Annales Mathematicae Silesianae **23** (2009), 103–117
Prace Naukowe Uniwersytetu Śląskiego nr 2769, Katowice

Report of Meeting

The Ninth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Będlewo (Poland), February 4–7, 2009

The Ninth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held in the Mathematical Research and Conference Center Będlewo, Poland, from February 4 to 7, 2009. It was organized by the Institute of Mathematics of the Silesian University from Katowice.

24 participants came from the University of Debrecen (Hungary) and the Silesian University of Katowice (Poland) at 12 from each of both cities. The 25^{th} participant of the Seminar was Professor Peter Volkmann from the University of Karlsruhe who is at present a visting professor at the Silesian University of Katowice.

Professor Roman Ger opened the Seminar and welcomed the participants to Będlewo.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iteration theory, equations on algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers–Ulam stability, functional equations and inequalities involving mean values, generalized convexity. Interesting discussions were generated by the talks.

There was a profitable Problem Session.

The social program included a banquet and an excursion which consisted of visiting the Outdoor Museum of Miniature Buildings in Pobiedziska and the Gniezno Cathedral . The closing address was given by Professor Zsolt Páles. His invitation to the Tenth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in February 2010 in Hungary was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in Section 1, problems and remarks in chronological order in Section 2, and the list of participants in the final section.

1. Abstract of talks

Roman Badora: *Stability of some functional equations*

Let X be a group and let Λ be a finite subgroup of the automorphism group $Aut(X)$ of X (the action of $\lambda \in \Lambda$ on $x \in X$ is denoted by λx). We study the stability of the following functional equations

$$\frac{1}{N}\sum_{\lambda\in\Lambda} f(x+\lambda y) = f(x)g(y) + h(y), \quad x, y \in X,$$
$$\frac{1}{N}\sum_{\lambda\in\Lambda} f(x+\lambda y) = f(y)g(x) + h(x), \quad x, y \in X$$

($N = \mathrm{card}\Lambda$, $f, g, h\colon X \to \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), which cover Jensen's functional equation, Cauchy's functional equation, the exponential functional equation, d'Alembert's functional equation and the functional equation of the square of the norm.

The stability problem concerning these functional equations with $\Lambda = \{Id_X\}$ was solved by L. Székelyhidi in *Stability properties of functional equations describing the scientific laws*, J. Math. Anal. Appl. 150 (1990), 151–158.

Szabolcs Baják: *Computer aided solution of the invariance equation for two-variable Stolarsky means* (Joint work with Zsolt Páles)

We solve the so-called invariance equation in the class of two-variable Stolarsky means $\{S_{p,q}\colon p, q \in \mathbb{R}\}$, i.e., we find necessary and sufficient conditions on the 6 parameters a, b, c, d, p, q such that the identity

$$S_{p,q}\big(S_{a,b}(x, y), S_{c,d}(x, y)\big) = S_{p,q}(x, y), \quad x, y \in \mathbb{R}_+,$$

be valid. We recall that, for $pq(p-q) \neq 0$ and $x \neq y$, the Stolarsky mean $S_{p,q}$ is defined by

$$S_{p,q}(x, y) := \Big(\frac{q(x^p - y^p)}{p(x^q - y^q)}\Big)^{\frac{1}{p-q}}.$$

In the proof first we approximate the Stolarsky mean, and then we use the computer algebra system Maple V Release 9 to compute the Taylor expansion of the approximation up to 12th order, which enables us to describe all the cases of the equality.

Karol Baron: *On the convergence in law of iterates of random-valued functions*

Given a probability space $(\Omega, \mathcal{A}, P)$, a separable and complete metric space X with the σ–algebra $\mathcal{B}$ of all its Borel subsets and a $\mathcal{B}\otimes\mathcal{A}$–measurable $f\colon X\times\Omega\to X$ we consider its iterates $f^n, n\in\mathbb{N}$, defined on $X\times\Omega^{\mathbb{N}}$ by $f^1(x,\omega)=f(x,\omega_1)$ and $f^{n+1}(x,\omega)=f(f^n(x,\omega),\omega_{n+1})$ and provide a simple criterion for the convergence in law of $(f^n(x,\cdot))_{n\in\mathbb{N}}$ to a random variable independent of $x\in X$.

Mihály Bessenyei: *On a class of single variable functional equations*

Motivated by a solving method of certain functional equations known from competition exercises, we investigate a class of functional equations giving representations or existence theorems for the solution. The main tools of the proofs are Cramer's rule and the inverse-function theorem.

References

[1] Brodszkij V.Sz., Szlipenko A.K., *Functional Equations*, Visa Skola, Kiev, 1986 [in Russian].
[2] Lajkó K., *Functional Equations in Exercises*, University Press of Debrecen, 2005 [in Hungarian].

Zoltán Boros: *Weakly affine functions on the plane*

Let k denote a positive integer, X be a linear space over $\mathbb{R}$, and D be a convex subset of X. We call a function $f\colon D\to\mathbb{R}$ *weakly k-affine* if, for every $x_0, x_1, x_2, \ldots, x_k\in X$ such that the system $(x_1-x_0, x_2-x_0, \ldots, x_k-x_0)$ is linearly independent, there exist $t_j\in]0,1[$ $(j=1,\ldots,k)$ satisfying $t_1+t_2+\cdots+t_k<1$ and

$$f\left((1-t_1-t_2-\cdots-t_k)x_0+t_1x_1+t_2x_2+\ldots t_kx_k\right)$$
$$=(1-t_1-t_2-\cdots-t_k)f(x_0)+t_1f(x_1)+t_2f(x_2)+\cdots+t_kf(x_k)\,.$$

We call f *affine* if

$$f\left(tx+(1-t)y\right)=tf(x)+(1-t)f(y)$$

holds for every $t\in[0,1]$ and $x,y\in D$.

As it is well known ([1], Theorem 89, Sec. 3.7, p. 73), if I is a real interval and $f\colon I \to \mathbb{R}$ is continuous and weakly 1-affine, then f is affine. In this talk we consider a convex set $D \subset \mathbb{R}^2$. We prove that every continuous and weakly 2-affine function $f\colon D \to \mathbb{R}$ has to be affine as well.

Reference

[1] Hardy G.H., Littlewood J.E., Pólya G., *Inequalities*, University Press, Cambridge, 1934 (1st ed.), 1952 (2nd ed.).

Pál Burai: *Bernstein–Doetsch type results for s-convex functions* (Joint work with Attila Házy and Tibor Juhász)

We introduce (H, s)-convexity as a possible generalization of the concept of s-convexity. Some simple facts and regularity theorems on (H, s)-convex functions are presented.

Włodzimierz Fechner: *On a composite functional equation on abelian groups*

We discuss the following composite functional equation:

$$f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y)$$

on an abelian group. We obtain some results on an arbitrary uniquely divisible by 2 abelian group and then we determine all solutions f of this equation on the group of integers and all real solutions which are continuous at zero or monotonic.

Roman Ger: *On a problem of Roger Cuculière*

We are going to present a solution of the (generalized) Problem 11345 posed by Roger Cuculière in Amer. Math. Monthly 115 (2008), no. 2:

Find all nondecreasing solutions $f\colon \mathbb{R} \to \mathbb{R}$ of the functional equation

$$(*) \qquad f(x + f(y)) = f(f(x)) + f(y), \quad x, y \in \mathbb{R}.$$

Actually, we have obtained a description of all Lebesgue measurable solutions to the functional equation $(*)$ getting the monotonic solutions as a straightforward corollary.

Since there exist discontinuous (and hence nonmeasurable) additive functions satisfying equation $(*)$, it seems reasonable to look for the general solution of this equation. The description of such solutions may easily be obtained from the fact that equation $(*)$ is simply a Cauchy equation whose domain has been restricted to a “cylinder” $\mathbb{R} \times Z$ where $(Z, +)$ stands for a subgroup of the additive group of reals.

Attila Gilányi: *On linear functional equations modulo* $\mathbb{Z}$ (Joint work with Agata Nowak)

In the present talk, motivated by some results contained in the papers [1–7] we consider functions $f_1, \dots, f_n \colon V \to \mathbb{R}$ which satisfy the property

$$\sum_{i=1}^{n} f_i(p_i x + q_i y) \in \mathbb{Z}, \quad x, y \in V,$$

where V is a linear space over $\mathbb{Q}$ and $p_1, \dots, p_n, q_1, \dots, q_n$ are fixed rational numbers. We establish conditions under which the functions f_i have the form $f_i = g_i + h_i$, $i = 1, \dots, n$, where h_i, $i = 1, \dots, n$, are integer valued functions defined on V and $g_i \colon V \to \mathbb{R}$, $i = 1, \dots, n$, fulfil the linear functional equation

$$\sum_{i=1}^{n} g_i(p_i x + q_i y) = 0, \quad x, y \in V.$$

References

[1] Baker J.A., *On some mathematical characters*, Glas. Mat. Ser. III **25(45)** (1990), 319–328.

[2] Baron K., Sablik M., Volkmann P., *On decent solutions of a functional congruence*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 27–39.

[3] Baron K., Volkmann P., *On the Cauchy equation modulo* $\mathbb{Z}$, Fund. Math. **131** (1988), 143–148.

[4] Brzdęk J., *The Cauchy and Jensen differences on semigroups*, Publ. Math. Debrecen **48** (1996), 117–136.

[5] van der Corput J.G., *Goniometrische functies gekarakteriseerd door een functionaalbetrekking*, Euclides **17** (1940), 55–75.

[6] Sablik M., *A functional congruence revisited*, Grazer Math. Ber. **316** (1992), 181–200.

[7] Száz Á., Száz G., *Additive relations*, Publ. Math. Debrecen **20** (1973), 259–272.

Eszter Gselmann: *On the stability of the modified entropy equation*

In this talk we investigate the Hyers–Ulam stability of the functional equation

$$f(x, y, z) = f(x, y + z, 0) + (y + z)^{\alpha} f\Big(0, \frac{y}{y + z}, \frac{z}{y + z}\Big), \tag{1}$$

where x, y, z are positive real numbers and α is a given real number. Equation (1) is a special case of the so–called *modified entropy equation*, that is,

$$f(x, y, z) = f(x, y + z, 0) + \mu(y + z) f\Big(0, \frac{y}{y + z}, \frac{z}{y + z}\Big), \tag{2}$$

where μ is a given multiplicative function defined on the positive cone of $\mathbb{R}^k$ and (2) is supposed to hold for all elements x, y, z of the above mentioned cone and all operations on vectors are to be understood componentwise.

During the proof of our main result, we will use that the parametric fundamental equation of information is stable and we make also use of the stability of a simple associativity equation. These results will be also presented.

Attila Házy: *On approximately s-convex functions* (Joint work with Pál Burai and Tibor Juhász)

A real valued function $f\colon D \to \mathbb{R}$ defined on an open convex subset D of a normed space X is called *Breckner rationally* (s,d)-*convex* (or briefly $(\mathbb{Q}, s, d)$-*convex*) if it satisfies

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) + d(x,y)$$

for all $x, y \in D$ and $t \in \mathbb{Q} \cap [0,1]$, where $d\colon X \times X \to \mathbb{R}$ is a given function and $s \in]0,1]$ is a fixed parameter.

We prove some regularity and convexity properties for this type functions.

Tomasz Kochanek: *Erdős–Kac theorem for approximately additive functions*

Let $\omega(n)$ stand for the number of different prime factors of a number $n \in \mathbb{N}$. The spectacular theorem of Paul Erdős and Marek Kac asserts, roughly speaking, that for large $N \in \mathbb{N}$ the function

$$\{1, \dots, N\} \ni n \mapsto \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}$$

has a distribution close to the Gaussian distribution given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Trying to answer the general Ulam stability problem with reference to the Erdős–Kac theorem, we define a function $f\colon \mathbb{N} \to \mathbb{R}$ to be *approximately additive* if for relatively prime x, y the difference $f(xy) - f(x) - f(y)$ behaves well enough. Under appropriate assumptions on f (guaranteeing its "similarity" to ω) we show that there are functions

$$\mu(n) = \log\log n + o(\log\log n)\,, \quad \sigma(n) = \sqrt{\log\log n} + o(\sqrt{\log\log n})$$

such that for every $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty}\frac{1}{n}\#\Big\{m\le n:\ \frac{f(m)-\mu(n)}{\sigma(n)}\le x\Big\}=\Phi(x).$$

Barbara Koclęga-Kulpa: *On a class of equations stemming from various quadrature rules* (Joint work with Tomasz Szostok)

We deal with functional equations of the form

$$F(y)-F(x)=(y-x)\sum_{k=1}^{n}a_k f(\lambda_k x+(1-\lambda_k)y), \tag{1}$$

which are connected to quadrature rules of approximate integration. We prove that every function $f\colon \mathbb{R}\to\mathbb{R}$ satisfying the equation (1) with some function $F\colon \mathbb{R}\to\mathbb{R}$, where $\sum_{k=1}^{n}a_k\neq 0$, is a polynomial of degree at most $2n-1$. In our results we do not assume any specific form of the coefficients occuring at the right-hand side of (1) and we allow n to be any positive integer. Thus we extend previous results concerning equations of this type.

Michał Lewicki: *Measurability of (M,N)-Wright convex functions*

Let $I\subset\mathbb{R}$ be an open interval and $M,N\colon I^2\to I$ be means on I satisfying

$$M(x,y)+N(x,y)=x+y,\quad x,y\in I.$$

We give sufficient conditions on M,N such that every Lebesgue measurable solution $f\colon I\to\mathbb{R}$ of the functional inequality

$$f(M(x,y))+f(N(x,y))\le f(x)+f(y),\quad x,y\in I,$$

is convex.

Gyula Maksa: *Remarks to the comparison of weighted quasi-arithmetic means*

The comparison of two weighted quasi-arithmetic means with the same weights is known from the work G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1952. In this talk, we present a comparison theorem for the weighted quasi-arithmetic means – without supposing in advance that the weights are the same. In the proof we use the properties of the (λ,μ)-convex functions.

Janusz Matkowski: *On composition operators*

For nonempty sets X,Y denote by Y^X the family of all functions $\varphi\colon X\to$

Y. Given a set Z and an arbitrary function $h\colon X \times Y \to Z$, the mapping $H\colon Y^X \to Z^X$ defined by

$$H(\varphi)(x) := h(x, \varphi(x)), \quad \varphi \in Y^X J,$$

is called a *composition* (*superposition* or *Nemytskij*) *operator* of a generator h.

Let X is be a metric space, $Y = Z = \mathbb{R}$, $J \subset \mathbb{R}$ be an interval, and let $(\mathcal{F}_1(X,Y), \|\cdot\|_1) \subset Y^X$, and $(\mathcal{F}_2(X,Z), \|\cdot\|_2) \subset Z^X$, be some function normed spaces. By $\mathcal{F}_1(X,J)$ denote the set of all $\varphi \in \mathcal{F}_1(X,Y)$ such that $\varphi\colon X \to J$. Assume that $H\colon \mathcal{F}_1(X,Y) \to \mathcal{F}_2(X,Z)$. We present some properties of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ for which the inequality

$$\|H(\varphi) - H(\psi)\|_2 \leq \gamma(\|\varphi - \psi\|_1), \quad \varphi, \psi \in \mathcal{F}_1(X,J),$$

for an increasing function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ implies that

$$h(x,y) = a(x)y + b(x), \quad x \in X,\ y \in J,$$

for some functions $a, b\colon X \to \mathbb{R}$. The main theorems generalize some earlier results for Lipschitzian and uniformly continuous composition operators in some normed function spaces.

Fruzsina Mészáros: *Functional equations arising from the characterization problems of probability theory*

Functional equations have many interesting applications in characterization problems of probability theory. In my talk I should speak about characterizations of univariate distributions, for example we will see characterizations for the gamma, normal and beta distributions by the help of functional equations. Then characterization of joint distributions by means of conditional distributions will also be considered.

Janusz Morawiec: *Remarks on a generalized refinement equation*

Given a probability space $(\Omega, \mathcal{A}, P)$ and random variables $L, M\colon \Omega \to \mathbb{R}$ we consider the existence of non-trivial L^1-solutions of the following generalized refinement equation

$$\phi(x) = \int_\Omega |L(\omega)| \phi(L(\omega)x + M(\omega)) dP(\omega).$$

Gergő Nagy: *Some preserver problems on quantum structures*

Preserver problems appear in many fields of mathematics, such as functional analysis, geometry and algebra. This presentation is about some selected problems of that kind concerning the mathematical description of quantum theory. Probably the most fundamental result in this area is Wigner's famous unitary-antiunitary theorem on the structure of symmetry transformations. These maps are bijections on the set of the rank-1 projections of a Hilbert space which preserve the so-called transition probability which is a very important concept in quantum information theory. Recent results concerning commutativity preserving transformations on certain quantum structures are also presented. Commutativity is an important relation in quantum mechanics which appears, e.g. in the so-called no-cloning theorem, a well-known statement of quantum information theory. Maps on the set of density operators (or in other words states) which preserve a certain measure of commutativity are also discussed.

Zsolt Páles: *Ng's Theorem revisited*

Che Tat Ng's celebrated result on Wright convex functions states that they can be represented as the sum of a convex and an additive function. In the standard proofs, usually the additive summand is constructed first and then the convex part. We present a new approach to the proof of this theorem by constructing the convex part first. We apply this idea also to the proof of the decomposition of higher-order Wright convex functions obtained jointly with Gyula Maksa.

Justyna Sikorska: *On a D-exponential functional equation*

We study the exponential functional equation

$$f(x+y) = f(x)f(y), \quad (x,y) \in D,$$

with D having some properties. Regardless of the solutions of this equation, which in many special cases are already known, we investigate its stability and consider the pexiderized version of it. Our intention is to give quite general approach to the studies of this subject as well as to describe the properties of D so that the results include those ones concerning orthogonal and some other conditional exponential equations.

Adrienn Varga: *On Daróczy's problem for additive functions*

In this talk we investigate the functional equation

$$\sum_{i=1}^{n} \alpha_i A(\beta_i x) = 0$$

which holds for all $x \in \mathbb{R}$ with an unknown additive function $A\colon \mathbb{R} \to \mathbb{R}$ and fixed real parameters α_i, β_i, where $i = 1, \dots, n$.

Peter Volkmann: *On Hyers–Ulam stability of the differential equation* $f'(t) = f(t)$

Given an interval $I \subset \mathbb{R}, \mathrm{int} I \neq \emptyset$ and a Banach space E, let $f\colon I \to E$ satisfy $\|f'(t) - f(t)\| \leq \varepsilon,\ t \in I$. Then there exists $c \in E$ such that

$$\|f(t) - ce^t\| \leq \varepsilon, \quad t \in I. \tag{1}$$

With 3ε instead of ε in (1) the result is known from Włodzimierz Fechner and Roman Ger (to appear), the special case $E = \mathbb{R}$ of this going back to Claudi Alsina and Roman Ger, J. Inequalities Appl. 2 (1998), 373–380 .

Wirginia Wyrobek: *Joint continuity of biadditive functions with measurable sections* (Joint work with Tomasz Kochanek)

Let G be an Abelian topological group, H be a separable Abelian metric group. We deal with biadditive functions $f\colon G \times G \to H$ whose sections are measurable with respect to an abstract σ-algebra satisfying a Steinhaus-type condition. We show that some natural assumptions on G force such a function f to be jointly continuous. Our result may be applied in particular to Baire and Christensen measurability.

2. Problems and Remarks

1. Remark. *(Completeness of normed spaces as a consequence of the stability of some functional equations.)*

Theorem. *Suppose* $n \in \mathbb{N} = \{1, 2, \dots\}$ *and let* E *be a normed space such that for every* $f\colon \mathbb{N} \to E$ *satisfying*

$$\sup_{\mu,\nu\in\mathbb{N}} \|\Delta_\mu^n f(\nu) - n! f(\mu)\| < \infty$$

there exists $c \in E$ *yielding*

$$\sup_{\mu\in\mathbb{N}} \|f(\mu) - \mu^n c\| < \infty. \tag{1}$$

Then E *is a Banach space.*

Proof. Let $\hat{E}$ be the completion of E, and define $L\colon \hat{E}^{\mathbb{N}} \to \hat{E}^{\mathbb{N}\times\mathbb{N}}$ by $(Lh)(\mu,\nu) = \Delta_\mu^n h(\nu) - n! h(\mu)$ (for $h\colon \mathbb{N} \to \hat{E}, (\mu,\nu) \in \mathbb{N} \times \mathbb{N}$). Then L is linear and its restriction to the Banach space $B(\mathbb{N}, \hat{E})$ of bounded functions $h\colon \mathbb{N} \to \hat{E}$ is bounded.

We consider $\hat{c} \in \hat{E}$. We define a monomial function $m \in \hat{E}^{\mathbb{N}}$ by $m(\mu) = \mu^n \hat{c}$, $\mu \in \mathbb{N}$, hence $Lm = 0$. The space E being dense in $\hat{E}$, we find $f\colon \mathbb{N} \to E$ such that

$$\|f(\mu) - \mu^n \hat{c}\| \le 1, \quad \mu \in \mathbb{N}. \tag{2}$$

Then $Lf = L(f - m) + Lm = L(f - m)$ yields the boundedness of Lf, thus there is $c \in E$ satisfying (1). Inequalities (1), (2) easily imply $\hat{c} = c$, therefore $\hat{c} \in E$. We have shown $\hat{E} \subset E$, i.e. E is complete. □

For $n = 2$ our result gives a positive answer to a question of Mohammad Sal Moslehian, Problem 3, p. xlii of: *Inequalities and Applications*, edited by Catherine Bandle et al., Birkhäuser, Basel, 2009.

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*Supported by Grant OTKA NK 68040

2. Problem. The Matkowski–Sutô problem for two-variable quasi-arithmetic means is to determine the continuous strictly monotone solutions $f, g : I \to \mathbb{R}$ for the functional equation

$$f^{-1}\Big(\frac{f(x) + f(y)}{2}\Big) + g^{-1}\Big(\frac{g(x) + g(y)}{2}\Big) = x + y, \quad x, y \in I. \tag{3}$$

The solutions were first described by O. Sutô [7, 8] in the class of analytic functions. Matkowski [5] obtained the same solutions by assuming only that the functions are twice continuously differentiable. After some preliminary results ([1, 2]), the above equation was solved without any regularity conditions in [3]. This approach was extended in some recent papers, e.g., in [4].

The analogous problem for three-variable quasi-arithmetic means is to determine the continuous strictly monotone solutions $f, g, h\colon I \to \mathbb{R}$ for the functional equation

$$\begin{aligned} f^{-1}\Big(\frac{f(x) + f(y) + f(z)}{3}\Big) &+ g^{-1}\Big(\frac{g(x) + g(y) + g(z)}{3}\Big) \\ &+ h^{-1}\Big(\frac{h(x) + h(y) + h(z)}{3}\Big) = x + y + z, \quad x, y, z \in I. \end{aligned} \tag{4}$$

In the paper [5], Matkowski also solved equation (4) (and the analogous higher-order ones) under twice continuous differentiability of the data. The solution set of (4) is much simpler than that of (3): All the (strictly increasing twice continuously differentiable) functions f, g, and h satisfying (4) have to be affine, i.e., the three quasi-arithmetic means are equal to the arithmetic mean.

The *open problem* is to show that the solution set of functional equation (4) is not bigger in the class of strictly increasing continuous functions. In other words, if f, g, and h are strictly increasing continuous functions satisfying (4) then they are also twice (and consequently infinitely many times) continuously differentiable.

As a preliminary step toward this regularity improving result, we present the following

Proposition. *Let f, g, and h be strictly increasing continuous functions satisfying* (4). *Then f, g, and h and also their inverses are locally Lipschitzian functions.*

Proof. By the symmetry, it suffices to show that f and f^{-1} are locally Lipschitzian. Rewrite (4) in the following form:

$$g^{-1}\Big(\frac{g(x)+g(y)+g(z)}{3}\Big)+h^{-1}\Big(\frac{h(x)+h(y)+h(z)}{3}\Big) \\ = x+y+z-f^{-1}\Big(\frac{f(x)+f(y)+f(z)}{3}\Big), \quad x,y,z\in I.$$

Now observe that, for fixed elements $y, z \in I$, the left hand side of this equation is a strictly increasing function of the variable $x \in I$, therefore, for fixed $y, z \in I$, the function

$$x \mapsto x - f^{-1}\Big(\frac{f(x)+f(y)+f(z)}{3}\Big)$$

is strictly increasing on the interval I. Assume (without loss of generality) that f is increasing. Substituting $x := f^{-1}(u)$ and $y := z := f^{-1}(v)$ (where $u, v \in f(I)$), we can see that, for fixed $v \in f(I)$, the map

$$u \mapsto f^{-1}(u) - f^{-1}\Big(\frac{u+2v}{3}\Big)$$

is strictly increasing on $f(I)$. From this monotonicity property, by the results of the paper [6], it follows that f^{-1} and its inverse function f are locally Lipschitzian. □

References

[1] Daróczy Z., Maksa Gy., Páles Zs., *Extension theorems for the Matkowski–Sutô problem*, Demonstratio Math. **33** (2000), no. 3, 547–556.

[2] Daróczy Z., Páles Zs., *On means that are both quasi-arithmetic and conjugate arithmetic*, Acta Math. Hungar. **90** (2001), no. 4, 271–282.
[3] Daróczy Z., Páles Zs., *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen **61** (2002), no. 1–2, 157–218.
[4] Jarczyk J., *Invariance of weighted quasi-arithmetic means with continuous generators*, Publ. Math. Debrecen **71** (2007), no. 3–4, 279–294.
[5] Matkowski J., *Invariant and complementary quasi-arithmetic means*, Aequationes Math. **57** (1999), no. 1, 87–107.
[6] Páles Zs., *A regularity theorem for composite functional equations*, Acta Sci. Math. (Szeged) **69** (2003), 591–604.
[7] Sutô O., *Studies on some functional equations I*, Tôhoku Math. J. **6** (1914), 1–15.
[8] Sutô O., *Studies on some functional equations II*, Tôhoku Math. J. **6** (1914), 82–101.

Zsolt Páles

3. Problem. The Hyers-Ulam stability of the Cauchy equation (for functions acting on $\mathbb{R}$) may be stated in the following way, if function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$f(x+y) - f(x) - f(y) \in [-\varepsilon, \varepsilon]$$

then there exists a unique function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$g(x+y) - g(x) - g(y) \in \{0\}$$

which satisfies $|g - f| \leq \varepsilon$. Therefore it is natural to state the following problem, as a possible generalization of the above result. Prove that if $A, B \subset \mathbb{R}$ are such sets that $d_H(A,B) \leq \varepsilon$ and $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x+y) - f(x) - f(y) \in A$$

then there exists function g satisfying

$$g(x+y) - g(x) - g(y) \in B$$

such that $|f - g| \leq \varepsilon$. Of course d_H means here the Hausdorff distance.

Tomasz Szostok

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