $\mathcal{C}_0$-semigroup.

Rafał Kucharski: *Two remarks on n-additive and symmetric functions*

The solutions of the Fréchet equation $\Delta_h^{n+1} f(x) = 0$ are of the form $\sum_{k=0}^{n} a_k^*$, where a_0^* is a constant and $a_k^*(x) = g_k(x, \cdots, x)$, with a_k being symmetric and additive with respect to each variable. Functions a_k^* play the role of monomials in generalized polynomial function f, thus are called generalized monomials. The following basic question arises: how are they related to the ordinary monomials?

First we will show a construction of additive function $\mathcal{A}$ such that every n-additive and symmetric function $a\colon \mathbb{R}^n \to \mathbb{R}$ can be factorized as

$$a(x) = \gamma(\mathcal{A}(x)^n)$$

for some additive function γ, of course depending on a. As a consequence we also get similar factorization for generalized monomials.

Second, we will show example being the negative answer for the following question, related to the theorem of Maksa and Rätz [1]: are two generalized monomials proportional if their counterimages of the set $(0,\infty)$ are equal?

However, it occurs that symmetric and n-additive functions can be characterised in this way, what will be our final presented result.

Reference

[1] Maksa Gy., Rätz J., Remark 5 in: *Proceedings of the 19th ISFE, Nantes–La Turballe*, France 1981, Centre for Information Theory, Univ. of Waterloo, 1981.

Michał Lewicki: *On Beckenbach families admitting discontinuous Jensen affine functions*

In the talk we discuss a characterization of Beckenbach families admitting discontinuous Jensen affine functions and we put problems connected with this theme.

Judit Makó: *Strengthening of approximate convexity* (Joint work with Zsolt Páles)

Let X be a real linear space and $D \subseteq X$ be a nonempty convex subset. Given an error function $E\colon [0,1] \times (D - D) \to \mathbb{R}$ and an element $t \in]0,1[$, a function $f\colon D \to \mathbb{R}$ is called (E,t)-*convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + E(t; x - y)$$

for all $x, y \in D$. Our main result states that, for all $a, b \in (\mathbb{N} \cup \{0\}) + \{0, t, 1-t\}$, such that $\{a, b, a+b\} \cap \mathbb{N} \neq \emptyset$, every (E,t)-convex function is also $(F, \frac{a}{a+b})$-convex, where

$$F(s,u) := \frac{(a+b)^2 s(1-s)}{t(1-t)} E(t, \frac{u}{a+b}) \quad (u \in (D-D), s \in]0;1[).$$

As a consequence of this result, under further assumptions on E, the strong and approximate convexity properties of (E,t)-convex functions can be strengthened.

Gyula Maksa: *Remarks on real derivations*

As it is widely known in our community, a real derivation is a function $d\colon \mathbb{R} \to \mathbb{R}$ (the reals) satisfying both equations

$$d(x+y) = d(x) + d(y) \quad (x, y \in \mathbb{R}), \quad \text{and} \tag{1}$$

$$d(xy) = xd(y) + yd(x) \quad (x, y \in \mathbb{R}). \tag{2}$$

In this talk, we discuss the following type of problems related to real derivations.

Problem A. Suppose that the function $d\colon \mathbb{R} \to \mathbb{R}$ satisfies (1). What has to be supposed additionally that the function d be real derivation? There are several known answers to this question in the literature. We intend to contribute to this area.

Problem B. Suppose that the function $d\colon \mathbb{R} \to \mathbb{R}$ satisfies (2). What has to be supposed additionally that the function d be real derivation? We give some answers to this question, as well.

Lajos Molnár: *Transformations of the unitary group on a Hilbert space* (Joint work with Peter Šemrl)

Let H be an infinite dimensional separable complex Hilbert space and $\mathcal{U}$ be the group of all unitary operators on H. Motivated by the algebraic properties of surjective isometries of $\mathcal{U}$ that have recently been revealed, and also by some classical results related to automorphisms of the unitary groups of operator algebras, we determine the structures of bijective transformations of $\mathcal{U}$ that respect certain algebraic operations. These are, among others, the usual product of operators, the Jordan triple product, the inverted Jordan triple product, and the multiplicative commutator. Our basic approach to obtain these results is the use of commutativity preserving transformations on the unitary group.

Gergő Nagy: *Maps on sets of density operators preserving the Holevo quantity* (Joint work with Lajos Molnár)

The Holevo quantity is an important notion in quantum information theory. It is defined as the difference between the von Neumann entropy of a convex combination of density operators and the convex combination of the entropies of these operators with the same coefficients. Let H be a finite dimensional complex Hilbert space and denote by $S(H)$ the set of all density operators on H. In the main result of this talk, we describe the structure of those maps from a dense subset of $S(H)$ to $S(H)$ which preserve the Holevo quantity for a given system of coefficients. Some related theorems will also be discussed.

Agata Nowak: *On some non-symmetric convexities*

Let $I \subseteq \mathbb{R}$ be an interval and let $f, \varphi, \psi\colon I \to \mathbb{R}$. Inspired by the last Winter Seminar we consider two non-symmetric inequalities:

$$(1)\quad \varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right) \geq \psi^{-1}\left(\mu\psi(x) + (1-\mu)\psi(y)\right) + (\lambda-\mu)(x-y)$$

$(x, y \in I)$, and

$$(2) \qquad f(py + (1-p)x) \le qf(y) + (1-q)f(x) \quad (x, y \in I,\ x < y).$$

The first one occurred in the talk of prof. Pales, the second one, which defines $(p, q)-$halfconvex functions, was dealt with in [1]. We remind origins of these inequalities and present some necessary conditions for functions φ, ψ and f under which they satisfy respectively (1) and (2).

Reference

[1] Daroczy Z., *On the equality and comparison problem of a class of mean values*, Aequationes Math. **81** (2011), 201–208.

Andrzej Olbryś: *Separation theorem for delta-convexity*

Let $(X, ||\cdot||)$ and $(Y, ||\cdot||)$ be two real Banach spaces and let D be a nonempty open and convex subset of X. In [2] Veselý and Zajiček have introduced the following

Definition. A map $F\colon D \to Y$ is called *delta-convex* if there exists a continuous and convex functional $f\colon D \to \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member y^* of the space Y^* dual to Y with $||y^*|| = 1$. If this is the case then we say that F is a delta-convex mapping with a *control function* f.

In our talk we establish the necessary and sufficient conditions under which two maps can be separated in some sense by delta-convex map. These results are related to the theorem on separation by convex functions presented in [1].

References

[1] Baron K., Matkowski J., Nikodem K., *A sandwich with convexity*, Math. Pannon. **5** (1994), 139–144.
[2] Veselý L., Zajiček L., *Delta-convex mappings between Banach spaces and applications*, Dissertationes Math. 289, Polish Scientific Publishers, Warszawa, 1989.

Zsolt Páles: *Comparison of the geometric mean with Gini means*

Gini means are classical generalizations of the Hölder or power means. In the class of Hölder means, the comparison means for two-variable or for higher number of variables are equivalent problems. The comparison problem of Gini means is a natural extension of this problem. It is known to be characterized for two variables and arbitrarily many variables by old results of the author and Daróczy–Losonczi. However, for fixed number of variables the conditions for this comparison problem form an open problem. In this talk, we only describe the necessary and sufficient conditions of the comparison of the geometric mean with an arbitrary Gini mean for fixed number of variables bigger than 2.

Barbara Przebieracz: *A proof of the Mazur–Orlicz theorem via the Markov–Kakutani common fixed point theorem, and vice versa*

We present a new proof of the Mazur–Orlicz theorem using the Markov-‑Kakutani common fixed point theorem, and a new proof of the Markov-‑Kakutani common fixed point theorem, using the Mazur–Orlicz theorem

Justyna Sikorska: *Stability of some exponential equations in Banach algebras*

The Baker's result on superstability of the exponential functional equation [1] was discussed and generalized by Roman Ger and Peter Šemrl [2].

In the present talk we investigate the generalized stability of the exponential equation as well as the stability of some of its conditional forms for functions with values in Banach algebras.

References

[1] Baker J.A., *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80** (1980), 411–416.
[2] Ger R., Šemrl P., *The stability of the exponential equation*, Proc. Amer. Math. Soc. **124** (1996), 779–787.

László Székelyhidi: *On exponential polynomials*

Exponential polynomials form a basic function class in the theory of functional equations. Solution spaces of a great number of classical functional equations are included in this class. Exponential polynomials serve also as basic building bricks of spectral analysis and spectral synthesis. Recently new characterizations of exponential polynomials have been found using ring theoretical tools. Here we present some new results in this respect, which make it possible to define and study exponential polynomials on hypergroups, too.

Patrícia Szokol: *Maps on density operators preserving f-divergences* (Joint work with Lajos Molnár and Gergő Nagy)

Classical f-divergences between probability distributions were introduced by Csiszár and by Ali and Silvey, independently. They are widely used concepts in classical information theory and statistics to measure distance or difference between probability distributions.

Recently, Hiai, Petz et al. have introduced and studied the corresponding concept of quantum f-divergence for quantum states (or density operators) in the place of probability distributions.

Let H denote a given finite dimensional Hilbert space. In this talk we present a Wigner-type result for transformations that preserve the f-divergence. Namely, for an arbitrary strictly convex function f defined on the nonnegative real line we show that every transformation on the space of all density operators on H which preserves the quantum f-divergence is implemented either by a unitary or by an antiunitary operator on H.

Adrienn Varga: *On a special class of functional equations containing weighted arithmetic means*

We study equation

$$\sum_{i=0}^{n} a_i f(b_i x + (1-b_i)y) = 0 \quad (x, y \in I) \tag{1}$$

$$\text{where } a_0, a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}, \quad \sum_{i=0}^{n} a_i = 0, \quad 0 \le b_0 < b_1 < \ldots < b_n \le 1$$

are given real parameters and $I \subset \mathbb{R}$ is a nonempty open interval. If $f\colon I \to \mathbb{R}$ is a solution of (1), then f is a a generalized polynomial of degree at most $n-1$. We give some sufficient and necessary conditions for the existence of nontrivial solutions. These conditions are based on the algebraic properties of the parameters.

References

[1] Székelyhidi L., *On a class of linear functional equations*, Publ. Math. (Debrecen) **29** (1982), 19–28.

[2] Laczkovich M., Székelyhidi G., *Harmonic analysis on discrete abelian groups*, Proc. Amer. Math. Soc. **133** (2005), 1581–1586.

[3] Varga A., Vincze Cs., *On a functional equation containing weighted arithmetic means*, International Series of Numerical Mathematics **157** (2009), 305–315.

[4] Varga A., Vincze Cs., *On Daróczy's problem for additive functions*, Publ. Math. Debrecen **75** (2009), 299–310.

[5] Varga A., *On additive solutions of a linear equation*, Acta Math Hungar. **128** (2010), 15–25.

Paweł Wójcik: *Self-adjoint operators on real Banach spaces*

The aim of this report is to discuss a functional equation

$$\rho'_+(f(x), y) = \rho'_+(x, f(y))$$

for all $x, y \in X$. If a mapping $f\colon X \to X$ satisfies this functional equation, then f must be a linear continuous operator. We will talk about of the solution of this equation in the case $X = C(M)$. Moreover, we give some new characterization of inner product spaces.

2. Problems and Remarks

1. Remark. *(On the superiority of Jensen convexity over the usual one - examples)*

(1) Each convex function is Jensen convex.
(2) Symmetry.
(3) Definition of delta convexity in the sense of L. Veselý and L. Zajiček: $F\colon D \longrightarrow Y$ is termed *delta-convex* provided that there exists a continuous convex functional $f\colon D \longrightarrow \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member y^* of the space Y^* dual to Y with $\| y^* \| = 1$.
    With the aid of Jensen differences:
    $F\colon D \longrightarrow Y$ is termed *delta-convex* provided that there exists a functional $f\colon D \longrightarrow \mathbb{R}$ such that F and f have a joint point of continuity and
$$\Big\| F\Big(\frac{x+y}{2}\Big) - \frac{F(x)+F(y)}{2} \Big\| \le \frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big).$$
(4) Note that for any continuous increasing function $i\colon \mathbb{R} \longrightarrow \mathbb{R}$ the antiderivative map
$$c(x) := \int_0^x i(t)\,dt, \quad x \in \mathbb{R},$$
yields a convex selfmapping of $\mathbb{R}$ because of the monotonicity of c'. It seems to be less known that the continuity requirement upon i is inessential here.

Proposition. *For every increasing map $i\colon \mathbb{R} \longrightarrow \mathbb{R}$ a function $c\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by the formula*
$$c(x) := \int_0^x i(t)\,dt, \quad x \in \mathbb{R},$$
is convex.

PROOF. Plainly, c is continuous; therefore, it suffices to prove that c is Jensen convex. To this end, fix arbitrarily real numbers $x, y, x < y$, to observe that due to the monotonicity of i we get

$$\int_{\frac{x+y}{2}}^{y} i(t)\,dt \geq \frac{1}{2}(y-x)\,i\Big(\frac{x+y}{2}\Big) \geq \int_{x}^{\frac{x+y}{2}} i(t)\,dt\,.$$

Consequently,

$$\begin{aligned} c(y)+c(x) &= 2\int_0^x i(t)\,dt + \int_x^{\frac{x+y}{2}} i(t)\,dt + \int_{\frac{x+y}{2}}^{y} i(t)\,dt \\ &\geq 2\int_0^x i(t)\,dt + 2\int_x^{\frac{x+y}{2}} i(t)\,dt = 2\int_0^{\frac{x+y}{2}} i(t)\,dt = 2\,c\Big(\frac{x+y}{2}\Big), \end{aligned}$$

which is the desired conclusion. □

(5) PROBLEM 11641 proposed in American Mathematical Monthly by Nicolae Bourbăcuţ (Sarmizegetusa, Romania)

Let f be a convex function from $\mathbb{R}$ into $\mathbb{R}$ and suppose that

$$f(x+y) + f(x-y) - 2f(x) \leq y^2 \tag{$*$}$$

for all real x and y.
(a) Show that f is differentiable.
(b) Show that for all real x and y,

$$|f'(x) - f'(y)| \leq |x-y|.$$

As a matter of fact, the occurrence of the quadratic function at the right hand side of inequality $(*)$ as well as the requirement that the domain spoken of is the entire real line are slightly misleading. Actually, I have proved the following much more general result.

THEOREM. *Let $\varphi\colon (a,b) \longrightarrow \mathbb{R}$ be a given differentiable function. Then each convex solution $f\colon (a,b) \longrightarrow \mathbb{R}$ of the functional inequality*

$$\frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big) \leq \frac{\varphi(x)+\varphi(y)}{2} - \varphi\Big(\frac{x+y}{2}\Big), \quad x, y \in (a,b),$$

is differentiable and the inequality

$$|f'(x) - f'(y)| \leq |\varphi'(x) - \varphi'(y)|$$

holds true for all $x, y \in (a,b)$.

Obviously, Problem 11641 refers to the case where $(a,b) = \mathbb{R}$ and $\varphi(x) = \frac{1}{2}x^2, x \in \mathbb{R}$.

Roman Ger

2. Problem. Given an open interval I, functions $f\colon I \to \mathbb{R}$ of the form $f = g - h$, where $g, h\colon I \to \mathbb{R}$ are nondecreasing functions, are characterized by the property that they are of bounded variation on any compact subinterval of I, that is, for any $[a,b] \subseteq I$,

$$V_{[a,b]}f := \sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : (t_0, t_1, \ldots, t_n) \in \mathcal{P}_{[a,b]}\right\}$$

is finite. Here $\mathcal{P}_{[a,b]}$ denotes the set of partitions of the interval $[a,b]$ defined by

$$\mathcal{P}_{[a,b]} := \bigcup_{n=1}^{\infty} \{(t_0, t_1, \ldots, t_n) : a = t_0 < t_1 < \cdots < t_n = b\}$$

This remarkable result of Camille Jordan was extended to convex differences by Frigyes Riesz. He proved that $f\colon I \to \mathbb{R}$ is of the form $f = g - h$, where $g, h\colon I \to \mathbb{R}$ are convex functions if and only if f has bounded second-order variation on any compact subinterval of I, that is, for any $[a,b] \subseteq I$,

$$V^2_{[a,b]}f := \sup\left\{\sum_{i=1}^{n-1} \left|\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} - \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}\right| : (t_0, \ldots, t_n) \in \mathcal{P}_{[a,b]}\right\}.$$

Now let us consider the problem of characterizing Jensen convex differences, i.e., functions of the form $f = g - h$, where $g, h\colon I \to \mathbb{R}$ are Jensen convex functions. For these functions, $V^2_{[a,b]}f$ is not finite in general. However, one can verify that if f is a Jensen convex difference, then the following second-order $\mathbb{Q}$-variation is finite:

$$V^{2,\mathbb{Q}}_{[a,b]}f := \sup\left\{\sum_{i=1}^{n-1} \left|\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} - \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}\right| : (t_0, \ldots, t_n) \in \mathcal{P}^{\mathbb{Q}}_{[a,b]}\right\},$$

where $\mathcal{P}^{\mathbb{Q}}_{[a,b]}$ denotes the set of $\mathbb{Q}$-partitions of the interval $[a,b]$ defined by

$$\mathcal{P}^{\mathbb{Q}}_{[a,b]} := \left\{(t_0, t_1, \ldots, t_n) \in \mathcal{P}_{[a,b]} : \frac{t_i - a}{b - a} \in \mathbb{Q},\ (i = 1, \ldots, n-1)\right\}.$$

The open problem is to show the reversed implication, that is, the finiteness of $V_{[a,b]}^{2,\mathbb{Q}} f$ for every $[a,b] \subseteq I$ implies that f is of the form $f = g - h$, where $g, h \colon I \to \mathbb{R}$ are Jensen convex functions.

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3. Remark. *(A characterization of strongly Jensen-convex functions of higher order via the Dinghas derivative)*

Motivated by some results on strongly convex and strongly Jensen-convex functions by R. Ger and K. Nikodem published in [3] and related to the talk presented by A. Gilányi during this meeting, we characterize higher order strong Jensen-convexity via the Dinghas interval derivative.

Based on the definition of E. Hopf [5] and T. Popoviciu [6], we call a function $f \colon I \to \mathbb{R}$ *Jensen-convex of order n* if it satisfies the inequality

$$\Delta_h^{n+1} f(x) \geq 0$$

for all $x \in I$, $h > 0$ such that $x + (n+1)h \in I$, where n is a positive integer, $I \subseteq \mathbb{R}$ is an interval and Δ_h denotes the well-known difference operator of increment h.

According to R. Ger and K. Nikodem [3], if c is a positive real number, a function $f \colon I \to \mathbb{R}$ is said to be *strongly Jensen-convex of order n with modulus c* if it fulfills

$$\Delta_h^{n+1} f(x) \geq c(n+1)! h^{n+1}$$

for all $x \in I$, $h > 0$ such that $x + (n+1)h \in I$.

Obviously, if $c = 0$, the second definition above gives the concept of higher order Jensen-convex functions. In the case when $n = 1$, it gives the notion of strongly Jensen-convex functions with modulus c. (Cf., e.g., [1] and [3].) The connection between strongly Jensen-convex and Jensen-convex functions of higher order was described by R. Ger and K. Nikodem in [3] in the following form.

Theorem 1. *Let n be a positive integer, c be a positive real number and $I \subset \mathbb{R}$ be an interval. A function $f \colon I \to \mathbb{R}$ is strongly Jensen-convex of order n with modulus c if and only if the function $g(x) = f(x) - cx^{n+1}$, $(x \in I)$ is Jensen-convex of order n.*

In the following, we give a characterization of strongly Jensen-convex functions of higher order via a generalized derivative introduced by A. Dinghas in [2]. The n^{th} *order lower Dinghas interval derivative* of a function $f\colon I \to \mathbb{R}$ at a point $\xi \in I$ is defined by

$$\underline{\mathrm{D}}^n f(\xi) = \liminf_{\substack{(x,y)\to(\xi,\xi)\\ x\le\xi\le y}} \left(\frac{n}{y-x}\right)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(\frac{(n-k)x+ky}{n}\right).$$

If the limit

$$\lim_{\substack{(x,y)\to(\xi,\xi)\\ x\le\xi\le y}} \left(\frac{n}{y-x}\right)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(\frac{(n-k)x+ky}{n}\right)$$

exists, we call it n^{th} *order lower Dinghas interval derivative* of f at a point $\xi \in I$ and we denote it by $\mathrm{D}^n f(\xi)$.

It is easy to see that, in the case when f is n times differentiable at ξ, we have $\mathrm{D}^n f(\xi) = f^{(n)}(\xi)$. On the other hand, there exist functions, which are not n times differentiable, but their n^{th} order Dinghas derivative exists. This means that D is a generalized derivative, indeed.

As a consequence of Corollary 1 in [4], we obtain that, for an arbitrary positive integer n, a function $f\colon I \to \mathbb{R}$ is Jensen-convex of order n on I if and only if $\underline{\mathrm{D}}^{n+1} f(\xi) \ge 0$ for all $\xi \in I$. Based on this result and Theorem 1, we can prove the following statement.

Theorem 2. *Let n be a positive integer, c be a positive real number, and $I \subseteq \mathbb{R}$ be an interval. A function $f\colon I \to \mathbb{R}$ is strongly Jensen-convex of order n with modulus c if and only if*

$$\underline{\mathrm{D}}^{n+1} f(\xi) \ge c(n+1)!$$

for all $\xi \in I$.

Finally, we formulate a simple consequence of the Theorem 2.

Corollary. *Let n be a positive integer, c be a positive real number, $I \subseteq \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$ be a function and suppose that f is $n+1$ times differentiable on I. Then f is strongly Jensen-convex of order n with modulus c if and only if $f^{(n+1)} f(\xi) \ge c(n+1)!$ for all $\xi \in I$.*

We note, that the corollary above can also be obtained as a consequence of a characterization of continuous strongly convex functions of higher order via derivatives given in Theorem 6 in [3], and the fact that in the case of continuous functions, the classes of Jensen-convex functions of order n and continuous convex functions of order n coincide.

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