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THE GENERALIZED INFIMAL CONVOLUTION CAN BE USED TO NATURALLY PROVE SOME DOMINATED MONOTONE ADDITIVE EXTENSION THEOREMS

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Abstract. By using a particular case of the generalized infimal convolution, we provide an instructive proof for a particular case of a dominated monotone additive extension theorem of Benno Fuchssteiner.

Introduction

In [15], by using a particular case of the infimal convolution [29]

$$h(x) = (f * g)(x) = \inf\big\{ f(u) + g(v) : (u, v) \in D_f \times D_g : x = u + v \big\},$$

we have naturally proved the following classical Hahn–Banach theorem [7].

Theorem 1. *If p is a positively homogeneous subadditive function of a real vector space X to $\mathbb{R}$ and φ is linear function of a subspace V of X to $\mathbb{R}$ such that φ is dominated by p, then there exists a linear function f of X to $\mathbb{R}$ that extends φ and is dominated by p.*

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Now, by using a particular case of the generalized infimal convolution [32]

$$h(x) = (f * g)(x) = \inf\big\{ f(u) + g(v) : (u, v) \in D_f \times D_g : x \leq u + v \big\}$$

and its homogenization

$$h^*(x) = \inf_{n \in \mathbb{N}} n^{-1} h(nx),$$

we shall naturally prove the following particular case of a dominated monotone additive extension theorem of Fuchssteiner [10].

Theorem 2. *If p is an increasing subadditive function of a commutative preordered group X to $\mathbb{R}$ and φ is and additive function of a subgroup V of X to $\mathbb{R}$ such that φ is dominated by p, then there exists an increasing additive function f of X to $\mathbb{R}$ that extends φ and is dominated by p.*

This theorem can be used to easily prove the following straightforward generalization of a weakening of a monotone linear extension theorem of Bauer, Bonsall and Namioka. (See [19, p. 24].)

Theorem 3. *If φ is an increasing additive function of a cofinal subgroup V of a commutative preordered group X to $\mathbb{R}$, then there exists an increasing additive function f of X to $\mathbb{R}$ that extends φ.*

A detailed examination of an example of Jameson [19, p. 25] will show that even a strictly increasing linear function of a non-cofinal subspace of a two dimensional partially ordered vector space need not have an increasing additive extension to the whole space.

1. Hahn–Banach extensions and the infimal convolution

Notation 1.1. *Suppose that X is a commutative preordered group and p is an increasing subadditive function of X to $\mathbb{R}$ such that $p(0) = 0$.*

Moreover, assume that V is a subgroup of X and φ is an additive function of V to $\mathbb{R}$ such that φ is dominated by p in the sense that $\varphi(v) \leq p(v)$ for all $v \in V$.

Remark 1.2. Note that thus the inequality relation in X is only assumed to be reflexive and transitive. Moreover, to guarantee the compatibility of the addition and inequality in X, it is enough to assume only that $x \leq y$ implies $x+z \leq y+z$ for all $z \in X$.

In this respect it also worth noticing that, by the assumed subadditivity and real-valuedness of p, we necessarily have $p(0) = p(0+0) \leq p(0) + p(0)$, and thus $0 \leq p(0)$. Therefore, to guarantee the equality $p(0) = 0$, it is enough to assume only that $p(0) \leq 0$.

Remark 1.3. Note that if $x \in X$ such that $x \leq 0$, then $p(x) \leq p(0) = 0$. Therefore, if in particular p is nonnegative, then we necessarily have $p(x) = 0$.

Quite similarly, we can note that if $0 \leq x$, and thus $-x \leq 0$, then $0 = p(0) \leq p(x)$ and $p(-x) \leq p(0) = 0$. Therefore, if in particular p is even, then we necessarily have $p(x) = 0$.

This shows that the increasingness of p is, in general, a rather restrictive property. However, note that the equality relation on X is always a compatible partial order relation on X for which p is increasing.

Moreover, if following [28, Definition 1.9] of S. Simons, for any $x, y \in X$, we define $x \preccurlyeq y$ if $p(x-y) \leq 0$, then $\preccurlyeq$ is the largest compatible preorder relation on X for which p is still increasing.

In connection with our former assumptions on p and φ, we can also easily establish the following counterparts of [2, Lemma 1.7 and Corollary 1.8] of B. Anger and J. Lembcke.

Theorem 1.4. *If σ is a subadditive function of X to $\mathbb{R}$, then the following assertions are equivalent:*

(1) *σ is increasing and $\sigma(x) \leq 0$;*
(2) *$\sigma(x) \leq 0$ for all $x \in X$ with $x \leq 0$;*
(3) *$\sigma \leq \rho$ for some increasing function ρ of X to $\mathbb{R}$ with $\rho(0) \leq 0$.*

Proof. If (1) holds, then (3) trivially holds with $\rho = \sigma$. While, if (3) holds and $x \in X$ such that $x \leq 0$, then we can at once see that $\sigma(x) \leq \rho(x) \leq \rho(0) \leq 0$. Therefore, (2) also holds.

Now, it remains to show only that (2) also implies (1). For this, note that if $x, y \in X$ such that $x \leq y$, then $x - y \leq 0$. Hence, if (2) holds, we can infer that $\sigma(x-y) \leq 0$. Now, by the subadditivity of σ, it is clear that

$$\sigma(x) = \sigma(x-y+y) \leq \sigma(x-y) + \sigma(y) \leq 0 + \sigma(y) = \sigma(y).$$

Therefore, σ is increasing. Moreover, if (2) holds, then because of $0 \leq 0$ we also have $\sigma(0) \leq 0$. Therefore, (1) also holds. □

From our former assumptions on p and φ, by using this theorem, we can immediately derive

Corollary 1.5. *φ is increasing.*

Proof. Note that Theorem 1.4 can be applied by taking V in place of X, φ in place of σ and $p|V$ in place of ρ. $\square$

Definition 1.6. If U is a subgroup of X such that $V \subset U$, then an additive function ψ of U to $\mathbb{R}$, that extends φ and is dominated by p, will be called a *Hahn–Banach extension* of φ to U.

Remark 1.7. In the sequel, we shall actually be interested in the Hahn–Banach extensions f of φ to the whole of X. However, to prove the existence of total Hahn–Banach extensions, we shall need that of some partial ones.

Definition 1.8. For any $x \in X$, we define

$$\Gamma(x) = \big\{(u, v) \in X \times V : x \le u + v\big\}.$$

Remark 1.9. Thus, X is the domain of Γ. Namely, if $x \in X$, then because of $x \le x = x + 0$ and $0 \in V$, we have $(x, 0) \in \Gamma(x)$. Thus, in particular $\Gamma(x) \neq \emptyset$.

Moreover, it is also worth noticing that Γ is decreasing, $\mathbb{N}$-superhomogeneous and superadditive in the sense that $\Gamma(y) \subset \Gamma(x)$ for all $x, y \in X$ with $x \le y$,

$$n\Gamma(x) \subset \Gamma(nx) \quad \text{and} \quad \Gamma(x) + \Gamma(y) \subset \Gamma(x + y)$$

for all $n \in \mathbb{N}$ and $x, y \in X$. Note that the $\mathbb{N}$-superhomogeneity is a consequence of the superadditivity.

Definition 1.10. For any $x \in X$, we define

$$q(x) = (p * \varphi)(x) = \inf\big\{p(u) + \varphi(v) : (u, v) \in \Gamma(x)\big\}.$$

Remark 1.11. Note that if $x \in X$, then because of $\Gamma(x) \neq \emptyset$, we have $q(x) \neq +\infty$. In the next section, we shall show that $q(x) \neq -\infty$ also holds.

The function q is a generalized infimal convolution of p and φ such that

$$\begin{aligned} q(x) &= \inf\big\{p(u) + \varphi(v) : (u, v) \in X \times V : x \le u + v\big\} \\ &\le \inf\big\{p(u) + \varphi(v) : (u, v) \in X \times V : x = u + v\big\} \\ &= \inf_{v \in V} \big(p(x - v) + \varphi(v)\big) \end{aligned}$$

for all $x \in X$.

For the origins, generalizations and applications of the infimal convolution, see [21, 29, 15, 32] and the references therein.

Definition 1.12. For any function ρ of X to $\mathbb{R}$ and $x \in X$, we define

$$\check{\rho}(x) = \rho(-x) \quad \text{and} \quad \hat{\rho}(x) = -\rho(-x).$$

Remark 1.13. Clearly, ρ is even if and only if $\rho = \check{\rho}$, and ρ is odd if and only if $\rho = \hat{\rho}$. Thus, in particular we have $\varphi = \hat{\varphi}$.

Moreover, the mapping $\rho \mapsto \check{\rho}$ is increasing and the mapping $\rho \mapsto \hat{\rho}$ is decreasing. Furthermore, we have $\hat{\rho} = -\check{\rho}$, and $\check{\check{\rho}} = \rho$ and $\hat{\hat{\rho}} = \rho$.

Now, the close relationship that exists between the Hahn–Banach extensions and the infimal convolution can be nicely clarified by the following

Theorem 1.14. *If U is a subgroup of X such that $V \subset U$, and ψ is a Hahn–Banach extension of φ to U, then ψ is increasing and for any $u \in U$ we have*

$$\hat{q}(u) \leq \psi(u) \leq q(u).$$

Proof. By Corollary 1.5, it is clear that ψ is also increasing. Moreover, if $u \in U$ and $(s,t) \in \Gamma(u)$, then we can note that $s \in X$ and $t \in V$ such that $u \leq s + t$. Hence, we can infer $u - t \leq s$, and thus $p(u-t) \leq p(s)$. Moreover, since $u - t \in U - V \subset U - U \subset U$, we can also easily see that

$$\psi(u) = \psi(u - t + t) = \psi(u-t) + \psi(t) \leq p(u-t) + \varphi(t) \leq p(s) + \varphi(t).$$

Therefore,

$$\psi(u) \leq \inf\big\{p(s) + \varphi(t) : (s,t) \in \Gamma(u)\big\} = q(u).$$

Thus, we have proved that $\psi \leq q$. Hence, by using Remark 1.13, we can already see that $\hat{q}(u) \leq \hat{\psi}(u) = \psi(u)$ also holds. □

Remark 1.15. In the next section, we shall show that $q \leq p$ and q is also increasing and subadditive. Therefore, q is, in general, a better control function for ψ than p.

Now, as an immediate consequence of Theorem 1.14, we can also state

Corollary 1.16. *If ψ is as in Theorem 1.14 and q is odd on U, then q is an extension of ψ.*

Proof. In this case, for any $u \in U$, we have $\hat{q}(u) = q(u)$. Therefore, by Theorem 1.14, we also have $\psi(u) = q(u)$, and thus the required assertion also holds. $\square$

Hence, it is clear that in particular we also have

Corollary 1.17. *If U is a subgroup of X such that $V \subset U$ and q is odd on U, then there exists at most one Hahn–Banach extension ψ of φ to U.*

Definition 1.18. For any function ρ of X to $\mathbb{R}$ and $x \in X$, we define

$$\bar{\rho}(x) = \max\big\{\rho(x),\ \check{\rho}(x)\big\}.$$

Remark 1.19. Note that thus $\bar{\rho}$ is just the smallest even function of X to $\mathbb{R}$ such that $\rho \leq \bar{\rho}$. Therefore, ρ is even if and only if $\rho = \bar{\rho}$.

Moreover, it is also worth noticing that the mapping $\rho \mapsto \bar{\rho}$ is an algebraic closure on the partially ordered set $\mathbb{R}^X$ of all functions of X to $\mathbb{R}$.

Now, from Theorem 1.14, we can also immediately derive the following

Theorem 1.20. *If ψ is as in Theorem 1.14, then $|\psi|$ is dominated by $\bar{q}$.*

Proof. If $u \in U$, then by using Theorem 1.14, we can see that

$$\psi(u) \leq q(u) \leq \max\big\{q(u),\ \check{q}(u)\big\} = \bar{q}(u)$$

and

$$-\psi(u) \leq -\hat{q}(u) = \check{q}(u) \leq \max\big\{q(u),\ \check{q}(u)\big\} = \bar{q}(u).$$

Therefore,

$$|\psi|(u) = |\psi(u)| \leq \bar{q}(u),$$

and thus the required assertion is also true. $\square$

Hence, it is clear that in particular we also have

Corollary 1.21. *If ψ is as in Theorem 1.14 and q is even on U, then $|\psi|$ is also dominated by q.*

2. Further inequalities for the function q

THEOREM 2.1. *$q \le p$.*

PROOF. If $x \in X$, then by Remark 1.9 we have $(x, 0) \in \Gamma(x)$. Hence, it is clear that

$$q(x) = \inf\big\{p(u) + \varphi(v) : (u, v) \in \Gamma(x)\big\} \le p(x) + \varphi(0) = p(x).$$

Therefore, the required equality is also true. □

THEOREM 2.2. *$q(0) = 0$.*

PROOF. By Theorem 2.1, we have $q(0) \le p(0) = 0$. Moreover, from the $\varphi = \psi$ particular case of Theorem 1.14, we can see that $0 = \varphi(0) \le q(0)$. Therefore, the required equality is also true. □

THEOREM 2.3. *q is increasing.*

PROOF. If $x, y \in X$ such that $x \le y$, then by Remark 1.9 we have $\Gamma(y) \subset \Gamma(x)$. Therefore, if $(u, v) \in \Gamma(y)$, then we also have $(u, v) \in \Gamma(x)$. Hence, it is clear that

$$q(x) = \inf\big\{p(s) + \varphi(t) : (s, t) \in \Gamma(x)\big\} \le p(u) + \varphi(v),$$

and thus

$$q(x) \le \inf\big\{p(u) + \varphi(v) : (u, v) \in \Gamma(y)\big\} = q(y).$$

Therefore, the required assertion is true. □

The increasingness of q can also be derived from Theorem 1.4, by using Theorem 2.1 and the following

THEOREM 2.4. *q is subadditive.*

PROOF. If $x, y \in X$, then

$$q(x) = \inf\big\{p(u) + \varphi(v) : (u, v) \in \Gamma(x)\big\}$$

and

$$q(y) = \inf\big\{p(s) + \varphi(t) : (s, t) \in \Gamma(y)\big\}.$$

Therefore, for any $\alpha, \beta \in \mathbb{R}$, with

$$q(x) < \alpha \quad \text{and} \quad q(y) < \beta,$$

there exist $(u, v) \in \Gamma(x)$ and $(s, t) \in \Gamma(y)$ such that

$$p(u) + \varphi(v) < \alpha \quad \text{and} \quad p(s) + \varphi(t) < \beta.$$

Now, by using Remark 1.9, we can see that

$$(u + s, v + t) = (u, v) + (s, t) \in \Gamma(x) + \Gamma(y) \subset \Gamma(x + y).$$

Therefore,

$$\begin{aligned} q(x + y) &= \inf\big\{p(\tau) + \varphi(\omega) : (\tau, \omega) \in \Gamma(x + y)\big\} \\ &\leq p(u + s) + \varphi(v + t) \leq p(u) + p(s) + \varphi(v) + \varphi(t) < \alpha + \beta \end{aligned}$$

Hence, by letting α and β tend to $q(x)$ and $q(y)$, respectively, we can already infer that

$$q(x + y) \leq q(x) + \beta, \quad \text{and thus} \quad q(x + y) \leq q(x) + q(y). \qquad \square$$

THEOREM 2.5. *q is real-valued.*

PROOF. If $x \in X$, then from Remark 1.11 we know $q(x) \neq +\infty$. Moreover, by using Theorems 2.2 and 2.4, we can see that

$$0 = q(0) = q(x - x) \leq q(x) + q(-x).$$

Hence, it is clear that $q(x) \neq -\infty$ also holds. Namely, if $q(x) = -\infty$ were true, then because of the above inequality and $q(-x) \neq +\infty$, we would have $0 \leq -\infty$. $\square$

THEOREM 2.6. *$\hat{q} \leq q$.*

PROOF. Now, if $x \in X$, then from the inequality $0 \leq q(x) + q(-x)$, we can also infer that $\hat{q}(x) = -q(-x) \leq q(x)$. Therefore, $\hat{q} \leq q$ also holds. $\square$

REMARK 2.7. Note that for a function ρ of X to $\mathbb{R}$ we have $\hat{\rho} \leq \rho$ if and only if ρ is superodd in the sense that $-\rho(x) \leq \rho(-x)$ for all $x \in X$.

THEOREM 2.8. *For any $n \in \mathbb{N}$ and $x \in X$, we have*

$$n\hat{q}(x) \le q(nx) \le nq(x).$$

PROOF. If $x \in X$ and $n \in \mathbb{N}$ such that $q(nx) \le nq(x)$, then by Theorem 2.4 we also have

$$q\big((n+1)x\big) = q(nx+x) \le q(nx) + q(x) \le nq(x) + q(x) = (n+1)q(x).$$

Therefore, the second part of the required assertion is true.

Hence, by using Theorem 2.6 and Remark 2.7, we can already infer that

$$-q(nx) \le q(-nx) = q\big(n(-x)\big) \le nq(-x),$$

and thus

$$n\hat{q}(x) = -nq(-x) \le q(nx)$$

for all $n \in \mathbb{N}$ and $x \in X$. Therefore, the first part of the required assertion is also true. □

Now, as a useful consequence of Theorems 2.8, 2.2 and 2.5, we can also state

THEOREM 2.9. *For any $x \in X$ and $k \in \mathbb{Z}$, with $k \le 0$, we have*

$$kq(x) \le q(kx) \le k\hat{q}(x).$$

PROOF. Note that if $k < 0$, then by writing $-k$ in place n and $-x$ in place of x in Theorem 2.8 we get

$$kq(x) = -k\hat{q}(-x) \le q(kx) \le -kq(-x) = k\hat{q}(x).$$

Moreover, by Theorem 2.2 and 2.5, we also have

$$q(0x) = q(0) = 0 = 0q(x) = 0\hat{q}(x). \qquad \square$$

Finally, we note that now, in addition to Theorem 2.1, we can also state

THEOREM 2.10. $\hat{p} \le q$.

PROOF. By Theorem 2.1 we have $q \le p$. Hence, by using Remark 1.13, we can infer that $\hat{p} \le \hat{q}$. Moreover, by Theorem 2.6, we have $\hat{q} \le q$. Therefore, $\hat{p} \le q$ also holds. □

Now, as an immediate consequence of Theorems 2.1 and 2.10, we can also state

Corollary 2.11. *If p is odd, then $q = p$.*

Moreover, analogously to Theorem 1.20, we can also prove the following

Theorem 2.12. $|q| \leq \bar{p}$.

Hence, by Remark 1.19, it is clear that in particular we also have

Corollary 2.13. *If p is even, then $|q| \leq p$.*

Moreover, from Theorems 2.1 and 2.10, we can also immediately derive

Theorem 2.14. $\hat{p} \leq \hat{q} \leq p$.

Proof. By Theorem 2.1 and 2.10, we have $q \leq p$ and $\hat{p} \leq q$. Hence, by using Remark 1.13, we can infer that $\hat{p} \leq \hat{q}$ and $\hat{q} \leq \hat{\hat{p}} = p$. □

3. Further additivity and homogeneity properties of q

In addition to Theorem 2.4, we can also easily prove the following

Theorem 3.1. *For any $x \in X$ and $v \in V$, we have*

$$q(x + v) = q(x) + \varphi(v).$$

Proof. If $(s, t) \in \Gamma(x + v)$, then $s \in X$ and $t \in V$ such that $x + v \leq s + t$. Hence, we can infer that $t - v \in V$ and $x \leq s + t - v$. Therefore, $(s, t - v) \in \Gamma(x)$, and thus

$$\begin{aligned} q(x) &= \inf\big\{ p(\tau) + \varphi(\omega) : (\tau, \omega) \in \Gamma(x) \big\} \\ &\leq p(s) + \varphi(t - v) = p(s) + \varphi(t) - \varphi(v). \end{aligned}$$

Hence, we can infer that

$$q(x) + \varphi(v) \leq p(s) + \varphi(t),$$

and thus

$$q(x) + \varphi(v) \leq \inf\big\{p(s) + \varphi(t) : (s,t) \in \Gamma(x+v)\big\} = q(x+v).$$

Now, we can easily see that

$$q(x+v) = q(x+v) + \varphi(0) = q(x+v) + \varphi(-v) + \varphi(v) \leq q(x) + \varphi(v),$$

and thus the required equality also holds. □

From this theorem, by using Theorem 2.2, we can immediately derive

Corollary 3.2. *q is an extension of φ.*

Proof. By Theorems 3.1 and 2.2, for any $v \in V$, we have

$$q(v) = q(0) + \varphi(v) = \varphi(v).$$

Therefore, the required assertion is true. □

Hence, it is clear that in particular we also have

Corollary 3.3. *For any $x \in X$ and $v \in V$, we have*

$$q(x+v) = q(x) + q(v).$$

In view of Theorem 2.8, we may naturally introduce the following

Definition 3.4. For any function ρ of X to $\mathbb{R}$, and $n \in \mathbb{N}$ and $x \in X$, we define

$$\rho_n(x) = n^{-1}\rho(nx).$$

Remark 3.5. Note that thus ρ is n-homogeneous if and only if $\rho = \rho_n$. Moreover, for each $n \in \mathbb{N}$, the mapping $\rho \mapsto \rho_n$ has several useful properties.

For instance, we can easily see that this mapping is increasing, $\hat{\rho}_n = \widehat{\rho_n}$, and $\rho_{nm} = (\rho_n)_m$ for all $n, m \in \mathbb{Z}$.

Theorem 3.6. *If $n, m \in \mathbb{N}$ such that n divides m, then $q_m \leq q_n$.*

Proof. In this case, we have $m = kn$ for some $k \in \mathbb{N}$. Hence, by Theorem 2.8, we can see that

$$q_m(x) = m^{-1}q(mx) = (kn)^{-1}q(knx) \le (kn)^{-1}kq(nx) = n^{-1}q(nx) = q_n(x)$$

for all $x \in X$. Therefore, the required inequality is also true. $\square$

Corollary 3.7. *If $(k_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ such that k_n divides k_{n+1} for all $n \in \mathbb{N}$, then $\left(q_{k_n}\right)_{n=1}^{\infty}$ is a decreasing sequence in $\mathbb{R}^X$.*

Remark 3.8. Important particular cases are when either $k_n = 2^n$ for all $n \in \mathbb{N}$ or $k_n = n!$ for all $n \in \mathbb{N}$.

Definition 3.9. For any function ρ of X to $\mathbb{R}$ and $x \in X$, we define

$$\rho^*(x) = \inf_{n\in\mathbb{N}} \rho_n(x) \quad \text{and} \quad \rho^{\#}(x) = \sup_{n\in\mathbb{N}} \rho_n(x).$$

Remark 3.10. Note that thus $\rho^* \le \rho_1 = \rho$. Moreover, if σ is an $\mathbb{N}$-superhomogeneous function of X to $\mathbb{R}$ such that $\sigma \le \rho$, then for any $n \in \mathbb{N}$ and $x \in X$, we have

$$\sigma(x) = n^{-1}n\sigma(x) \le n^{-1}\sigma(nx) \le n^{-1}\rho(nx) = \rho_n(x),$$

Hence, we can infer that

$$\sigma(x) \le \inf_{n\in\mathbb{N}} \rho_n(x) = \rho^*(x),$$

and thus $\sigma \le \rho^*$ also holds.

Moreover, from the proof of the forthcoming Theorem 4.3, we can see that ρ^* is always $\mathbb{N}$-superhomogeneous. Therefore, if ρ^* is real-valued, then ρ^* is the largest $\mathbb{N}$-superhomogeneous function of X to $\mathbb{R}$ such that $\rho^* \le \rho$.

However, it is now more important to note that, by using Theorem 2.8, we can easily prove the following

Theorem 3.11. $\hat{q} \le q^* \le q^{\#} \le q$.

Proof. By Theorem 2.8, for any $n \in \mathbb{N}$ and $x \in X$, we have

$$\hat{q}(x) \le n^{-1}q(nx) = q_n(x) \quad \text{and} \quad q_n(x) = n^{-1}q(nx) \le q(x).$$

Hence, we can already infer that

$$\hat{q}(x) \leq \inf_{n\in\mathbb{N}} q_n(x) = q^*(x) \quad \text{and} \quad q^{\#}(x) = \sup_{n\in\mathbb{N}} q_n(x) \leq q(x).$$

Therefore, the required equalities are also true. □

Now, as an immediate consequence of Theorems 3.11, 2.2 and 2.5, we can also state

COROLLARY 3.12. *q^* and $q^{\#}$ are real-valued and $q^*(0) = 0$ and $q^{\#}(0) = 0$.*

Moreover, from Theorem 3.11, by Remark 1.13, it is clear that we also have

COROLLARY 3.13. *If q is odd then $q = q^* = q^{\#}$.*

However, it is now more important to note that from Theorem 1.14 we can easily derive the following

THEOREM 3.14. *If U is a subgroup of X containing V, and ψ is a Hahn–Banach extension of φ to U, then for any $u \in U$ we have*

$$\widehat{q^*}(u) \leq \psi(u) \leq q^*(u).$$

PROOF. By Theorem 1.14, for each $u \in U$, we have

$$\hat{q}(u) \leq \psi(u) \quad \text{and} \quad \psi(u) \leq q(u).$$

Hence, by using the $\mathbb{N}$-homogeneity of ψ, we can infer that

$$\hat{q}_n(u) \leq \psi_n(u) = \psi(u) \quad \text{and} \quad \psi(u) = \psi_n(u) \leq q_n(u)$$

for all $n \in \mathbb{N}$. Therefore,

$$\hat{q}^{\#}(u) = \sup_{n\in\mathbb{N}} \hat{q}_n(u) \leq \psi(u) \quad \text{and} \quad \psi(u) \leq \inf_{n\in\mathbb{N}} q_n(u) = q^*(u).$$

Moreover, we can note that

$$\begin{aligned} \hat{q}^{\#}(x) &= \sup_{n\in\mathbb{N}} n^{-1}\hat{q}(nx) = \sup_{n\in\mathbb{N}} -n^{-1}q(-nx) \\ &= -\inf_{n\in\mathbb{N}}\ n^{-1}q\big(n(-x)\big) = -q^*(-x) = \widehat{q^*}(x). \end{aligned}$$

for all $x \in X$. Therefore, the required inequalities are also true. □

Remark 3.15. In the next section, we shall see that q^* is also increasing and subadditive. Therefore, q^* is, in general, a better control function for ψ than q.

Now, improving Corollaries 1.16 and 1.17, we can also state

Corollary 3.16. *If ψ is as in Theorem 3.14 and q^* is odd on U, then q^* is an extension of ψ.*

Corollary 3.17. *If U is a subgroup of X such that $V \subset U$ and q^* is odd on U, then there exists at most one Hahn–Banach extension ψ of φ to U.*

Moreover, improving Theorem 1.20 and Corollary 1.21, we can also state

Theorem 3.18. *If ψ is as in Theorem 3.14, then $|\psi|$ is dominated by $\overline{q^*}$.*

Corollary 3.19. *If ψ is as in Theorem 3.14 and q^* is even on U, then $|\psi|$ is also dominated by q^*.*

4. Further important properties of the function q^*

Counterparts of the following basic facts on q^* are frequently used in connections with subadditive and convex functions.

Theorem 4.1. *q^* is increasing.*

Proof. If $x, y \in X$ such that $x \leq y$, then for each $n \in \mathbb{N}$ we have $nx \leq ny$. Hence, by Theorem 2.3, it follows that $q(nx) \leq q(ny)$, and thus

$$q_n(x) = n^{-1}q(nx) \leq n^{-1}q(ny) = q_n(y).$$

Therefore,

$$q^*(x) = \inf_{k\in\mathbb{N}} q_k(x) \leq q_n(x) \leq q_n(y),$$

and thus

$$q^*(x) \leq \inf_{n\in\mathbb{N}} q_n(y) = q^*(y).$$

This shows that q^* is increasing. $\square$

The increasingness of q^* can also be derived from Theorem 1.4, by using the inequality $q^* \leq p$ and the following

THEOREM 4.2. *q^* is subadditive.*

PROOF. If $x, y \in X$, then we can note that

$$q^*(x) = \inf_{n \in \mathbb{N}} n^{-1} q(nx) \quad \text{and} \quad q^*(y) = \inf_{m \in \mathbb{N}} m^{-1} q(my).$$

Therefore, for any $\alpha, \beta \in \mathbb{R}$, with

$$q^*(x) < \alpha \quad \text{and} \quad q^*(y) < \beta,$$

there exist $n, m \in \mathbb{N}$ such that

$$n^{-1} q(nx) < \alpha \quad \text{and} \quad m^{-1} q(my) < \beta.$$

Now, by using Theorems 2.4 and 2.8, we can see that

$$q\big(nm(x+y)\big) \leq q(nmx) + q(nmy) \leq mq(nx) + nq(my).$$

and thus

$$(nm)^{-1} q\big(nm(x+y)\big) \leq n^{-1} q(nx) + m^{-1} q(my) < \alpha + \beta.$$

Therefore,

$$q^*(x+y) = \inf_{k \in \mathbb{N}} k^{-1}\big(k(x+y)\big) \leq (nm)^{-1} q\big(nm(x+y)\big) < \alpha + \beta$$

also holds. Hence, by letting α and β tend to $q^*(x)$ and $q^*(y)$, respectively, we can already infer that

$$q^*(x+y) \leq q^\star(x) + \beta, \quad \text{and thus} \quad q^*(x+y) \leq q^*(x) + q^*(y).$$

Therefore, q^* is subadditive. $\square$

THEOREM 4.3. *q^* is $\mathbb{N}$-homogeneous.*

Proof. If $x \in X$ and $n \in \mathbb{N}$, then we can note that

$$q^*(nx) = \inf_{m \in \mathbb{N}} \; m^{-1} q(nmx).$$

Therefore, for each $\alpha \in \mathbb{R}$, with $q^*(nx) < \alpha$, there exists $m \in \mathbb{N}$ such that

$$m^{-1} q(nmx) < \alpha, \quad \text{and thus} \quad (nm)^{-1} q(nmx) < n^{-1}\alpha.$$

Hence, we can infer that

$$q^*(x) = \inf_{k \in \mathbb{N}} k^{-1} q(kx) \leq (nm)^{-1} q(nmx) < n^{-1}\alpha.$$

Now, by letting α tend to $q^*(x)$, we can already see that

$$q^*(x) \leq n^{-1} q^*(nx) \quad \text{and thus} \quad nq^*(x) \leq q^*(nx).$$

Moreover, by Theorem 4.2, it is clear that $q^*(nx) \leq nq^*(x)$ also holds. Therefore, the corresponding equality is also true, and thus q^* is $\mathbb{N}$–homogeneous. □

Now, as a useful consequence of Theorems 4.2 and 4.3 and Corollary 3.12, we can also state

Theorem 4.4. *For any $k \in \mathbb{Z}$ and $x \in X$, we have*

$$\widehat{q^*}(kx) \leq kq^*(x) \leq q^*(kx).$$

Proof. If $x \in X$, then by Corollary 3.12 we have

$$0q^*(x) = 0 = q^*(0) = q^*(0x).$$

Moreover, by Theorem 4.3, we have $kq^*(x) = q^*(kx)$ for all $k \in \mathbb{Z}$ with $k > 0$.

Furthermore, from Theorem 4.2, by Corollary 3.12, we can see that q^* is also superodd. That is, $-q^*(x) \leq q^*(-x)$ for all $x \in X$. Now, by using Theorem 4.3, we can see that, for any $x \in X$ and $k \in \mathbb{Z}$, with $k < 0$, we have

$$kq^*(x) = (-k)\big(-q^*(x)\big) \leq (-k)q^*(-x) = q^*\big((-k)(-x)\big) = q^*(kx).$$

Therefore, $kq^*(x) \leq q^*(kx)$ holds for all $k \in \mathbb{Z}$ and $x \in X$. Hence, by writing $-k$ in place of k, we can see that $-kq^*(x) \leq q^*(-kx)$, and thus

$$\widehat{q^*}(kx) = -q^*(-kx) \leq kq^*(x)$$

also holds for all $k \in \mathbb{Z}$ and $x \in X$. □

Remark 4.5. Now, in addition to Theorem 4.3, we can also state that q^* is $\mathbb{Z}$-superhomogeneous.

Moreover, by writing $-x$ in place of x in the first statement of Theorem 4.4, we can see that $-q^*(kx) \leq kq^*(-x)$ also holds for all $k \in \mathbb{Z}$ and $x \in X$.

However, it is now more important to note that, by the corresponding definitions and the equality $\hat{q}^{\#} = \widehat{q^*}$, we also have the following

Theorem 4.6. *If $x \in X$ and $y \in \mathbb{R}$ such that*

$$\widehat{q^*}(x) \leq y \leq q^*(x),$$

then for all $n \in \mathbb{N}$, we have

$$\hat{q}(nx) \leq ny \leq q(nx).$$

Proof. To check the first statement of the theorem, note that by the proof of Theorem 3.14 we have

$$\sup_{n \in \mathbb{N}} n^{-1} \hat{q}(nx) = \hat{q}^{\#}(x) = \widehat{q^*}(x) \leq y.$$

Thus, in particular, we also have

$$n^{-1} \hat{q}(nx) \leq y, \quad \text{and hence} \quad \hat{q}(nx) \leq ny$$

for all $n \in \mathbb{N}$. □

Now, as a useful consequence of Theorems 4.6 and 2.2, we can also state

Theorem 4.7. *If x and y are as in Theorem 4.6, then for all $k \in \mathbb{Z}$, we have*

$$\hat{q}(kx) \leq ky \leq q(kx).$$

Proof. By Theorem 2.2, we have $0y = 0 = q(0) = q(0x)$. Moreover, by Theorem 4.6, we also have $ky \leq q(kx)$ for all $k \in \mathbb{Z}$ with $k > 0$.

On the other hand, if $k \in \mathbb{Z}$ such that $k < 0$, then by writing $-k$ in place of n in the first statement of Theorem 4.6 we can see that $\hat{q}(-kx) \leq -ky$, and thus

$$ky \leq -\hat{q}(-kx) = q(kx).$$

Therefore, we have $ky \leq q(kx)$ for all $k \in \mathbb{Z}$. Hence, by writing $-k$ in place of k, we can see that $-ky \leq q(-kx)$, and thus

$$\hat{q}(kx) = -q(-kx) \leq ky$$

also holds for all $k \in \mathbb{Z}$. □

Finally, we note that by using Theorem 3.6, we can prove the following

THEOREM 4.8. *If $(k_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that n divides k_m, then for any $x \in X$ we have*

$$q^*(x) = \inf_{n \in \mathbb{N}} q_{k_n}(x).$$

PROOF. By the corresponding definitions, it is clear that

$$q^*(x) = \inf_{n \in \mathbb{N}} q_n(x) \leq \inf_{n \in \mathbb{N}} q_{k_n}(x).$$

Moreover, for each $\alpha \in \mathbb{R}$, with $q^*(x) < \alpha$, there exist $n \in \mathbb{N}$ such that $q_n(x) < \alpha$. Furthermore, by the hypothesis, there exists $m \in \mathbb{N}$ such that n divides k_m. Hence, by Theorem 3.6, we can see that $q_{k_m}(x) \leq q_n(x)$. Therefore, $q_{k_m}(x) < \alpha$, and thus

$$\inf_{n \in \mathbb{N}} q_{k_n}(x) < \alpha$$

also holds. Hence, by letting α tend to $q^*(x)$, we can already infer that

$$\inf_{n \in \mathbb{N}} q_{k_n}(x) \leq q^*(x),$$

and thus the required equality is also true. □

COROLLARY 4.9. *For any $x \in X$, we have*

$$q^*(x) = \lim_{n \to \infty} q_{n!}(x).$$

PROOF. By Theorem 4.8, we have

$$q^*(x) = \inf_{n \in \mathbb{N}} q_{n!}(x).$$

Moreover, by Corollary 3.7, the sequence $(q_{n!}(x))_{n=1}^{\infty}$ is decreasing. Therefore, we also have

$$\inf_{n\in\mathbb{N}} q_{n!}(x) = \lim_{n\to\infty} q_{n!}(x).$$

Thus, the required equality is also true. □

REMARK 4.10. This corollary allows of an easy derivation of several properties of q^* from those of q.

For instance, from Theorem 3.1, by using Corollary 4.9, we can immediately derive the following

THEOREM 4.11. *For any $x \in X$ and $v \in V$, we have*

$$q^*(x+v) = q^*(x) + \varphi(v).$$

Hence, by Corollary 3.12, it is clear that in particular we can also state

COROLLARY 4.12. *q^* is an extension of φ.*

5. One-step Hahn–Banach extensions of φ

NOTATION 5.1. *Suppose that $a \in X$, and define*

$$L = \big\{k \in \mathbb{Z} : ka \in V\big\}$$

and

$$U = \mathbb{Z}a + V = \big\{ka + v : k \in \mathbb{Z},\ v \in V\big\}.$$

REMARK 5.2. Then, it can be easily seen that L is an ideal in $\mathbb{Z}$. Moreover, U is the smallest subgroup of X such that $a \in U$ and $V \subset U$.

THEOREM 5.3. *If $L \neq \{0\}$, then there exists a unique Hahn–Banach extension ψ of φ to U. Moreover, we have*

$$\psi(ka+v) = kl^{-1}\varphi(la) + \varphi(v)$$

for all $k \in \mathbb{Z}$, $v \in V$ and $l \in L$ with $l \neq 0$.

Proof. If ψ an additive extension of φ to U, then for any $k \in \mathbb{Z}$ and $v \in V$ we have

$$\psi(ka+v) = k\psi(a) + \psi(v) = k\psi(a) + \varphi(v).$$

Moreover, by choosing $l \in L$ such that $l \neq 0$, then we can also note that

$$l\psi(a) = \psi(la) = \varphi(la), \quad \text{and thus} \quad \psi(a) = l^{-1}\varphi(la).$$

Therefore, the unicity part of the theorem is true.

To prove the existence part of the theorem, note that now, because of $L \neq \{0\}$, there exists $l \in \mathbb{Z}$, with $l \neq 0$, such that $la \in V$. Moreover, since $-L \subset L$, we may assume that $l > 0$. Define

$$b = l^{-1}\varphi(la).$$

Then, $lb = \varphi(la)$. Moreover, for any $k \in L$, we also have

$$klb = k\varphi(la) = \varphi(kla) = l\varphi(ka), \quad \text{and thus} \quad kb = \varphi(ka).$$

The latter observation allows us to easily show that, for any $k_1, k_2 \in \mathbb{Z}$ and $v_1, v_2 \in V$,

$$k_1 a + v_1 = k_2 a + v_2 \quad \text{implies} \quad k_1 b + \varphi(v_1) = k_2 b + \varphi(v_2).$$

Namely, if $k_1 a + v_1 = k_2 a + v_2$ holds, then we also have

$$(k_1 - k_2)a = -(v_1 - v_2) \in V, \quad \text{and thus} \quad k_1 - k_2 \in L.$$

Hence, we can already infer that

$$(k_1 - k_2)b = \varphi\big((k_1 - k_2)a\big) = \varphi\big(-(v_1 - v_2)\big) = -\varphi(v_1 - v_2),$$

and thus

$$k_1 b + \varphi(v_1) - \big(k_2 b + \varphi(v_2)\big) = (k_1 - k_2)b + \varphi(v_1 - v_2) = 0.$$

Now, we may unambiguously define a function ψ of U to $\mathbb{R}$ such that

$$\psi(ka+v) = kb + \varphi(v)$$

for all $k \in \mathbb{Z}$ and $v \in V$. Hence, it is clear that ψ is an additive extension of φ to U such that $\psi(a) = b$. Moreover, for any $k \in \mathbb{Z}$ and $v \in V$, we have

$$\begin{aligned}\psi(ka+v) &= kb + \varphi(v) = kl^{-1}\varphi(la) + l^{-1}l\varphi(v) = l^{-1}\big(\varphi(kla) + \varphi(lv)\big)\\ &= l^{-1}\varphi\big(l(ka+v)\big) \le l^{-1}p\big(l(ka+v)\big) \le l^{-1}lp(ka+v) = p(ka+v).\end{aligned}$$

Therefore, ψ is also dominated by p. □

Remark 5.4. By using an extension of Definition 3.4, we can write

$$\psi(ka+v) = \varphi_l(ka+v) = k\varphi_l(a) + \varphi(v)$$

for all $k \in \mathbb{Z}$, $v \in V$ and $l \in L$ with $l \neq 0$.

Theorem 5.5. *If $L = \{0\}$, then for any $b \in \mathbb{R}$ with*

$$\widehat{q^*}(a) \le b \le q^*(a),$$

there exists a unique Hahn–Banach extension ψ of φ to U such that $\psi(a) = b$. Moreover, we have

$$\psi(ka+v) = kb + \varphi(v)$$

for all $k \in \mathbb{Z}$ and $v \in V$.

Proof. If ψ an additive extension of φ to U, then as in the proof of Theorem 5.3 we have

$$\psi(ka+v) = k\psi(a) + \varphi(v)$$

for all $k \in \mathbb{Z}$ and $v \in V$. Therefore, ψ is uniquely determined by $\psi(a)$. Moreover, by Theorem 3.14, we also have

$$\widehat{q^*}(a) \le \psi(a) \le q^*(a).$$

To prove the existence part of the theorem, we first note that now, for any $k_1, k_2 \in \mathbb{Z}$ and $v_1, v_2 \in V$,

$$k_1a + v_1 = k_2a + v_2 \quad \text{implies} \quad k_1 = k_2, \quad v_1 = v_2.$$

Namely, if $k_1a + v_1 = k_2a + v_2$ holds, then we have

$$(k_1 - k_2)a = -(v_1 - v_2) \in V.$$

Therefore, we can only have $k_1 - k_2 = 0$, and thus also $v_1 - v_2 = 0$.

Now, we may unambiguously define a function ψ of U to $\mathbb{R}$ such that

$$\psi(ka + v) = kb + \varphi(v)$$

for all $k \in \mathbb{Z}$ and $v \in V$. Moreover, we can also note that ψ is an additive extension of φ to U such that $\psi(a) = b$. Furthermore, by using Theorems 4.7, 3.1 and 2.1, we can easily see that

$$\psi(ka + v) = kb + \varphi(v) \le q(ka) + \varphi(v) = q(ka + v) \le p(ka + v)$$

for all $k \in \mathbb{Z}$ and $v \in V$. Therefore, ψ is also dominated by p. □

Remark 5.6. Note that now, by Theorem 1.14, ψ is increasing. Moreover, by Theorem 3.14, we also have

$$\widehat{q^*}(ka + v) \le \psi(ka + v) \le q^*(ka + v)$$

for all $k \in \mathbb{Z}$ and $v \in V$.

Theorem 5.7. *If q^* is odd at a, then the restriction of q^* to U is the unique Hahn–Banach extension of φ to U.*

Proof. From Theorems 5.3, 4.4 and 5.5, we know that φ always has a Hahn–Banach extension ψ to U. Moreover, by the hypothesis and Remark 5.6, we necessarily have

$$q^*(a) = \widehat{q^*}(a) \le \psi(a) \le^* (a),$$

and thus $\psi(a) = q^*(a)$. Hence, since

$$\psi(ka + v) = k\psi(a) + \varphi(v) = kq^*(a) + \varphi(v)$$

for all $k \in \mathbb{Z}$ and $v \in V$, it is clear ψ is uniquely determined.

Moreover, by using Corollary 3.12 and Theorem 4.3, we can easily see that

$$kq^*(a) = q^*(ka)$$

for all $k \in \mathbb{Z}$. Namely, if $k < 0$, then by the hypothesis and Theorem 4.3 we also have

$$kq^*(a) = (-k)\big(-q^*(a)\big) = (-k)q^*(-a) = q^*\big((-k)(-a)\big) = q^*(ka).$$

Now, by Theorem 4.11, it is clear that

$$\psi(ka+v) = kq^*(a) + \varphi(v) = q^*(ka) + \varphi(v) = q^*(ka+v)$$

for all $k \in \mathbb{Z}$ and $v \in V$. Therefore, the required assertion is also true. □

THEOREM 5.8. *If $L \neq \{0\}$, then q^* is odd at a.*

PROOF. Again, we can note that there exists $k \in \mathbb{Z}$, with $k > 0$, such that $ka \in V$. Hence, by using Theorem 4.3 and Corollary 4.12, we can see that

$$kq^*(a) = q^*(ka) = \varphi(ka)$$

and

$$k\widehat{q^*}(a) = -kq^*(-a) = -q^*(-ka) = \widehat{q^*}(ka) = \hat{\varphi}(ka) = \varphi(ka).$$

Therefore, $kq^*(a) = k\widehat{q^*}(a)$, and hence $q^*(a) = \widehat{q^*}(a)$. Thus, the required assertion is also true. □

Now, as an immediate consequence of our former results, we can also state

THEOREM 5.9. *The following assertions are equivalent:*

(1) *q^* is odd at a;*
(2) *q^* is odd on U;*
(3) *there exists a unique Hahn–Banach extension ψ of φ to U;*
(4) *there exists at most one Hahn–Banach extension ψ of φ to U;*
(5) *the restriction of q^* to U is a Hahn–Banach extension of φ to U.*

PROOF. If (1) holds, then by Theorem 5.7 it is clear that the assertions (3), (4) and (5) also hold. Moreover, we can note that (3) and (4) are equivalent and (5) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

Therefore, we need only show that (4) also implies (1). For this, note that if (1) does not hold, then by Theorem 5.8 we have $L = \{0\}$. Moreover, by Theorem 4.4, we have $\widehat{q^*}(a) < q^*(a)$. Thus, there exist $b_1, b_2 \in \mathbb{R}$ such that

$$\widehat{q^*}(a) \leq b_1 < b_2 \leq q^*(a).$$

Moreover, by Theorem 5.5, there exist Hanh–Banach extensions ψ_1 and ψ_2 of φ to U such that $\psi_1(a) = b_1$ and $\psi_2(a) = b_2$. Therefore, (4) does not also hold. □

6. Total Hahn–Banach extensions of φ

THEOREM 6.1. *There exists a Hahn–Banach extension f of φ to X.*

PROOF. Denote by Ψ the family of all Hahn–Banach-extensions ψ of φ. Then, it is clear Ψ is a nonvoid partially ordered set with the ordinary set inclusion. Namely, $\varphi \in \Psi$.

Moreover, if Φ is a totally ordered subset of Ψ, then it can be easily seen that $\phi = \cup\Phi$ is an upper bound of Φ in Ψ. Thus, by Zorn's lemma, there exists a maximal element f of Ψ.

Thus, it remains only to show that the domain D_f of f is X. For this, note that if for some $a \in X$ we have $a \notin D_f$, then by Theorems 5.3, 4.4 and 5.5 there exists a Hahn–Banach extension ψ of f to the subgroup $U = \mathbb{R}a + D_f$. However, this contradicts the maximality of f. □

REMARK 6.2. Note that if f is as in the above theorem, then by Theorem 1.14 we can state that f is increasing. Moreover, by Theorem 3.14, we have

$$\widehat{q^*} \leq f \leq q^*.$$

Therefore, if in particular q^* is odd, then $f = q^*$.

Now, as an immediate consequence of our former results, we can also state the following

THEOREM 6.3. *The following assertions are equivalent:*

(1) *q^* is odd;*
(2) *q^* is a Hahn–Banach extension of φ;*
(3) *there exists a unique Hahn–Banach extension f of φ to X;*
(4) *there exists at most one Hahn–Banach extension f of φ to X.*

PROOF. By Remark 6.2, it is clear that (1) implies (4). Moreover, from Theorem 6.1, we know that there exists a Hahn–Banach extension f of φ to X. Therefore, (4) implies (3). Moreover, if (1) holds, then by Remark 6.2 we necessarily have $f = q^*$. Therefore, (1) also implies (2).

Now, since the implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (4) trivially hold, we need only show that (4) also implies (1). For this, note that if (1) does not hold, then there exists $a \in X$ such that q^* is not odd at a. Then, by the proof of Theorem 5.9, there exist two Hahn–Banach extensions ψ_1 and ψ_2 of φ to $U = \mathbb{Z}a + V$. Moreover, by Theorem 6.1, we can state that there exist Hahn–Banach extensions f_1 and f_2 of ψ_1 and ψ_2 to X, respectively. Thus, (4) does not also holds. This proves the required implication. □

Remark 6.4. Sections 7 and 11 of [23] and [7], respectively, show that the question of the uniqueness of the Hahn–Banach extension has also been intensively studied by several authors. However, the above simple convolutional characterization seems to be new.

Next, we show that Theorem 6.1 can be used to prove a straightforward generalization of a weakening of [19, 1.6.1. Theorem] of Bauer, Bonsall and Namioka. For this, we must slightly change our former assumptions.

Notation 6.5. *Suppose that X is a commutative preordered group and V is a cofinal subgroup of X in the sense that for each $x \in X$ there exists $v \in V$ such that $x \leq v$.*

Moreover, assume that φ is an increasing additive function of V to $\mathbb{R}$, and for any $x \in X$ define

$$p(x) = \inf\big\{\varphi(v) : x \leq v \in V\big\}.$$

Remark 6.6. Note that our present definition of cofinality is, in general, stronger than that of Jameson [19, p. 8], but it coincides with the usual one. Of course, if in particular the nonegativity domain $P = \{x \in X : x \geq 0\}$ of X is cofinal in X in the sense that for each $x \in X$ there exists $y \in P$ such that $x \leq y$, then the two definitions coincide.

Note that an arbitrary subset Y of X is cofinal in X if and only if $X = Y - P$. Thus, in particular P is cofinal in X if and only if X is generated by P in the sense that $X = P - P$. Moreover, by [19, 1.1.3], we can also state that P is cofinal in X if and only if X is direted by its preordering in the sense that for any $x, y \in X$ there exists $z \in X$ such that $x \leq z$ and $y \leq z$.

However, it is now more important to note that, by the above definitions, we also have the following

Theorem 6.7. *p is an increasing subadditive function of X to $\mathbb{R}$ extending φ.*

Proof. If $x \in X$, then by the cofinality of V in X there exist $z, w \in V$ such that $x \leq z$ and $-x \leq w$, and thus $-w \leq x$. Hence, it is clear that

$$p(x) = \inf\big\{\varphi(v) : x \leq v \in V\big\} \leq \varphi(z) < +\infty.$$

Moreover, if $v \in V$ such that $x \leq v$, then by using that $-w \leq v$, $-w \in V$ and φ is increasing we can see that $\varphi(-w) \leq \varphi(v)$. Therefore, we also have

$$-\infty < \varphi(-w) \leq \inf\big\{\varphi(v) : x \leq v \in V\big\} = p(x).$$

Furthermore, by using quite similar arguments as in the proofs of Theorems 2.3 and 2.4, we can see that p is increasing and subadditive. Moreover, if $v \in V$, then by using the increasingness of φ we can see that $\varphi(v) \le \varphi(s)$ for all $s \in V$ with $v \le s$. Therefore,

$$\varphi(v) = \min\big\{\varphi(s) : v \le s \in V\big\} = \inf\big\{\varphi(s) : v \le s \in V\big\} = p(v). \qquad \square$$

Remark 6.8. Note that if ρ is an increasing function of X to $\mathbb{R}$ such that $\rho(v) \le \varphi(v)$ for all $v \in V$, then

$$\rho(x) = \inf\big\{\rho(v) : x \le v \in V\big\} \le \inf\big\{\varphi(v) : x \le v \in V\big\} = p(x)$$

for all $x \in X$. Therefore, p is actually the largest increasing function of X to $\mathbb{R}$ such that $p(v) \le \varphi(v)$ for all $v \in V$.

Now, we can readily prove the promised application of Theorem 6.1.

Theorem 6.9. *φ can be extended to an increasing additive function f of X to $\mathbb{R}$.*

Proof. By Theorems 6.7 and 6.1, we can state that there exists a Hahn–Banach extension f of φ to X. Thus, by Definition 1.6, f is an additive function of X to $\mathbb{R}$ that extends φ and is dominated by p. Moreover, by Theorem 1.14, f is also increasing. Therefore, the required assertion is also true. $\square$

Remark 6.10. In the present particular case, by Theorem 3.14, we can also state a counterpart of Remark 6.2.

However, now the infimal convolution $q = p * \varphi$ need not actually be used. Namely, because of the particular choice of p, we have the following

Theorem 6.11. $p = q$.

Proof. If $x \in X$, then by the corresponding definitions

$$q(x) = \inf\big\{p(u) + \varphi(v) : (u, v) \in X \times V : x \le u + v\big\}.$$

Therefore, for each $\alpha \in \mathbb{R}$, with $q(x) < \alpha$ there exist $u \in X$ and $v \in V$, with $x \le u + v$, such that

$$p(u) + \varphi(v) < \alpha.$$

Hence, we can infer that

$$\inf\big\{\varphi(s) : u \leq s \in V\big\} = p(u) < \alpha - \varphi(v).$$

Therefore, there exists $s \in V$, with $u \leq s$, such that

$$\varphi(s) < \alpha - \varphi(v), \quad \text{and thus} \quad \varphi(s+v) = \varphi(s) + \varphi(v) < \alpha.$$

Hence, since $s + v \in V$ and $x \leq u + v \leq s + v$, we can already infer that

$$p(x) = \inf\big\{\varphi(t) : x \leq t \in V\big\} < \alpha.$$

Now, by letting α tend to $q(x)$, we can also state that $p(x) \leq q(x)$. Hence, by Theorem 2.1, it is clear that $p(x) = q(x)$, and thus $p = q$ also holds. □

7. Some very particular illustrating examples

The following example shows that, even in the particular case considered in Notation 6.5, the homogenization q^* of $q = p * \varphi$ may differ from q.

EXAMPLE 7.1. Take $X = \mathbb{R}$, and consider X to be equipped with the usual addition, multiplication and inequality. Moreover, define $V = \mathbb{Z}$, and $\varphi(v) = v$ for all $v \in V$. Then, it is clear that X is a totally ordered vector space over $\mathbb{R}$. Moreover, V is a cofinal subgroup of X and φ is a strictly increasing additive function of V to $\mathbb{R}$.

Furthermore, we can easily see that

$$\begin{aligned} p(x) &= \inf\big\{\varphi(v) : x \leq v \in V\big\} = \inf\big\{v : x \leq v \in \mathbb{Z}\big\} \\ &= \min\big\{v : x \leq v \in \mathbb{Z}\big\} = -\max\big\{-v : x \leq v \in \mathbb{Z}\big\} \\ &= -\max\big\{-v : v \in \mathbb{Z},\ -v \leq -x\big\} = -\max\big\{k \in \mathbb{Z} : k \leq -x\big\} \end{aligned}$$

for all $x \in X$. Hence, by using the entire part function defined by

$$e(x) = [x] = \max\big\{k \in \mathbb{Z} : k \leq x\big\}$$

for all $x \in \mathbb{R}$, we can note that

$$p(x) = -\max\big\{k \in \mathbb{Z} : k \leq -x\big\} = -e(-x) = \hat{e}(x)$$

for all $x \in X$, and thus $p = \hat{e}$.

Hence, by Theorem 6.7, we can see that $\hat{e}$ is an increasing subadditive function of $\mathbb{R}$ to $\mathbb{Z}$ such that $\hat{e}(k) = \varphi(k) = k$ for all $k \in \mathbb{Z}$. Thus, in particular $\hat{e}$ is a retraction of the linearly ordered set $\mathbb{R}$ to $\mathbb{Z}$. Moreover, from Theorem 6.11, for the function $q = p * \varphi$, we can see that $p = q$. Hence, by using Corollary 4.9 and Remark 3.5, we can infer that

$$\begin{aligned} q^*(x) = p^*(x) &= \lim_{n\to\infty} p_{n!}(x) = \lim_{n\to\infty} \hat{e}_{n!}(x) \\ &= \lim_{n\to\infty} \widehat{e_{n!}}(x) = \lim_{n\to\infty} -e_{n!}(-x) = -\lim_{n\to\infty} e_{n!}(-x) \end{aligned}$$

for all $x \in X$.

By the definition of the function e, it is clear that $e(nx) \leq nx < e(nx)+1$, and thus

$$e_n(x) = n^{-1}e(nx) \leq x < n^{-1}e(nx) + n^{-1} = e_n(x) + n^{-1},$$

and thus $0 \leq x - e_n(x) < n^{-1}$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Hence, we can see that

$$x = \lim_{n\to\infty} e_n(x)$$

for all $x \in \mathbb{R}$, and moreover the convergence is uniform. The above useful equality shows, in particular, that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Moreover, we can note that

$$q^*(x) = -\lim_{n\to\infty} e_{n!}(-x) = -\lim_{n\to\infty} e_n(-x) = -(-x) = x$$

for all $x \in X$. Therefore, $q \neq q^*$. Now, by Theorem 6.3 and Definition 1.6, we can also state that q^* is the unique additive function of X to $\mathbb{R}$ that extends φ and is dominated by p.

Remark 7.2. In connection with the above example, note that if f is an additive function of X to $\mathbb{R}$ such that $f(1) = \varphi(1)$, then by the $\mathbb{Q}$-homogeneity of f we necessarily have

$$f(r) = rf(1) = r\varphi(1) = r = q^*(r)$$

for all $r \in \mathbb{Q}$. However, the equality $f = q^*$ need not be true.

The latter fact can only be proved with the help of the Hamel bases of X [20, pp. 78–85]. Note that if for instance $a = \sqrt{2}$, then $a \in X \setminus \mathbb{Q}$. Therefore, $\{1, a\}$ is a linearly independent subset of X as a vector space over $\mathbb{Q}$. Hence, by using Zorn's lemma, it can be easily seen that there exists a maximal linearly independent subset E of X such that $\{1, a\} \subset E$. Thus, in particular, for each $x \in X$, there exists a unique function $\tilde{x}$ of E to $\mathbb{Q}$ such that the support

$$E_x = \{e \in E : \tilde{x}(e) \neq 0\}$$

of $\tilde{x}$ is finite and

$$x = \sum_{e \in E_x} \tilde{x}(e)e.$$

Now, by taking

$$\sigma(1) = 1, \quad \sigma(a) = 2, \quad \text{and} \quad \sigma(e) \in \mathbb{R} \quad \text{for} \quad e \in E \setminus \{1, a\},$$

we may naturally define a function ρ of X to $\mathbb{R}$ such that

$$\rho(x) = \sum_{e \in E_x} \tilde{x}(e)\sigma(e)$$

for all $x \in X$. Then, it can be easily seen that ρ is additive. Moreover, we can note that

$$\rho(r) = r\rho(1) = r\sigma(1) = r = q^*(r)$$

for all $r \in \mathbb{Q}$, but

$$\rho(a) = 2 \neq \sqrt{2} = a = q^*(a).$$

Thus, in particular ρ is an additive extension of φ to X such that $\rho \neq q^*$.

The following example, mentioned by Jameson [19, p. 25], shows that even a strictly increasing linear functional of a non-cofinal subspace of a two dimensional partially ordered vector space need not have an increasing additive extension to the whole space. A more complicated three-dimensional example, for the same purposes, has formerly been offered by Peressini [25, p. 85].

EXAMPLE 7.3. Take $X = \mathbb{R}^2$, and consider X to be equipped with the usual coordinate-wise linear operations. Then, X is a vector space over $\mathbb{R}$.

Moreover, consider X to be equipped with the lexicographic ordering. Thus, for any $(x, y), (z, w) \in X$, we write

$$(x, y) \leq (z, w) \quad \text{if either} \quad x < z \quad \text{or} \quad (x = z \quad \text{and} \quad y \leq w).$$

Then, it can be easily seen that $\leq$ is a total ordering on X that is compatible with addition and multiplication by nonnegative scalars. Thus, X is a totally ordered vector space.

Moreover, define

$$V = \big\{(0,y) : y \in \mathbb{R}\big\} \quad \text{and} \quad \varphi(0,y) = y \quad \text{for all} \quad y \in \mathbb{R}.$$

Then, it is clear that V is a subspace of X and φ is a linear function of V to $\mathbb{R}$. Moreover, we can note that φ is strictly increasing, but V is not cofinal in X. Namely, for instance, we have $(0,y) < (1,0)$ for all $y \in \mathbb{R}$. In this respect, it is also worth noticing that, for any $(x,y) \in X$, we have

$$\begin{aligned} p(x,y) &= \inf\big\{\varphi(0,z) : (x,y) \leq (0,z)\big\} \\ &= \inf\big\{z \in \mathbb{R} : x < 0 \text{ or } (x = 0, y \leq z)\big\} = \begin{cases} -\infty & \text{if} \quad x < 0; \\ y & \text{if} \quad x = 0; \\ +\infty & \text{if} \quad 0 < x. \end{cases} \end{aligned}$$

Next, we show that φ does not have an increasing additive extension to X. For this, assume on the contrary that f is such an extension of φ to X. Then, in particular we have

$$\begin{aligned} f(x,y) &= f\big((x,0) + (0,y)\big) = f(x,0) + f(0,y) \\ &= f(x,0) + \varphi(0,y) = f(x,0) + y \end{aligned}$$

for all $(x,y) \in X$. Moreover, we can note that the mapping $x \mapsto f(x,0)$, where $x \in \mathbb{R}$, is also increasing and additive. Thus, by a classical result of the theory of functional equations [1, Corollary 5, p. 15], there exists $c \in \mathbb{R}$ such that

$$f(x,0) = cx$$

for all $x \in \mathbb{R}$. Therefore, we actually have

$$f(x,y) = cx + y$$

for all $(x,y) \in X$. However, if this is true, then taking $n \in \mathbb{N}$ and noticing that $(0,0) < (1,-n)$, we can infer that

$$0 = c0 + 0 = f(0,0) \leq f(1,-n) = c - n.$$

Hence, it follows that $n \leq c$ for all $n \in \mathbb{N}$. This contradiction proves the required assertion.

Remark 7.4. Note that if we consider X to be equipped with the more usual coordinatewise partial ordering instead of the total lexicographic one, then φ is still strictly increasing and V is not cofinal in X. However, for any $(x, y) \in X$, we have

$$\begin{aligned} p(x,y) &= \inf\big\{\varphi(0,z) : (x,y) \le (0,z)\big\} \\ &= \inf\big\{z \in \mathbb{R} : x \le 0,\ y \le z\big\} = \begin{cases} y & \text{if} \quad x \le 0; \\ +\infty & \text{if} \quad 0 < x. \end{cases} \end{aligned}$$

Moreover, for any $c \in \mathbb{R}$, with $c \ge 0$, the function f, defined by $f(x,y) = cx+y$ for all $(x,y) \in X$, is an increasing linear extension of φ to X that is dominated by p.

The following example is a modification of [15, Example 5.1]. For some closely related examples, which can also be well adjusted to the present setting, see also [14].

Example 7.5. Suppose now that $X = \mathbb{R}^2$ is again equipped with the lexicographic ordering as in Example 7.3, but we have

$$V = \big\{(x,x) : x \in \mathbb{R}\big\} \quad \text{and} \quad \varphi(x,x) = x \quad \text{for all} \quad x \in \mathbb{R}.$$

Then, it is clear that V is again a subspace of X and φ is again a linear function of V to $\mathbb{R}$. Moreover, we can note that φ is again strictly increasing, but in contrast to Example 7.3 and Remark 7.4 the subspace V is now cofinal in X.

Furthermore, for any $(x, y) \in X$, we have

$$\begin{aligned} p(x,y) &= \inf\big\{\varphi(z,z) : (x,y) \le (z,z)\big\} \\ &= \inf\big\{z : x < z \text{ or } (x = z,\ y \le z)\big\} = x. \end{aligned}$$

Namely, in each of the above cases we have $x \le z$. Therefore, $x \le p(x,y)$. Moreover, if $z \in \mathbb{R}$ such that $x < z$, then $p(x,y) \le z$. Hence, by letting z tend to x, we can see that $p(x,y) \le x$ also holds.

Now, in addition to Theorem 6.7, we can state that p is an increasing, linear extension of φ. Moreover, by Theorem 6.11, for the function $q = p * \varphi$, we have $p = q = q^*$. Hence, by Theorem 6.11, we can see that p is actually the unique additive extension of φ to X that is dominated by p.

On the other hand, we can also note that if f is an increasing additive extension of φ to $\mathbb{R}^2$, then in particular we have

$$\begin{aligned} f(x,y) &= f\big((x,0)+(y,y)-(y,0)\big) \\ &= f(x,0)+f(y,y)-f(y,0) = f(x,0)+y-f(y,0) \end{aligned}$$

for all $(x,y)\in X$. Moreover, as in Example 7.1, we can also state that there exists $c\in\mathbb{R}$ such that

$$f(x,0)=cx$$

for all $x\in\mathbb{R}$. Therefore, we actually have

$$f(x,y)=cx+(1-c)y$$

for all $(x,y)\in X$. Hence, by noticing that $(0,0)<(1,0)$ and $(0,0)<(0,1)$ we can infer that

$$0=f(0,0)\le f(1,0)=c \quad \text{and} \quad 0=f(0,0)\le f(0,1)=1-c$$

and thus $0\le c\le 1$. Moreover, by taking $n\in\mathbb{N}$ and noticing that $(0,0)<(1,-n)$, we can infer that

$$0=f(0,0)\le f(1,-n)=c-(1-c)n.$$

Hence, it follows that $(1-c)n\le c$ for all $n\in\mathbb{N}$. Therefore, $1-c=0$, and thus $c=1$. Consequently, $f=p$ is the unique increasing additive extension of φ to X.

Remark 7.6. Note that if we consider in X the more usual coordinatewise partial ordering instead of the total lexicographic one, then φ is still strictly increasing and V is cofinal in X. However, for any $(x,y)\in X$, we have

$$\begin{aligned} p(x,y) &= \inf\big\{\varphi(z,z):(x,y)\le(z,z)\big\} \\ &= \inf\big\{z\in\mathbb{R}: x\le z,\ y\le z\big\} = \max\{x,y\}. \end{aligned}$$

Moreover, for any $c\in[0,1]$, the function f defined by $f(x,y)=cx+(1-c)y$ for all $x,y\in\mathbb{R}$ is also an increasing linear extension of φ that is dominated by p.

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