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ON THE ORBIT OF AN A-m-ISOMETRY

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Abstract. An A-m-isometry is a bounded linear operator T on a Hilbert space $\mathbb{H}$ satisfying an identity of the form $\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}AT^{k}=0$, where A is a positive (semi-definite) operator. In this paper, we show that the results for the supercyclicity and the hypercyclicity of m-isometries described in [6, 8] remain true for A-m-isometries.

1. Introduction and Preliminaries

A few years ago, the class of m-isometric operators attracted much attention. They have been the object of some intensive studies. The theory of these operators was investigated especially by Agler and Stankus [1, 2, 3]. Recently, in [13], A. Saddi and O.A.M. Sid Ahmed generalized the concept of m-isometry on a Hilbert space when an additional semi-inner product is considered.

In this framework, we show that many results from [6, 8] remain true if we consider an additional semi-inner product defined by a positive semi-definite operator A. We are interested in studying the orbit of an A-m-isometry.

The contents of the paper are the following. In Section 1, we give notation and results about the concept of A-m-isometries that will be useful in the sequel. In Section 2, some results about the behavior of the orbit of

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an A-m-isometry are shown. In Section 3, we focus on the supercyclicity and more generally, the N-supercyclicity of A-m-isometries. In particular it is shown that no power-bounded A-isometry is supercyclic and for invertible A, an A-m-isometry which is not an A-$(m-1)$-isometry with m is even cannot be N-supercyclic. In the closing section, we prove that an A-m-isometry is never weakly hypercyclic.

In the following, we shall introduce much of the notation of the paper and give some basic properties of A-m-isometries. For more details on such a class of operators, we refer the readers to [13].

Throughout this paper $\mathbb{H}$ represents a complex separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$. By $\mathcal{L}(\mathbb{H})$, we denote the Banach algebra of all linear operators on $\mathbb{H}$. $\mathcal{L}(\mathbb{H})^+$ represents the cone of positive (semi-definite) operators on $\mathbb{H}$ (i.e. $\mathcal{L}(\mathbb{H})^+ := \{A \in \mathcal{L}(\mathbb{H}) : \langle Au, u\rangle \geq 0, \text{ for all } u \in \mathbb{H}\}$). The null space and the range of an operator $T \in \mathcal{L}(\mathbb{H})$ are denoted respectively by $N(T)$ and $R(T)$. If $V \subset \mathbb{H}$ is a closed subspace, P_V is the orthogonal projection onto V. Finally let the set $\mathcal{L}_A(\mathbb{H})$ given by

$$\mathcal{L}_A(\mathbb{H}) := \{T \in \mathcal{L}(\mathbb{H}) : R(T^*A) \subset R(A)\}.$$

Any $A \in \mathcal{L}(\mathbb{H})^+$ defines a positive semi-definite sesquilinear form:

$$\langle\cdot,\cdot\rangle_A : \mathbb{H}\times\mathbb{H} \longrightarrow \mathbb{C}, \quad \langle u, v\rangle_A = \langle Au, v\rangle.$$

By $\|\cdot\|_A$ we denote the semi-norm induced by $\langle\cdot,\cdot\rangle_A$, i.e. $\|u\|_A = \langle u, u\rangle_A^{\frac{1}{2}}$. Observe that $\|u\|_A = 0$ if and only if $u \in N(A)$. Then $\|.\|_A$ is a norm if and only if A is an injective operator. For $T \in \mathcal{L}_A(\mathbb{H})$, set

$$\|T\|_A = \sup_{u\in\overline{R(A)},\, u\neq 0} \frac{\|Tu\|_A}{\|u\|_A} \ \ (< \infty).$$

It is straightforward to notice that

$$\|T\|_A = \sup\{|\langle Tu, v\rangle_A| : u, v \in \mathbb{H} \text{ and } \|u\|_A \leq 1, \|v\|_A \leq 1\}.$$

$\mathbb{N}$ denotes the set of non-negative integers ($\mathbb{N} = \{1, 2, 3, \ldots\}$), I is the identity operator on $\mathbb{H}$, from now A represents a nonzero ($A \neq 0$) positive operator on $\mathbb{H}$ and denote B its square root (i.e. $B = \sqrt{A}$).

Let $T \in \mathcal{L}(\mathbb{H})$, an operator $W \in \mathcal{L}(\mathbb{H})$ is called an A-adjoint of T if $AW = T^*A$. By the Douglas Theorem ([13, Theorem 1.1]), an operator $T \in \mathcal{L}(\mathbb{H})$ admits an A-adjoint if and only if $T \in \mathcal{L}_A(\mathbb{H})$. Moreover, there exists a distinguished A-adjoint operator $T^\sharp$ of T, namely, the reduced solution of the equation $AX = T^*A$, i.e. $T^\sharp = A^\dagger T^*A$, where $A^\dagger$ is the Moore–Penrose inverse of A. Recall that given $A \in \mathcal{L}(\mathbb{H})$ the Moore–Penrose inverse of A, denoted by $A^\dagger$, is defined as the unique linear extension of $\mathcal{A}^{-1}$ to $D(A^\dagger) := R(A) + R(A)^\perp$ with $N(A^\dagger) = R(A)^\perp$ where $\mathcal{A}$ is the isomorphism $A|_{N(A)^\perp} : N(A)^\perp \longrightarrow R(A)$. Moreover, $A^\dagger$ is the unique solution of the four Moore–Penrose equations: $AXA = A$, $XAX = X$, $XA = P_{N(A)^\perp}$, $AX = P_{\overline{R(A)}}|_{D(A^\dagger)}$ (for more details we refer the reader to [11]).

Definition 1.1 ([13]). An operator $T \in \mathcal{L}_A(\mathbb{H})$ is said to be *A-power bounded* if

$$\sup_{n\in\mathbb{N}} \|T^n\|_A < +\infty.$$

An I-power bounded operator is called *power-bounded*.

Since we are interested in studying the class of A-m-isometric operators, we recall the following definition.

Definition 1.2 ([13]). For $m \in \mathbb{N}$, $A \in \mathcal{L}(\mathbb{H})^+$, an operator $T \in \mathcal{L}(\mathbb{H})$ is called an *A-m-isometry* if

$$T^{*m}AT^m - \binom{m}{1} T^{*m-1}AT^{m-1} + \ldots + (-1)^{m-1}\binom{m}{m-1} T^*AT + (-1)^m A = 0,$$

where T^* denotes the adjoint operator of T.

Equivalently, for all $u \in \mathbb{H}$

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^{m-k}u\|_A^2 = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k u\|_A^2 = 0. \tag{1}$$

Remark 1.1. 1. If T satisfies (1), then it is an m-isometry with respect to the semi-norm on $\mathbb{H}$ induced by A.

2. If $A = I$, then an A-m-isometry is an m-isometry.
3. An A-1-isometry will be called an A-isometry.

For $n, k = 0, 1, 2, \ldots$, we denote

$$n^{(k)} = \begin{cases} 1, & \text{if } n = 0 \text{ or } k = 0; \\ n(n-1)(n-2)\ldots(n-k+1), & \text{otherwise.} \end{cases}$$

For $T \in \mathcal{L}(\mathbb{H})$ and $k = 0, 1, 2, \ldots$, we consider the operator

$$\beta_k(T) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} T^{*j}AT^j.$$

The symbol $S_T(n) := T^{*n}AT^n$ of T can be written

$$S_T(n) = \sum_{k=0}^{\infty} n^{(k)}\beta_k(T).$$

Observe that $\beta_0(T) = A$ and if T is an A-m-isometry, then $\beta_k(T) = 0$ for every $k \geq m$. Hence,

$$S_T(n) = \sum_{k=0}^{m-1} n^{(k)}\beta_k(T)$$

and consequently

$$\|T^n u\|_A^2 = \sum_{k=0}^{m-1} n^{(k)} \langle \beta_k(T)u, u\rangle, \quad \text{for all } u \in \mathbb{H}. \tag{2}$$

The A-covariance operator Δ_T is defined by

$$\Delta_T := \beta_{m-1}(T) = \frac{1}{(m-1)!}\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j} T^{*j}AT^j. \tag{3}$$

THEOREM 1.1 ([13]). *Let $T \in \mathcal{L}(\mathbb{H})$. If T is an A-m-isometry, then the following properties hold.*

1. *Δ_T is positive and for all $u \in \mathbb{H}$,*

$$\langle \Delta_T u, u\rangle = \sum_{k=0}^{m-1}(-1)^{m-k-1}\frac{1}{k!(m-k-1)!}\|T^k u\|_A^2.$$

2. *The null space $N(\Delta_T)$ of Δ_T is an invariant subspace for T. Moreover, if $N(\Delta_T)$ is invariant for A and $A_0 = A|_{N(\Delta_T)}$, then the restriction operator $T|_{N(\Delta_T)}$ is an A_0-$(m-1)$-isometry.*
3. *If $\mathbb{M} \subset \mathbb{H}$ is an invariant subspace for T and A such that $T|_{\mathbb{M}}$ is an $A|_{\mathbb{M}}$-$(m-1)$-isometry, then $\mathbb{M} \subset N(\Delta_T)$.*

Given $T \in \mathcal{L}(\mathbb{H})$, the orbit of a subspace $E \subset \mathbb{H}$ under T is defined by

$$Orb(T, E) := \{T^n u : u \in E,\ n \in \mathbb{N}\}.$$

In the particular case in which $E = \{u\}$ is a singleton, we write

$$Orb(T, u) := \{u, Tu, T^2u, \ldots\}.$$

DEFINITION 1.3 ([6, 8]). We say that an operator $T \in \mathcal{L}(\mathbb{H})$ is:

1. *Hypercyclic (weakly hypercyclic)* if there exists an element $u \in \mathbb{H}$ such that $Orb(T, u)$ is dense (weakly dense).
2. *Supercyclic (weakly supercyclic)* if there is an element $u \in \mathbb{H}$ such that the set

$$\mathbb{C}Orb(T, u) := \{\lambda u, \lambda Tu, \lambda T^2 u, \dots, \ \lambda \in \mathbb{C}\}$$

   (the closure with respect to the weak topology of $Orb(T, u)$) is dense.
3. N-*supercyclic* if there exists an N-dimensional subspace E of $\mathbb{H}$ such that $Orb(T, E)$ is dense.
4. *Finitely supercyclic* if there exists a finite subset $E = \{u_1, u_2, \dots, u_n\}$ such that $\bigcup_{i=1}^{n} \mathbb{C}Orb(T, u_i)$ is dense.

It is clear that a scalar multiple of a hypercyclic vector is a hypercyclic vector, but in general the sum of two hypercyclic vectors my fail to be a hypercyclic vector. Moreover, since the norm topology is strictly stronger than the weak topology, every hypercyclic operator is weakly hypercyclic, but not viceversa. The notion of hypercyclicity corresponds to the invariant subset problem as that cyclicity does to the invariant subspace problem. An operator T has no non-trivial closed invariant subset if and only if every vector $x \neq 0$ is hypercyclic for T.

2. Properties of A-m-isometries

In this section we establish some results concerning the family of A-m-isometric operators that will be useful in Section 3. We start with the following elementary result.

LEMMA 2.1. *Let $T \in \mathcal{L}(\mathbb{H})$ be invertible. If T is an A-m-isometry, then so is the operator T^{-1}.*

PROOF. If T is an A-m-isometry, then by replacing u by $T^{-m}u$ in (2), we get

$$0 = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^{-(m-k)}u\|_A^2 = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^{-k}u\|_A^2$$

which implies that T^{-1} is an A-m-isometry. $\square$

In [1, Proposition 1.23], J. Agler and M. Stankus have proved that, for an even integer m, every invertible m-isometry is also an $(m-1)$-isometry. The following result shows that this property is also satisfied by the class of A-m-isometries.

Proposition 2.1. *If T is an invertible A-m-isometry and m is even, then T is an A-$(m-1)$-isometry.*

Proof. Since T is invertible and m is even, for $u \in \mathbb{H}$

$$\begin{aligned} \langle -\Delta_T u, u\rangle &= \lim_{n \longrightarrow -\infty} -\frac{1}{n^{(m-1)}} \langle S_T(n)u, u\rangle \\ &= \lim_{n \longrightarrow -\infty} -\frac{1}{n^{(m-1)}} \|T^n u\|_A^2 \geq 0. \end{aligned}$$

This implies that $-\Delta_T \geq 0$. According to Theorem 1.1, the operator Δ_T is positive, that is $\Delta_T \geq 0$. Hence, $\Delta_T = 0$. This implies that

$$\sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} T^{*j} A T^j = 0$$

which means that T is an A-$(m-1)$-isometry. $\square$

Corollary 2.1. *Let $T \in \mathcal{L}(\mathbb{H})$ be an A-2-isometry. If T is invertible, then*

1.

$$T^{*p} A T^p = A \quad \textit{for all } p \geq 0. \tag{4}$$

2. *If T is such that $||T|| = 1$, then BT is hyponormal.*

Proof. 1. By hypothesis, T is an invertible A-2-isometry. According to Proposition 2.1, T is an A-isometry. Hence,

$$T^* A T = A.$$

An induction argument yields to $T^{*p} A T^p = A$ for all $p \geq 0$, which finishes the proof.

2. The relation (4) for $p = 1, 2$ implies that T is an A-quasi-isometry and [14, Corollary 3.6] allows to conclude.

$\square$

Remark 2.1. Combining Corollary 2.1 and [13, Lemma 3.3], we deduce that if T is an A-2-isometry, then
1. $T^{*p}AT^p = pT^*AT - (p-1)A,$ for all $p \geq 0.$
2. $T^{*p}AT^p = A,$ for all $p \geq 0,$ if T is invertible.

Moreover, the equality 2. when $A = I$ and $p = 1$ yields that every invertible 2-isometry is a unitary operator.

We have also the next result.

Proposition 2.2. *Let $T \in \mathcal{L}(\mathbb{H})$ be a surjective A-isometry. Then the following statements hold:*
1. *$R(A) = R(T^*A)$.*
2. *If T is injective and $T(R(A)) \subset \overline{R(A)}$, then T is A-normal (i.e. $T^\sharp T = TT^\sharp$).*

Proof. 1. Since T is an A-isometry, $T^*AT = A$. Thus, $R(T^*A) \subset R(A)$. Using Douglas Theorem ([13], Theorem 1.1), we obtain $R(A) \subset R(T^*A)$. So part one is proved.
2. We have proved that $R(T^*A) \subset R(A)$. Then T admits an A-adjoint operator $T^\sharp$, moreover, $T^\sharp T = A^\dagger T^*AT = A^\dagger A = P_{\overline{R(A)}}$, where $A^\dagger$ is the Moore-Penrose inverse of A. On the other hand, since T is invertible, $TT^\sharp = TA^\dagger T^*A = TA^\dagger AT^{-1} = TP_{\overline{R(A)}}T^{-1}$. If moreover $\overline{R(A)}$ is invariant for T, so $TT^\sharp = P_{\overline{R(A)}}TT^{-1} = P_{\overline{R(A)}} = T^\sharp T$ which yields the assertion. □

Faghih and Hedayatian ([8, Theorem 1]) proved that, for every vector u, the orbit $Orb(T, u)$ of an m-isometry T is norm increasing or norm decreasing, except possibly for a finite number of terms. Moreover, T. Bermúdez et al ([6, Proposition 2.2]) showed that the orbit is always eventually norm increasing. In the following we obtain that, under additional assumption, this property holds true for A-m-isometric operators.

Define the operators $(T_j)_{0\leq j\leq m-1}$, inductively, by

$$T_0 = T \text{ and } T_j = T_{j-1}|_{N(\beta_{m-j}(T_{j-1}))},\ j = 1, 2, \ldots, m-1.$$

Consider the following assumption

$$(\mathrm{H_i}): \quad \text{For all } j = 1, 2, \ldots, m-(i+1),\ N(\beta_{m-j}(T_{j-1})) \text{ is invariant for } A$$

where $1 \leq i \leq m-1$.

LEMMA 2.2. *Let $T \in \mathcal{L}_A(\mathbb{H})$ be an A-m-isometry with $m > 1$, $u \in \mathbb{H}$ and k be the largest integer with $1 \leq j \leq m-1$ such that $\langle \beta_j(T)u, u\rangle \neq 0$. If T satisfies (H_k), then $\langle \beta_k(T)u, u\rangle > 0$.*

PROOF. For $k = m-1$ the statement is true from Theorem 1.1-(1.). If $k \neq m-1$, for $j \in \{1, \ldots, m-(k+1)\}$, Theorem 1.1-(2.) implies that $N(\beta_{m-j}(T_{j-1}))$ is invariant for T_{j-1}, since by hypothesis it is also invariant for A, so T_j is an A-$(m-j)$-isometry which, in turn, implies that $\beta_{m-j-1}(T_j)$ is a positive operator. This coupled with the fact that

$$\langle \beta_{m-j}(T_{j-1})u, u\rangle = \langle \beta_{m-j}(T)u, u\rangle = 0$$

shows that $u \in N(\beta_{m-j}(T_{j-1}))$. In particular, T_{m-k-1} is an A-$(k+1)$-isometry, and so $\beta_k(T_{m-k-1})$ is a positive operator. Moreover, since the vector $u \in N(\beta_{k+1}(T_{m-k-2}))$ and we assumed that $\langle \beta_k(T)u, u\rangle \neq 0$, we get

$$\langle \beta_k(T)u, u\rangle = \langle \beta_k(T_{m-k-1})u, u\rangle > 0. \qquad \square$$

Now, we are in position to prove the following theorem.

THEOREM 2.2. *Let $T \in \mathcal{L}_A(\mathbb{H})$ be an A-m-isometry and $u \in \mathbb{H}$. Then except possibly for a finite number of terms, Orb(T,u) satisfies*

$$\|T^{n+1}u\|_A^2 - \|T^n u\|_A^2 \geq 0, \quad n \geq 0$$

provided that T satisfies (H_k), where k is the largest integer with $1 \leq j \leq m-1$ such that $\langle \beta_j(T)u, u\rangle \neq 0$.

PROOF. If $m = 1$, then the result is obvious. Assume that $m > 1$ and let $u \in \mathbb{H}$ such that $\langle \beta_j(T)u, u\rangle = 0$ for $j = 1, 2, ..., m-1$. Then T is an A-isometry and we have

$$\|T^{n+1}u\|_A^2 - \|T^n u\|_A^2 = 0, \quad n = 0, 1, 2, \ldots$$

Otherwise, for every positive integer n, using (2), we observe that

$$\begin{aligned}\|T^{n+1}u\|_A^2 - \|T^n u\|_A^2 &= \langle (S_T(n+1) - S_T(n))u, u\rangle \\ &= \sum_{i=0}^{k} [(n+1)^{(i)} - n^{(i)}]\langle \beta_i(T)u, u\rangle.\end{aligned}$$

Consequently, from Lemma 2.2, we get

$$\lim_{n\longrightarrow+\infty}\frac{\|T^{n+1}u\|_A^2-\|T^nu\|_A^2}{(n+1)^{(k)}-n^{(k)}}=\langle\beta_k(T)u,u\rangle>0.$$

Hence, there exists a positive integer n_0 so that the sequence $\left(\|T^nu\|_A^2\right)_{n\geq n_0}$ is strictly increasing. □

THEOREM 2.3. *Let $T\in\mathcal{L}_A(\mathbb{H})$ be an A-m-isometry. If for a strictly increasing sequence $(n_i)_{i\geq1}$ of positive integers, there exists a constant C such that for all $u\in\mathbb{H}$,*

$$\|T^{n_i}u\|_A\leq C,\quad i=1,2,\dots,$$

then T is an A-isometry.

PROOF. Let $u\in\mathbb{H}$. If $m=1$, then the result is obvious. Let $m>1$. According to (2), we have

$$\sum_{k=0}^{m-1}n_i^{(k)}\langle\beta_k(T)u,u\rangle=\|T^{n_i}u\|_A^2\leq C^2,\quad i=1,2,\dots \tag{5}$$

On the other hand, if $\langle\beta_j(T)u,u\rangle\neq0$ for some j with $1\leq j\leq m-1$, then

$$\lim_{i\longrightarrow+\infty}\sum_{k=0}^{m-1}n_i^{(k)}\langle\beta_k(T)u,u\rangle=\infty.$$

According to (5), this yields a contradiction. Hence, $\langle\beta_j(T)u,u\rangle=0$ for every j with $1\leq j\leq m-1$. This implies that T is an A-isometry. □

PROPOSITION 2.3. *Let $T\in\mathcal{L}_A(\mathbb{H})$ be an A-m-isometry. The following properties hold true.*

1. *Suppose that a subsequence of $\left(\|T^n\|_A\right)_{n\geq1}$ is bounded. Then T is an A-isometry. In particular, every A-power bounded A-m-isometry is an A-isometry.*
2. *If T is not an A-isometry, then,*
   (a) *$\|T^n\|_A>1$, for all $n\geq1$.*
   (b) *$\|T^{-n}\|_A>1$, for all $n\geq1$, if T is invertible.*

PROOF. 1. It is a consequence of Theorem 2.3.

2. (a) Suppose that there exists $k \in \mathbb{N}$ such that $\|T^k\|_A \leq 1$. Hence

$$\begin{aligned}\|T^{nk}\|_A &= \|\underbrace{T^k \circ T^k \circ \ldots \circ T^k}_{n \text{ times}}\|_A \\ &\leq \|T^k\|_A \ldots \|T^k\|_A \leq 1, \text{ for all } n \in \mathbb{N}.\end{aligned}$$

This implies according to 1., that T is an A-isometry which is a contradiction.

(b) Referring to Lemma 2.1 and using the same arguments used for the case of T, the assertion holds. □

Denote by B_A the closed set on $\mathbb{H}$ given by $B_A = \{x \in \mathbb{H} : \|x\|_A \leq 1\}$. In the next result, we show that if T is an A-m-isometry, then $\|T^n\|_A^2$ has the same behavior as n^{m-1}.

Proposition 2.4. *Let $T \in \mathcal{L}_A(\mathbb{H})$ be an A-m-isometry. The following properties are satisfied:*

1. $\frac{\|T^n u\|_A^2}{n^{m-1}}$ *converges uniformly on B_A to $\langle \Delta_T u, u\rangle$.*
2. $\frac{\|T^n\|_A^2}{n^{m-1}}$ *converges to $\sup\limits_{u \in B_A} \langle \Delta_T u, u\rangle$.*

Proof. 1. Note first that if $T \in \mathcal{L}_A(\mathbb{H})$, then $\|Tu\|_A \leq \|T\|_A \|u\|_A$ for all $u \in \mathbb{H}$. By (2) we have

$$\frac{\|T^n u\|_A^2}{n^{m-1}} - \langle \Delta_T u, u\rangle = \left(\frac{n^{(m-1)}}{n^{m-1}} - 1\right) \langle \Delta_T u, u\rangle + \sum_{k=0}^{m-2} \frac{n^{(k)}}{n^{m-1}} \langle \beta_k(T) u, u\rangle \to 0$$

as $n \to \infty$. Moreover, given $\varepsilon > 0$ and $u \in B_A$,

$$\begin{aligned}\left|\frac{\|T^n u\|_A^2}{n^{m-1}} - \langle \Delta_T u, u\rangle\right| &\leq \left(\frac{n^{(m-1)}}{n^{m-1}} - 1\right) |\langle \beta_{m-1}(T) u, u\rangle| \\ &\quad + \sum_{k=0}^{m-2} \frac{n^{(k)}}{n^{m-1}} |\langle \beta_k(T) u, u\rangle| \\ &\leq \left(\frac{n^{(m-1)}}{n^{m-1}} - 1\right) \sum_{k=0}^{m-1} \frac{1}{k!(m-k-1)!} \|T^k u\|_A^2 \\ &\quad + \sum_{k=0}^{m-2} \frac{n^{(k)}}{n^{m-1}} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} \|T^j u\|_A^2\end{aligned}$$

$$\leq \left(\frac{n^{(m-1)}}{n^{m-1}} - 1\right) \sum_{k=0}^{m-1} \frac{1}{k!(m-k-1)!} C$$
$$+ \sum_{k=0}^{m-2} \frac{n^{(k)}}{n^{m-1}} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} C < \varepsilon$$

for all sufficiently large n, where $C = \max_{0 \leq k \leq m-1} \|T^k\|_A^2$. Hence the convergence is uniform on B_A.

2. Since the convergence of $\frac{\|T^n u\|_A^2}{n^{m-1}}$ to $\langle \Delta_T u, u\rangle$ is uniform on B_A, we obtain

$$\lim_{n\longrightarrow\infty} \frac{\|T^n\|_A^2}{n^{m-1}} = \lim_{n\longrightarrow\infty} \sup_{u\in B_A} \frac{\|T^n u\|_A^2}{n^{m-1}} = \sup_{u\in B_A} \lim_{n\longrightarrow\infty} \frac{\|T^n u\|_A^2}{n^{m-1}}$$
$$= \sup_{u\in B_A} \langle \Delta_T u, u\rangle. \qquad \square$$

3. Supercyclicity and N-supercyclicity of A-m-isometries

The concept of supercyclicity was introduced by Hilden and Wallen in [10]. In this section, we will prove that the results established in [6, 8] for m-isometric operators remain true for A-m-isometries. It is not difficult to show that an operator $T \in \mathcal{L}(\mathbb{H})$ is supercyclic if and only if for each pair of non-empty open subsets U and V of $\mathbb{H}$, there is a nonzero scalar λ and a positive integer k such that $\lambda T^k(U) \cap V \neq \emptyset$. In particular, this characterization leads to the following property.

Proposition 3.1. *If $T \in \mathcal{L}(\mathbb{H})$ is an invertible supercyclic operator, then T^{-1} is also supercyclic.*

Proof. Let $T \in \mathcal{L}(\mathbb{H})$ be an invertible operator and U and V two non-empty open subsets of $\mathbb{H}$. If T is supercyclic then there is $\lambda \neq 0$ and an integer $k \geq 1$ such that $U \cap \lambda T^k(V) \neq \emptyset$. Since T is invertible then $T^{-k}(U) \cap \lambda V = T^{-k}(U \cap \lambda T^k(V)) \neq \emptyset$. This implies that $\frac{1}{\lambda} T^{-k}(U) \cap V \neq \emptyset$ and T^{-1} is supercyclic. $\square$

We start with the following result.

Theorem 3.1. *A power bounded A-isometry cannot be supercyclic.*

PROOF. Let $T \in \mathcal{L}(\mathbb{H})$ be a power bounded A-isometry. Suppose T is supercylic. Let $x \in \mathbb{H}$ be a supercyclic vector for T. Let $y \in \mathbb{H}$. There exists a sequence $(\lambda_i)_i \subset \mathbb{C}$ and a strictly increasing sequence $(n_i) \subset \mathbb{N}$ such that

$$\lim_{i\to+\infty} \lambda_i T^{n_i} x = y. \tag{6}$$

Hence $\lim\limits_{i\to+\infty} \lambda_i \sqrt{A} T^{n_i} x = \sqrt{A} y$ and $\lim\limits_{i\to+\infty} |\lambda_i| ||T^{n_i} x||_A = ||y||_A$. Since T is an A-isometry,

$$\lim_{i\to+\infty} |\lambda_i| ||x||_A = ||y||_A. \tag{7}$$

Note that x cannot be in $N(A)$, otherwise $A = 0$. If A is not injective then by choosing $y \in N(A) \setminus \{0\}$, it is easy to note from (7) that the sequence $(\lambda_i)_i$ converges to zero. From (6) it follows that $||T^{n_i} x|| \longrightarrow +\infty$ which yields a contradiction. If A is injective, set $y \neq 0$, from (7) the limit $\lim\limits_{i\to\infty} |\lambda_i|$ exists and is nonzero, it follows $(||T^{n_i} x||)_i$ converges and $T^{n_i} x \nrightarrow 0$. This contradicts [4, Theorem 2.2]. Hence the proof is achieved. $\square$

REMARK 3.1. 1. If A is invertible then an A-isometry is automatically power bounded. In which case Proposition 2.3 states that if T is an A-power bounded A-m-isometry, then T is not supercyclic.

2. Recently, it has been proven that finitely supercyclic operators are supercyclic (see [12]). In a way, by studying the supercyclicity we are giving a characterization of operators belonging to such a family.
3. It is known that the set of all supercyclic vectors of a supercyclic operator is dense. On the other hand, it is noted in the proof of Theorem 3.1 that a supercyclic vector (if it exists) of an A-isometry cannot be in $N(A)$. Hence if interior$(N(A)) \neq \phi$ then an A-isometry is never supercyclic.

The following example gives a non power-bounded A-isometry which is not supercyclic.

EXAMPLE 3.1. Let H be a separable Hilbert space with an orthonormal basis $\{e_n,\ n \geq 1\}$. Let $T \in \mathcal{L}(H)$ be the unilateral weighted shift defined by $Te_n = w_n e_{n+1}$ where $w_n = \sqrt{\frac{n+1}{n}}$, $n \geq 1$. Let $A \in \mathcal{L}(H)$ the positive operator given by $Ae_n = \frac{1}{n} e_n$, $n \geq 1$. It is not difficult to verify that T is a 2-isometry and not supercyclic. Moreover, T is a non power-bounded A-isometry.

This example and Theorem 3.1 lead naturally to the following question.

QUESTION 1. Is it true that an A-isometry cannot be supercyclic?

The theorem below shows that certain classes of A-m-isometries are not supercyclic.

THEOREM 3.2. *Let $T \in \mathcal{L}_A(\mathbb{H})$. If the following properties hold*
1. *T is an A-m-isometry,*
2. *for any $x \in \mathbb{H}$ there exists $n_0 \geq 0$ such that the sequence $\big(||T^n x||\big)_{n \geq n_0}$ is increasing,*

then T is not supercyclic.

PROOF. We argue by contradiction. Assume that T is a supercyclic A-m-isometry with $y \in \mathbb{H}$ a supercyclic vector. Then, for each $x \in \mathbb{H}$, we have

$$\lim_{j \longrightarrow +\infty} \mu_j T^{m_j} y = x$$

where $(m_j)_j$ is a sequence of positive integers and $(\mu_j)_j$ is a sequence of scalars. Using the condition 2. one gets

$$\lim_{j \longrightarrow +\infty} |\mu_j| \; \|T^{m_j} y\| \leq \lim_{j \longrightarrow +\infty} |\mu_j| \; \|T^{m_j+1} y\|.$$

This inequality gives

$$\|x\| \leq \|Tx\| \tag{8}$$

which implies that T is injective with closed range. Since T is supercyclic, T is invertible and T^{-1} is power bounded. Lemma 2.1 and Proposition 3.1 imply that the operator T^{-1} is a power bounded, supercyclic A-m-isometry. Replacing x by Bx in (8) one obtains $\|x\|_A \leq \|TBx\|$, thus $||T^{-n}||_A \leq ||TB||||I||_A||T^{-n}||$, $n = 1, 2, , ,$. Proposition 2.3 shows that T^{-1} is a supercyclic A-isometry, Theorem 3.1 yields a contradiction. Hence the result follows. □

In [7] Bourdon proves that a hyponormal operator cannot be supercyclic, then in [9] Feldman shows that a normal operator cannot be N-supercyclic. In [5] Bayart and Matheron give a generalization by proving that hyponormal operators are never N-supercyclic. In particular, if T is an invertible A-2-isometry with $||T|| = 1$, then Corollary 2.1 implies that the operator BT cannot be N-supercyclic. Here we discuss the N-supercyclicity of certain classes of A-m-isometries.

THEOREM 3.3. *If A is invertible, T is an A-m-isometry which is not an A-$(m-1)$-isometry with m even, then T is not N-supercyclic.*

PROOF. We reason with contradiction. Assume that T is an N-supercyclic A-m-isometry with m even. Then [13, Proposition 4.1] implies that T is an invertible A-m-isometry with m even. According to Proposition 2.1, T is an A-$(m-1)$-isometry, which is impossible and the proof is finished. □

To simplify, denote $X = (\mathbb{H}, ||.||)$, $Y = (\mathbb{H}, ||.||_A)$ and for $T \in \mathcal{L}_A(\mathbb{H})$ and $m \geq 1$, put

$$\Delta_T^{\sharp} := \frac{1}{(m-1)!} \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} T^{\sharp j} T^j. \tag{9}$$

It is known that Y is a Hilbert space if and only if A is invertible (see [14]). In what case according to Remark 1.1, an operator is A-m-isometry in X if and only if it is an m-isometry in Y. The next result generalizes [6, Theorem 3.4].

THEOREM 3.4. *Let $T \in \mathcal{L}_A(\mathbb{H})$ be an A-m-isometry. If A is invertible and Δ_T is injective, then T is not N-supercyclic.*

PROOF. Note first that the Adjoint of an operator $T \in \mathcal{L}_A(\mathbb{H})$ in Y is $T^{\sharp}$ (i.e. the A-adjoint in X). Moreover $A\Delta_T^{\sharp} = \Delta_T$, thus Δ_T is injective if and only if is so $\Delta_T^{\sharp}$. If T is an A-m-isometry in X, then T is an m-isometry in Y. Since $\Delta_T^{\sharp}$ is injective, [6, Theorem 3.4] implies that T is not N-supercyclic in Y then T is not N-supercyclic in X. □

COROLLARY 3.5. *Assume that A is invertible. Then*
1. *An A-isometry is never N-supercyclic.*
2. *An A-2-isometry is never N-supercyclic.*

PROOF. 1. It follows from Theorem 3.4, since in what case $\Delta_T = A$.
2. If T is an A-isometry, (1.) gives the result. If T is not an A-isometry, then it suffices to apply Theorem 3.3. □

Next we give an example of an A-3-isometry which is neither A-2-isometry nor supercyclic.

EXAMPLE 3.2. Let H be a separable Hilbert space with an orthonormal basis $\{e_n,\ n \geq 1\}$. Let $T,\ A \in \mathcal{L}(H)$ where T is the unilateral weighted shift defined by $Te_n = \sqrt{\frac{n+3}{n}} e_{n+1}$ and A is the invertible positive operator given by $Ae_n = \frac{1}{n+1}e_n$, $n \geq 1$. It is not difficult to prove that T is an A-3-isometry which is neither an A-2-isometry nor a 3-isometry. On the other hand T is a D-isometry ($De_n = \frac{1}{n(n+1)(n+2)}e_n$, $n \geq 1$) then it is not N-supercyclic.

This leads naturally to the following question.

QUESTION 2. Is it true that an A-3-isometry cannot be N-supercyclic?

4. Weak hypercyclicity of A-m-isometries

Several necessary conditions exist for an operator to be weakly hypercyclic. For example, if $||T|| \leq 1$ or $\sup\limits_n ||T^n|| < +\infty$, then every orbit is norm bounded, and hence can never be norm dense or weakly dense. Hence, if T is weakly hypercyclic, then either $||T|| > 1$ or $\sup\limits_n ||T^n|| = +\infty$. Another necessary condition for an operator T to be weakly hypercyclic is that its adjoint T^* has no eigenvalues. The purpose of this section is to see what about the weak hypercyclicity of A-m-isometries. An immediate consequence of Corollary 3.5, is that if A is invertible then neither A-isometries nor A-2-isometries are hypercyclic. In the next theorem, we show that they are not even weakly hypercyclic.

THEOREM 4.1. *Let $T \in \mathcal{L}(\mathbb{H})$. If T is an A-isometry or an A-2-isometry, then T cannot be weakly hypercyclic.*

PROOF. 1. We argue by contradiction. Suppose that T is a weakly hypercyclic A-isometry. There exists $x \in \mathbb{H} \setminus N(A)$, such that for any $y \in \mathbb{H}$ there exists an increasing sequence $(n_i) \subset \mathbb{N}$ such that

$$\langle T^{n_i}x, Ay\rangle \to \langle y, Ay\rangle, \ \ i \to +\infty$$

Since T is an A-isometry, then by Cauchy–Schwarz inequality, one gets $||x||_A \geq ||y||_A$ for all $y \in \mathbb{H}$. This is impossible.

2. Suppose that T is an A-2-isometry. Then

$$T^*(T^*AT - A)T = T^{*2}AT^2 - T^*AT = T^*AT - A \tag{10}$$

If $\triangle_T = 0$ then T is an A-isometry and if $\triangle_T \neq 0$ then from (10) and Theorem 1.1, T will be a $\triangle_T$-isometry. Hence in both cases T cannot be weakly hypercyclic.

□

We are now ready to state the main result of this section in which we generalize [8, Theorem 4].

Theorem 4.2. *Let $T \in \mathcal{L}(\mathbb{H})$ be an A-m-isometry. Then T is not weakly hypercyclic.*

Proof. We have already seen in Theorem 4.1 that the result holds for $m = 1, 2$. Let $m > 2$ and assume, to the contrary, that T is a weakly hypercyclic A-m-isometry with a weakly hypercyclic vector u. Then (2) holds for $n = 1, 2, , ,$. If $\beta_{m-1}(T)u \neq 0$, then the positivity of $\beta_{m-1}(T)$ shows that $\langle \beta_{m-1}(T)u, u\rangle > 0$. This, in turn, implies the convergence of the series $\sum_{n=1}^{\infty} \|T^n u\|_A^{-2}$ so is of the series $\sum_{n=1}^{\infty} \|T^n u\|^{-2}$. Thus, in view of [8, Proposition 1]) we get a contradiction. Hence, $\beta_{m-1}(T)u = 0$. Since for every n, $T^n u$ is also a weakly hypercyclic vector for T, we see that $\beta_{m-1}(T)T^n u = 0, n = 0, 1, 2, \ldots$ This along with the fact that $N(\beta_{m-1}(T))$ is weakly closed, implies that $\beta_{m-1}(T) = 0$. Hence, T is an A-$(m-1)$-isometry. Continuing the above process, we finally conclude that T is an A-2-isometry, which is impossible. □

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