Annales Mathematicae Silesianae

26 (2012)

WYDAWNICTWO UNIWERSYTETU ŚLĄSKIEGO

KATOWICE 2013

Annales Mathematicae Silesianae **26** (2012), 55–59
Prace Naukowe Uniwersytetu Śląskiego nr 3031, Katowice

GENERALIZATION OF TITCHMARSH'S THEOREM FOR THE BESSEL TRANSFORM IN THE SPACE $L_{p,\alpha}(\mathbb{R}_+)$

Mohamed El Hamma, Radouan Daher

Abstract. In this paper, we prove a generalization of Titchmarsh's theorem for the Bessel transform in the space $L_{p,\alpha}(\mathbb{R}_+)$ for functions satisfying the (ψ, p)-Bessel Lipschitz condition.

1. Introduction and preliminaries

In [2], we proved a generalization of Titchmarsh's theorem for the Bessel transform in the space $L_{2,\alpha}(\mathbb{R}_+)$. In this paper we prove this generalization in the space $L_{p,\alpha}(\mathbb{R}_+)$, where $1 < p \leq 2$ and $\alpha > -\frac{1}{2}$. For this purpose, we use a Bessel generalized translation.

$L_{p,\alpha}(\mathbb{R}_+)$, $1 < p \leq 2$, is the Banach space of measurable functions $f(t)$ on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p},$$

where α is a real number, $\alpha > -\frac{1}{2}$.

Received: 18.09.2012. Revised: 7.11.2012.
(2010) Mathematics Subject Classification: 42B10.
Key words and phrases: Bessel operator, Bessel transform, Bessel generalized translation.

Let

$$\mathrm{B} = \frac{d^2}{dx^2} + \frac{(2\alpha+1)}{x}\frac{d}{dx}$$

be the Bessel differential operator.

For $\alpha \geq -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{z}{2}\right)^{2n}, \tag{1}$$

where Γ is the gamma-function (see [4]). Moreover, from (1) we see that

$$\lim_{z\longrightarrow 0} \frac{j_\alpha(z)-1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Longrightarrow |j_\alpha(z) - 1| \geq c|z|^2. \tag{2}$$

The function $y = j_\alpha(z)$ satisfies the differential equation

$$\mathrm{B}y + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. $j_\alpha(z)$ is function infinitely differentiable, even, and, moreover entire analytic.

In $\mathrm{L}_{p,\alpha}(\mathbb{R}_+)$, consider the Bessel generalized translation T_h [4] defined by

$$\mathrm{T}_h f(t) = c_\alpha \int_0^\pi f(\sqrt{t^2+h^2-2th\cos\varphi})\sin^{2\alpha}\varphi d\varphi,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha}\varphi d\varphi\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}.$$

The Bessel transform we call the integral transform from [3, 4, 5]

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt, \ \lambda \in \mathbb{R}^+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^{\alpha}\Gamma(\alpha+1))^{-2}\int_0^{\infty}\widehat{f}(\lambda)j_{\alpha}(\lambda t)\lambda^{2\alpha+1}d\lambda.$$

The following relation connect the Bessel generalized translation and the Bessel transform, in [1] we have

$$(3)\qquad (\widehat{\mathrm{T}_h f})(\lambda) = j_{\alpha}(\lambda h)\widehat{f}(\lambda).$$

We have the Hausdorff–Young inequality

$$(4)\qquad \|\widehat{f}\|_{q,\alpha} \leq C\|f\|_{p,\alpha},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and C is a positive constant.

2. Main Result

In this section we give the main result of this paper. We need first to define (ψ, p)-Bessel Lipschitz class.

Definition 2.1. A function $f \in \mathrm{L}_{p,\alpha}(\mathbb{R}_+)$ is said to be in the (ψ, p)-*Bessel Lipschitz class*, denoted by *Lip*(ψ, α, p), if

$$\|\mathrm{T}_h f(t) - f(t)\|_{p,\alpha} = O(\psi(h)) \quad \text{as } h \longrightarrow 0,$$

where
1. $\psi(t)$ is a continuous increasing function on $[0,\infty)$,
2. $\psi(0)=0$,
3. $\psi(ts)=\psi(t)\psi(s)$ for all $t,s\in[0,\infty)$.

Theorem 2.2. *Let $f(t)$ belong to Lip(ψ,α,p). Then*

$$\int_r^{\infty}|\widehat{f}(\lambda)|^q\lambda^{2\alpha+1}d\lambda = O(\psi(r^{-q})) \quad \textit{as } r\longrightarrow +\infty.$$

Proof. Let $f \in Lip(\psi, \alpha, p)$. Then we have

$$\|\mathrm{T}_h f(t) - f(t)\|_{p,\alpha} = O(\psi(h)) \quad \text{as } h \longrightarrow 0.$$

From formulas (3) and (4), we obtain

$$\int_0^\infty |1 - j_\alpha(\lambda h)|^q |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda \leq C^q \|\mathrm{T}_h f(t) - f(t)\|_{p,\alpha}^q.$$

From (2), we have

$$\int_{\frac{\eta}{dh}}^{\frac{\eta}{h}} |1 - j_\alpha(\lambda h)|^q |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda \geq \frac{c^q \eta^{2q}}{d^{2q}} \int_{\frac{\eta}{dh}}^{\frac{\eta}{h}} |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda,$$

$d > 1$, $0 < h \leq 1$. It follows from the above consideration that there exists a positive constant K_d such that

$$\int_{\frac{\eta}{dh}}^{\frac{\eta}{h}} |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda \leq K_d \psi^q(h) = K_d \psi(h^q).$$

Then

$$\int_r^{dr} |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda \leq C_d \psi(r^{-q}) \quad \text{for all } d > 1$$

of course

$$\int_r^{d^n r} |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda = \left(\int_r^{dr} + \int_{dr}^{d^2 r} + \ldots + \int_{d^{n-1} r}^{d^n r} \right) |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda.$$

Therefore

$$\int_r^\infty |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda \leq C_d (1 + \psi(d^{-q}) + \psi^2(d^{-q}) + \ldots) \psi(r^{-q}).$$

For fixed d_0 such that $\psi(d_0^{-q}) < 1$ we have

$$\int_r^\infty |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda \leq C_1 \psi(r^{-q}),$$

where $C_1 = C_{d_0}(1 - \psi(d_0^{-q}))^{-1}$.

Finally, we get

$$\int_r^{\infty} |\widehat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda = O(\psi(r^{-q})) \quad \text{as } r \longrightarrow \infty.$$

Thus, the proof is finished. □

Acknowledgement. The authors would like to thank the referee for his valuable comments and suggestions.

References

[1] Abilov V.A., Abilova F.V., Kerimov M.K., *On Estimates for the Fourier-Bessel Integral Transform in the Space* $L_2(\mathbb{R}_+)$, Zh. Vychisl. Mat. Mat. Fiz. **49** (2009), no. 7, 1158–1166.
[2] El Hamma M., Daher R., *Generalization of Titchmarsh's Theorem for the Bessel transform*, Rom. J. Math. Comput. Sci. **2** (2012), no. 2, 17–22.
[3] Kipriyanov I.A., *Singular Elliptic Boundary Value Problems*, Nauka, Moscow, 1997 (in Russian).
[4] Levitan B.M., *Expansion in Fourier series and integrals over Bessel functions*, Uspekhi Mat. Nauk **6** (1951), no. 2, 102–143.
[5] Trimèche K., *Transmutation operators and mean-periodic functions associated with differential operators*, Math. Rep. **4** (1988), no. 1, 1–282.

Department of Mathematics
Faculty of Sciences Aïn Chock
University of Hassan II
Casablanca
Morocco
e-mail: m_elhamma@yahoo.fr
e-mail: rjdaher024@gmail.com