ON VARIATIONAL APPROACH TO ECONOMIC EQUILIBRIUM — TYPE PROBLEM

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Abstract. The paper deals with an optimization problem in which minima of a finite collection of objective functions satisfy some unilateral constraints and are linked together by a certain subdifferential law. The governing relations are variational inequalities defined on a nonconvex feasible set. By reducing the problem to a variational inequality involving nonmonotone multivalued mapping defined over a nonnegative orthant, the existence of solutions is established under the assumption that constrained functions are positive homogeneous of degree at most one.

1. Introduction

Consider the problem of finding the vectors $x_j \in \mathbb{R}^n_+$, $j = 1, \ldots, m$, which minimize a finite collection of convex objectives $V_j : \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$, $j = 1, \ldots, m$. The minimizers are subject to unilateral constraints $\langle A_j \pi, x_j \rangle \leq \phi_j(\pi)$ with given nonnegative continuous functions $\phi_j : \mathbb{R}^n_+ \to \mathbb{R}_+$. The problem is to find a price vector $\pi \in \mathbb{R}^n_+$ and a multivector $(x_j) \in (\mathbb{R}^n_+)^m$ which are linked together by a subdifferential relation of the form $\sum_{j=1}^m A_j^T x_j \in \partial \Phi_+(\pi)$, Φ_+ being a convex function. This problem has been studied in [9] under the assumption that $\phi_j(\tau) \geq \delta_j$, $\delta_j > 0$. In the presented approach we begin with establishing existence of solutions for the case when ϕ_j, $j = 1, \ldots, m$, are positive homogeneous of degree $\theta_j < 1$. We then extend this result toward the case when ϕ_j, $j = 1, \ldots, m$, are positive homogeneous of degree one. This is important for the study of equilibrium models of welfare economics.

The main feature of the problem is that the feasible set of the unknowns π, x_j, $j = 1, \ldots, m$, is nonconvex and, hence, the standard theory of variational inequalities (cf. [6], [3]) cannot be used directly to obtain solutions. The approach

Received: 4. 10. 2005. Revised: 12. 06. 2006.

(2000) Mathematics Subject Classification: Primary 49J40; Secondary 91B50.

Key words and phrases: Variational inequality, Duality, Walrasian equilibrium.

presented here does not involve the notion of Pareto optimum or its generalizations (cf. [11], [8], [7], [4] and the references therein) but, roughly speaking, is based on the analysis of objectives' parametrized constrained minima $\big(x_j(\pi)\big)$. Some ideas from [10] concerning of nonmonotone inequality problems are applied.

2. Formulation of the problem

Denote by $\mathbb{R}^n$ the Euclidean space of all vectors $x = \big[x_1,\ldots,x_n\big]$, $x_i \in \mathbb{R}$, $i = 1,\ldots,n$, equipped with the inner product $\big\langle\cdot,\cdot\big\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. By $\mathbb{R}^{n\times n}$ we denote all $n \times n$ real valued matrices. Moreover, the following notations will be used:

$$\begin{aligned}
&\mathbb{R}_+ = \{\alpha \in \mathbb{R} \colon \alpha \geq 0\},\\
&\mathbb{R}^n_+ = \big\{x = \big[x_1,\ldots,x_n\big] \in \mathbb{R}^n \colon x_i \geq 0,\ \ \forall i = 1,\ldots,n\big\},\\
&\mathbb{R}^{n\times n}_+ = \big\{A = (A_{ik}) \in \mathbb{R}^{n\times n} \colon A_{ik} \geq 0,\ \ \forall i,k = 1,\ldots,n\big\},\\
&\mathbb{R}^n_- = \big\{x = \big[x_1,\ldots,x_n\big] \in \mathbb{R}^n \colon x_i \leq 0,\ \ \forall i = 1,\ldots,n\big\}.
\end{aligned}$$

Throughout the paper it will be assumed that the functions

$$V_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \quad j = 1,\ldots,m,$$

are convex, proper and lower semicontinuous. Assume that the functions

$$\phi_j : \mathbb{R}^n_+ \to \mathbb{R}_+, \quad j = 1,\ldots,m,$$

are continuous, positive homogeneous of degree θ_j, $0 < \theta_j \leq 1$ and such that

$$\min\{\phi_j(\tau) \colon \tau \in \mathbb{R}^n_+,\ |\tau| = 1\} = \gamma_j,\ \gamma_j > 0.$$

Moreover, let the matrices $A_j \in \mathbb{R}^{n\times n}_+$ satisfy

$$\operatorname{Ker} A_j = \{0\}, \quad j = 1,\ldots,m,$$

where $\operatorname{Ker} A_j = \{\tau \in \mathbb{R}^n_+ \colon A_j\tau = 0\}$. Furthermore, let

$$\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$$

be a convex, proper, lower semicontinuous function.
Recall that if H is a Hilbert space and $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a convex function, the subdifferential $\partial\varphi \colon H \to 2^H$ is defined by

$$\partial\varphi(u) = \{w \in H \colon \varphi(v) - \varphi(u) \geq \langle w, v-u\rangle,\ \forall v \in H\},$$

provided that $\varphi(u) < +\infty$ or $\partial\varphi(u) = \emptyset$ if $\varphi(u) = +\infty$.

We are now in a position to formulate the main problem of the paper.

PROBLEM (P). Find $\pi \in \mathbb{R}^n_+$ and $x_j \in \mathbb{R}^n_+$, $j = 1, \dots, m$ that the following conditions are satisfied:

$$(PM) \qquad V_j(x_j) = \min\left\{V_j(x) \colon \langle A_j\pi, x\rangle \le \phi_j(\pi) \wedge x \in \mathbb{R}^n_+\right\}, j = 1, \dots, m,$$

$$(PE) \qquad \left\langle -\sum_{j=1}^{m} A_j^T x_j, \tau - \pi \right\rangle + \Phi(\tau) - \Phi(\pi) \ge 0, \quad \forall \tau \in \mathbb{R}^n_+.$$

3. The case ϕ_j positive homogeneous of degree $0 < \theta_j < 1$

In this section we assume that ϕ_j, $j = 1, \dots, m$, are positive homogeneous functions of degree $0 < \theta_j < 1$, respectively, i.e.

(H_4^1) $\phi_j(t\tau) = t^{\theta_j}\phi_j(\tau)$, $\forall \tau \in \mathbb{R}^n_+$, $\forall t \ge 0$, where $0 < \theta_j < 1$.

and

(H_0) $\min\{\phi_j(\tau) \colon \tau \in \mathbb{R}^n_+,\ |\tau| = 1\} = \gamma_j$, for some $\gamma_j > 0$.

To find the solution of the problem (P) we use the method analogous to that presented in [9]. We begin with establishing the sufficient conditions for the existence of a solution of the problem (PM) in dependence on π. Then, we define a multivalued operator $\mathcal{R}$ which will allow us to describe the problem $(PM) - (PE)$ in the form of a variational inequality.

Fix $j \in \{1, \dots, m\}$ and $\pi \in \mathbb{R}^n_+$ with $\pi \ne 0$.
Denote by ind_K the indicator function of a set K, i.e.,

$$\operatorname{ind}_K(y) = \begin{cases} 0 & \text{if } y \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

In order to reformulate the problem (PM) we introduce the functions $\overline{V}_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\overline{V}_j := V_j + \operatorname{ind}_{\mathbb{R}^n_+}.$$

Moreover, we define a linear operator $A_{j\pi} : \mathbb{R}^n \to \mathbb{R}$ by

$$A_{j\pi}x = \langle A_j\pi, x\rangle, \quad x \in \mathbb{R}^n.$$

The subdifferential of the indicator function of $\{t \in \mathbb{R} \colon t \le \phi_j(\pi)\}$ is denoted by $\partial\operatorname{ind}_{\le\phi_j(\pi)} : \mathbb{R} \to 2^{\mathbb{R}}$.

We are now ready to reformulate (PM) as follows

PROBLEM $(\overline{P}_{j\pi})$. $\overline{v}_{j\pi} := \inf\{\overline{V}_j(x) + \operatorname{ind}_{\le\phi_j(\pi)}(A_{j\pi}x) : x \in \mathbb{R}^n\}$.

According to the Fenchel theory (cf. [1]) the dual problem of $(\overline{P}_{j\pi})$ is

$$\underline{v}_{j\pi} := \inf\{\overline{V}_j^\star(-A_{j\pi}^\star\alpha) + \operatorname{ind}^\star_{\leq\phi_j(\pi)}(\alpha) : \alpha \in \mathbb{R}\},$$

with $A_{j\pi}^\star : \mathbb{R} \to \mathbb{R}^n$ being the transpose of $A_{j\pi}$. Since $A_{j\pi}^\star\alpha = \alpha A_j\pi$ and $\operatorname{ind}^\star_{\leq\phi_j(\pi)}(\alpha) = \alpha\phi_j(\pi) + \operatorname{ind}_{\mathbb{R}_+}(\alpha)$, $\alpha \in \mathbb{R}$, the dual problem $(\underline{P}_{j\pi})$ reads as

PROBLEM $(\underline{P}_{j\pi})$. $\underline{v}_{j\pi} := \inf\{\overline{V}_j^\star(-\alpha A_j\pi)+\alpha\phi_j(\pi)+\operatorname{ind}_{\geq 0}(\alpha) : \alpha \in \mathbb{R}\}$, where $\overline{V}_j^\star : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ stands for the Fenchel conjugate function of $\overline{V}_j$ defined by

$$\overline{V}_j^\star(\mu) := \sup_{x\in\mathbb{R}^n} \{\langle\mu, x\rangle - \overline{V}_j(x)\}, \quad \mu \in \mathbb{R}^n. \tag{1}$$

Let $\operatorname{Dom} U$ stands for the effective domain of U, $B(0,r)$ — for an open ball centered at the origin with radius $r > 0$, $\operatorname{Int} K$ — for the interior of K and $\operatorname{cl} K$- for the closure of $K \subset \mathbb{R}^n$.

From the Fenchel theorem (cf. [1]) we get

PROPOSITION 1. *Assume that*

$$0 \in \operatorname{cl}(\operatorname{Dom} \partial\overline{V}_j). \tag{2}$$

Then for any $\pi \in \mathbb{R}^n_+ \setminus \{0\}$,

$$\overline{v}_{j\pi} + \underline{v}_{j\pi} = 0 \tag{3}$$

and there exists a number $\alpha_j \in \mathbb{R}_+$ *such that*

$$\overline{V}_j^\star(-\alpha_j A_j\pi) + \alpha_j\phi_j(\pi) = \underline{v}_{j\pi} \tag{4}$$

where α_j *is a solution of* $(\underline{P}_{j\pi})$. *In addition, if*

$$(\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \operatorname{Int}\operatorname{Dom}\overline{V}_j^\star \quad \textit{for some } r_j > 0, \tag{5}$$

then there exists a vector $x_j \in \mathbb{R}^n_+$ *such that* $\langle A_j\pi, x_j\rangle - \phi_j(\pi) \leq 0$ *and*

$$V_j(x_j) = \overline{v}_{j\pi} \tag{6}$$

where x_j *is a solution of* $(\overline{P}_{j\pi})$. *Moreover, the following compatibility conditions hold*

$$-\alpha_j A_j\pi \in \partial\overline{V}_j(x_j) \tag{7}$$

$$\alpha_j \in \partial\operatorname{ind}_{\leq\phi_j(\pi)}(\langle A_j\pi, x_j\rangle), \tag{8}$$

and the α_j *satisfies the variational inequality*

$$\langle A_j\pi, -\partial\overline{V}_j^\star(-\alpha_j A_j\pi)\rangle(t - \alpha_j) + \phi_j(\pi)(t - \alpha_j) \geq 0, \quad \forall t \geq 0. \tag{9}$$

PROOF. Under (2) and (5) it is enough to verify that

$$0 \in \operatorname{Int}_{\mathbb{R}}\left[A_{j\pi}\operatorname{Dom}\left(\overline{V}_j\right) - \operatorname{Dom}\left(\operatorname{ind}_{\leq \phi_j(\pi)}\right)\right]$$

and (since $A_j\pi \in \mathbb{R}^n_+ \setminus \{0\}$),

$$0 \in \mathbb{R}_+ A_j\pi + (\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \operatorname{Int}_{\mathbb{R}^n}\left[A^\star_{j\pi}\operatorname{Dom}\left(\operatorname{ind}^\star_{\leq \phi_j(\pi)}\right) + \operatorname{Dom}\left(\overline{V}^\star_j\right)\right]$$

and then to invoke (Theorem 3.2, p. 38, [1]). □

We now get the following result whose proof is similar to that of Theorem 1, p. 149 [9].

THEOREM 1. *Assume that for $j = 1, \dots, m$ the following conditions hold:*

(H_1) $0 \in \operatorname{cl}(\operatorname{Dom}\partial\overline{V}_j)$, $(\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \operatorname{Int}\operatorname{Dom}\overline{V}^\star_j$ *for some* $r_j > 0$;

(H_2) $\left\{x \in \mathbb{R}^n_+ : \{\langle x^\star, x\rangle : x^\star \in \partial\overline{V}_j(x)\} \cap \mathbb{R}_- \neq \emptyset\right\} \subset B_{\mathbb{R}^n}(0, M_j)$ *for some* $M_j > 0$;

(H^1_4) $\phi_j(t\tau) = t^{\theta_j}\phi_j(\tau)$, $\forall \tau \in \mathbb{R}^n_+$, $\forall t \geq 0$, *where* $0 < \theta_j < 1$;

(H_0) $\min\{\phi_j(\tau) \colon \tau \in \mathbb{R}^n_+, |\tau| = 1\} =: \gamma_j$, $\gamma_j > 0$.

Then for any $\pi \in \mathbb{R}^n_+ \setminus \{0\}$ the optimization problem of finding a vector $x_j \in \mathbb{R}^n_+$ such that

$$V_j(x_j) = \min\{V_j(y) \colon \forall y \in \mathbb{R}^n_+ \text{ with } \langle A_j\pi, y\rangle \leq \phi_j(\pi)\} \tag{10}$$

has at least one solution.

Moreover, there exists a number $\alpha_j \in \Lambda_j(\pi)$ such that

$$x_j \in \partial\overline{V}^\star_j(-\alpha_j A_j\pi). \tag{11}$$

Here $\Lambda_j(\pi)$ stands for the set of all solutions of variational inequality

$$\left\langle A_j\pi, -\partial\overline{V}^\star_j(-\alpha_j A_j\pi)\right\rangle(t - \alpha_j) + \phi_j(\pi)(t - \alpha_j) \geq 0, \quad \forall t \geq 0. \tag{12}$$

Additionally, $\Lambda_j : \mathbb{R}^n_+ \to 2^{\mathbb{R}_+}$ is an upper semicontinuous mapping from $\mathbb{R}^n_+$ into $2^{\mathbb{R}_+}$ assuming nonempty, closed, convex and bounded values when extended to $\mathbb{R}^n_+$ by setting $\Lambda_j(0) := \{0\}$.

PROOF. Let $j \in \{1, \dots, m\}$. From Proposition 1 it follows that for any $\pi \in \mathbb{R}^n_+ \setminus \{0\}$ we have $\Lambda_j(\pi) \neq \emptyset$. Furthermore, $\Lambda_j(\pi)$ as a set of all solutions of variational inequality (12) involving maximal monotone mapping $G_j(t) := \left\langle A_j\pi, -\partial\overline{V}^\star_j\left(-tA_j\pi\right)\right\rangle$, $t \geq 0$, is convex and closed (see [3]).

Now let us notice that for sufficiently small $\delta > 0$ the condition $|\pi| \leq \delta$ implies $\Lambda_j(\pi) = \{0\}$. To the contrary, suppose that for every $\delta > 0$ there exist $\pi^\delta \in \mathbb{R}^n_+$, $x^\delta_j \in \mathbb{R}^n_+$, $\alpha^\delta_j > 0$ such that $0 < |\pi^\delta| \leq \delta$ and

$$-\alpha^\delta_j A_j\pi^\delta \in \partial\overline{V}_j(x^\delta_j), \; \left\langle A_j\pi^\delta, x^\delta_j\right\rangle - \phi_j(\pi^\delta) \in \partial\operatorname{ind}_{\geq 0}(\alpha^\delta_j),$$

(which means that $\alpha_j^\delta \in \Lambda_j(\pi^\delta)$). Hence we get that

$$0 \geq -\alpha_j^\delta \langle A_j \pi^\delta, x_j^\delta \rangle \in \langle \partial \overline{V}_j(x_j^\delta), x_j^\delta \rangle.$$

Therefore, by the hypothesis (H_2), we obtain that $|x_j^\delta| \leq M_j$. In the case $\alpha_j^\delta > 0$, the condition $\langle A_j \pi^\delta, x_j^\delta \rangle - \phi_j(\pi^\delta) \in \partial \mathrm{ind}_{\geq 0}(\alpha_j^\delta)$ takes the form $\langle A_j \pi^\delta, x_j^\delta \rangle = \phi_j(\pi^\delta)$. Using (H_0) and (H_4^1) we get

$$0 < \gamma_j = |\pi^\delta|^{1-\theta_j} \langle A_j \frac{\pi^\delta}{|\pi^\delta|}, x_j^\delta \rangle \leq |A_j| M_j \delta^{1-\theta_j},$$

which leads to a contradiction.

Therefore in some neighbourhood of 0 in $\mathbb{R}_+^n$, say O, we have $\Lambda_j(\tau) = \{0\}$ $\forall \tau \in O$. This justifies the extention of $\Lambda_j : \mathbb{R}_+^n \setminus \{0\} \to 2^{\mathbb{R}_+}$ to $\mathbb{R}_+^n$ by setting $\Lambda_j(0) := \{0\}$. The extension preserves the upper seminontinuity of $\Lambda_j(\cdot)$ at 0. Hence we have established that $\partial \overline{V}_j^\star(0) \neq \emptyset$, $\forall j = 1, \ldots, m$.

For the boundedness of $\alpha_j \in \Lambda_j(\pi)$, $\pi \neq 0$ we suppose to the contrary that there exists $\{\alpha_j^k\}_{k \in N} \subset \Lambda_j(\pi)$, $\alpha_j^k \to +\infty$, as $k \to +\infty$. There exists $x_j^k \in \mathbb{R}_+^n$ such that $-\alpha_j^k A_j \pi \in \partial \overline{V}_j(x_j^k)$, $\langle A_j \pi, x_j^k \rangle - \phi_j(\pi) \in \partial \mathrm{ind}_{\geq 0}(\alpha_j^k)$, $k \in N$. Hence we get

$$-\alpha_j^k \phi_j(\pi) = \langle -\alpha_j^k A_j \pi, x_j^k \rangle = V_j(x_j^k) + \overline{V}_j^\star(-\alpha_j^k A_j \pi).$$

Using the fact $\partial \overline{V}_j^\star(0) \neq \emptyset$ we obtain

$$V_j(y) \geq -c_j, \quad \forall y \in \mathrm{Dom}\, V_j, \text{ for some } c_j \in \mathbb{R}.$$

From the definiton Fenchel conjugate function we get the estimate

$$0 < \gamma_j |\pi|^{\theta_j} \leq \phi_j(\pi) \leq \frac{c_j + V_j(y)}{\alpha_j^k} + \langle A_j \pi, y \rangle, \quad \forall y \in \mathrm{Dom}\, V_j.$$

Letting $k \to \infty$ we get

$$0 < \gamma_j |\pi|^{\theta_j} \leq \langle A_j \pi, y \rangle, \quad \forall y \in \mathrm{Dom}\, V_j,$$

which contradicts the assumption $0 \in \mathrm{cl}\,(\mathrm{Dom}\, \partial \overline{V}_j)$.

Finally it remains to show the upper semicontinuity of Λ_j on $\mathbb{R}_+^n \setminus \{0\}$. To this end assume that $\{\pi^k\}_{k \in N} \subset \mathbb{R}_+^n$ and $\alpha_j^k \in \Lambda_j(\pi^k)$ are such that $\pi^k \to \pi^\star$ in $\mathbb{R}_+^n \setminus \{0\}$ and $\alpha_j^k \to \alpha_j^\star$ in $\mathbb{R}_+$, as $k \to +\infty$ for some $\pi^\star \in \mathbb{R}_+^n$ and $\alpha_j^\star \in \mathbb{R}_+$, respectively. We now aim to show that $\alpha_j^\star \in \Lambda_j(\pi^\star)$.

Using the fact that there exists $x_j^k \in \mathbb{R}_+^n$ with $-\alpha_j^k A_j \pi^k \in \partial \overline{V}_j(x_j^k)$, we arrive at $-\alpha_j^k \langle A_j \pi^k, x_j^k \rangle \in \langle \partial \overline{V}_j(x_j^k), x_j^k \rangle$. But the left hand side of this inclusion is nonpositive. Therefore, by the hypothesis (H_2), the boundedness of $\{x_j^k\}_{k \in N}$ in $\mathbb{R}_+^n$ results. Consequently, one can suppose that $x_j^k \to x_j^\star$ in $\mathbb{R}^n$, as $k \to +\infty$ for some $x_j^\star \in \mathbb{R}_+^n$ (by passing to a subsequence, if necessary). Taking into account the conditions

$$-\alpha_j^k A_j \pi^k \in \partial \overline{V}_j(x_j^k), \quad \langle A_j \pi^k, x_j^k \rangle - \phi_j(\pi^k) \in \partial \mathrm{ind}_{\geq 0}(\alpha_j^k),$$

we are allowed to take the limit as $k \to \infty$. By the continuity of $\phi_j(\cdot)$, the maximal monotonicity of $\partial \overline{V}_j(\cdot)$ and $\partial \mathrm{ind}_{\geq 0}(\cdot)$, we get

$$-\alpha_j^\star A_j \pi^\star \in \partial \overline{V}_j(x_j^\star), \quad \langle A_j \pi^\star, x_j^\star \rangle - \phi_j(\pi^\star) \in \partial \mathrm{ind}_{\geq 0}(\alpha_j^\star),$$

from which we deduce that $\alpha^\star \in \Lambda_j(\pi^\star)$. The proof is complete. □

Now the problem (PE) can be considered. Taking into account (11) we introduce a multivalued mapping $\mathcal{R} : \mathbb{R}^n_+ \to 2^{\mathbb{R}^n}$ by setting

$$\mathcal{R}(\pi) := -\sum_{j=1}^{m} A_j^T \partial \overline{V}_j^\star(-\Lambda_j(\pi) A_j \pi), \quad \pi \in \mathbb{R}^n_+, \tag{13}$$

where $y \in \mathcal{R}(\pi)$ if and only if there exist $\alpha_j \in \Lambda_j(\pi)$, $x_j \in \partial \overline{V}_j^\star(-\alpha_j A_j \pi)$, for any $j = 1, \dots, m$, such that $y = -\sum_{j=1}^m A_j^T x_j$.

It is easily seen that an equivalent reformulation of the problem (PE) is:

Find $\pi \in \mathbb{R}^n_+$ and $X \in \mathcal{R}(\pi)$ such that

$$\langle X, \tau - \pi \rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \tau \in \mathbb{R}^n_+. \tag{14}$$

As far as $\mathcal{R}$ is concerned we have the following result.

PROPOSITION 2. *Suppose that for any $j = 1, \dots, m$ the assumptions (H_0), (H_1), (H_2), (H_4^1) are satisfied. Then $\mathcal{R} : \mathbb{R}^n_+ \to 2^{\mathbb{R}^n}$ is an upper semicontinuous mapping with nonempty, closed, convex and bounded values.*

PROOF. As in the proof of Proposition 4, p. 150 [9] we get that the mulitvalued mapping $\mathcal{R}$: $\mathbb{R}^n_+ \to 2^{\mathbb{R}^n}$ assumes closed, convex and bounded valued. It remains to show the upper semicontinuity of $\mathcal{R}$ on $\mathbb{R}^n_+$.

Suppose that $\{\pi^k\}_{k \in N} \subset \mathbb{R}^n_+$, $\pi_k \to \pi$, $y_k \in \mathcal{R}(\pi_k)$, $y_k \to y$ in $\mathbb{R}^n$, as $k \to +\infty$. We wish to show that $y \in \mathcal{R}(\pi)$.

Using (13) we get

$$y_k = -\sum_{j=1}^{m} A_j^T x_j^k, \quad x_j^k \in \partial \overline{V}_j^\star(-\alpha_j^k A_j \pi^k), \ \alpha_j^k \in \Lambda_j(\pi^k), \ k \in N, j = 1, \dots, m.$$

Since $0 \geq \langle -\alpha_j^k A_j \pi^k, x_j^k \rangle \in \langle \partial \overline{V}_j(x_j^k), x_j^k \rangle$, from (H_2) we obtain the boundedness of $\{x_j^k\}_{k \in N}$, $j = 1, \dots, m$.

We shall now consider the bounds for $\{\alpha_j^k\}_{k \in N}$, $j = 1, \dots, m$.

If $\pi_k \to 0$ then from the proof of Theorem 1 we get that for sufficiently large k, $\alpha_j^k = 0$ for any $j = 1, \dots, m$.

If $\pi_k \to \pi$, $\pi \neq 0$, then to the contrary suppose that for some $j = 1, \dots, m$ $\alpha_j^k \to +\infty$, as $k \to +\infty$. As in the proof of Theorem 1 we get the estimate

$$|\pi^k|^{\theta_j} \gamma_j \leq \phi_j(\pi_k) \leq \frac{c_j + V_j(y)}{\alpha_j^k} + \langle A_j \pi_k, y \rangle, \quad \forall y \in \mathrm{Dom}\, V_j,$$

where $c_j \in \mathbb{R}$ such that $V_j(y) \geq -c_j$ for any $y \in \mathrm{Dom}\, V_j$.

Letting $k \to +\infty$ we obtain

$$0 < |\pi|^{\theta_j}\gamma_j \leq \langle A_j\pi, y\rangle, \quad \forall y \in \mathrm{Dom}\, V_j,$$

which contradicts $0 \in \mathrm{cl}\,(\mathrm{Dom}\, \partial\overline{V}_j)$.

This shows that $y \in \mathcal{R}(\pi)$ and completes the proof. □

We can now formulate the main result in the case when ϕ_j are positive homogeneous of degree $\theta_j < 1$.

THEOREM 2. *Suppose that for any $j = 1, \dots, m$ the following conditions hold:*

(H_1) $0 \in \mathrm{cl}\,(\mathrm{Dom}\, \partial\overline{V}_j)$, $(\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \mathrm{Int}\,\mathrm{Dom}\, \overline{V}_j^\star$ *for some* $r_j > 0$;

(H_2) $\{x \in \mathbb{R}^n_+ : \{\langle x^\star, x\rangle : x^\star \in \partial\overline{V}_j(x)\} \cap \mathbb{R}_- \neq \emptyset\} \subset B_{\mathbb{R}^n}(0, M_j)$ *for some* $M_j > 0$;

(H_4^1) $\phi_j(t\tau) = t^{\theta_j}\phi_j(\tau)$, $\forall \tau \in \mathbb{R}^n_+$, $\forall t \geq 0$, *where* $0 < \theta_j < 1$;

(H_0) $\min\{\phi_j(\tau) \colon \tau \in \mathbb{R}^n_+, |\tau| = 1\} =: \gamma_j$, $\gamma_j > 0$;

(H_5^0) $\{\tau \in \mathbb{R}^n_+ : \Phi(\tau) \leq \sum_{j=1}^m \phi_j(\tau) + \Phi(0)\} \subset B_{\mathbb{R}^n}(0, M)$ *for some* $M > 0$;

(H_6) $\partial\Phi_+(0) \neq \emptyset$, *where* $\Phi_+ := \Phi + \mathrm{ind}_{\mathbb{R}^n_+}$;

(H_9^1) $\sum_{j=1}^m A_j^T x_j \notin \partial\Phi_+(0)$ *for any* $x_j \in \partial\overline{V}_j^\star(0)$.

Then the problem of finding elements $\pi \in \mathbb{R}^n_+$ and $X \in \mathcal{R}(\pi)$ satisfying the variational inequality

$$\langle X, \tau - \pi\rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall \tau \in \mathbb{R}^n_+, \tag{15}$$

has at least one solution. Equivalently, there exists $(\pi, (x_j), (\alpha_j))$ such that

$$\begin{cases} -\alpha_j A_j\pi \in \partial\overline{V}_j(x_j), \\ \langle A_j\pi, x_j\rangle - \phi_j(\pi) \in \partial\mathrm{ind}_{\geq 0}(\alpha_j), \\ \Phi(\tau) - \Phi(\pi) \geq \left\langle \tau - \pi, \sum_{j=1}^m A_j^T x_j\right\rangle, \quad \forall \tau \in \mathbb{R}^n_+, \end{cases}$$

which means that (P) has nontrivial solutions.

REMARK 1. The nontrival solution for the problem (P) means an element $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}^n_+ \times (\mathbb{R}^n_+)^m \times (\mathbb{R}_+)^m$ fulfilling $(PM) - (PE)$ and such that $\pi \neq 0$.

PROOF OF THEOREM 2. It is well known that if (H_6) holds then $T_\lambda := \left((\partial\Phi_+)^{-1} + \lambda I\right)^{-1}$ (with $I : \mathbb{R}^n \to \mathbb{R}^n$ being the identity) is a maximal monotone, bounded operator with $\mathrm{Dom}\, T_\lambda = R^n$ for any $\lambda > 0$ (cf. [2], p. 280). It is easily seen that $\mathcal{R} + T_\lambda$ is a pseudomonotone mapping on $\mathbb{R}^n_+$ (see Definition 7,

p. 288, [2]). Thus applying (Theorem 15, p. 289, [2]) we deduce the existence of $\pi_\lambda \in B_{\mathbb{R}^n}(0,2M) \cap \mathbb{R}^n_+$ such that

$$\langle \tau - \pi_\lambda, \mathcal{R}(\pi_\lambda) + T_\lambda(\pi_\lambda) \rangle \geq 0, \quad \forall \tau \in \mathbb{R}^n_+ \cap B_{\mathbb{R}^n}(0,2M), \tag{16}$$

with M satisfying (H_5^0). Thus there exist $x_j^\lambda \in \mathbb{R}^n_+ \cap B_{\mathbb{R}^n}(0,M_j)$, $y_\lambda \in T_\lambda(\pi_\lambda)$ and $\mu_\lambda \in \mathbb{R}^n_+$ with the properties that

$$-\sum_{j=1}^m A_j^T x_j^\lambda \in \mathcal{R}(\pi_\lambda), \quad y_\lambda \in \partial\Phi_+(\mu_\lambda), \quad |\lambda y_\lambda| = |\pi_\lambda - \mu_\lambda|,$$

$$\langle \pi_\lambda - \mu_\lambda, \lambda y_\lambda \rangle = |\pi_\lambda - \mu_\lambda| |\lambda y_\lambda|$$

and

$$\Big\langle \tau - \pi_\lambda, -\sum_{j=1}^m A_j^T x_j^\lambda + y_\lambda \Big\rangle \geq 0, \quad \forall \tau \in \mathbb{R}^n_+ \cap B_{\mathbb{R}^n}(0,2M). \tag{17}$$

Substitution $\tau = 0$ into (17) leads to the estimate

$$\Big\langle \pi_\lambda, \sum_{j=1}^m A_j^T x_j^\lambda \Big\rangle \geq \langle \pi_\lambda - \mu_\lambda, y_\lambda \rangle + \langle \mu_\lambda, y_\lambda \rangle,$$

from which it follows that

$$\sum_{j=1}^m \phi_j(\pi_\lambda) \geq \frac{|\pi_\lambda - \mu_\lambda|^2}{\lambda} + \Phi(\mu_\lambda) - \Phi(0).$$

Taking into account that $\Phi(\mu_\lambda) \geq -a|\mu_\lambda| - b$, for some $a, b > 0$ as Φ_+ being proper, lower semicontinuous convex, and that $\pi_\lambda \in B_{\mathbb{R}^n}(0,2M)$, we get

$$C \geq -a|\pi_\lambda - \mu_\lambda| + \frac{|\pi_\lambda - \mu_\lambda|^2}{\lambda}$$

for some $C > 0$. Thus $|\pi_\lambda - \mu_\lambda| \to 0$ as $\lambda \to 0$. Since $\{\pi_\lambda\} \subset B_{\mathbb{R}^n}(0,2M)$ and $\mathcal{R}$ is a bounded $\mathbb{R}^n$ valued operator, we can extract a sequence $\lambda_k \to 0$, as $k \to +\infty$ and find $x_j^* \in B_{\mathbb{R}^n}(0,M_j) \cap \mathbb{R}^n_+$, $\pi^* \in B_{\mathbb{R}^n}(0,2M) \cap \mathbb{R}^n_+$ such that $x_j^{\lambda_k} \to x_j^*$ in $\mathbb{R}^n_+$ and $\pi_{\lambda_k} \to \pi^*$ in $\mathbb{R}^n_+$ as $k \to +\infty$. Note that $\mu_{\lambda_k} \to \pi^*$ when $k \to +\infty$, as well. Thus the upper semicontinuity of $\mathcal{R}$ yields

$$-\sum_{j=1}^m A_j^T x_j^* \in \mathcal{R}(\pi^*).$$

Finally, for any $\tau \in B_{\mathbb{R}^n}(0,2M)$ from (17) we obtain

$$\begin{aligned}
0 &\le \Big\langle -\sum_{j=1}^{m} A_j^T x_j^{\lambda_k} + y_{\lambda_k}, \tau - \pi_{\lambda_k} \Big\rangle \\
&= \Big\langle -\sum_{j=1}^{m} A_j^T x_j^{\lambda_k}, \tau - \pi_{\lambda_k} \Big\rangle + \langle y_{\lambda_k}, \tau - \mu_{\lambda_k}\rangle + \langle y_{\lambda_k}, \mu_{\lambda_k} - \pi_{\lambda_k}\rangle \\
&\le \Big\langle -\sum_{j=1}^{m} A_j^T x_j^{\lambda_k}, \tau - \pi_{\lambda_k} \Big\rangle + \Phi(\tau) - \Phi(\mu_{\lambda_k}) - \frac{\left|\pi_{\lambda_k} - \mu_{\lambda_k}\right|^2}{\lambda_k} \\
&\le \Big\langle -\sum_{j=1}^{m} A_j^T x_j^{\lambda_k}, \tau - \pi_{\lambda_k} \Big\rangle + \Phi(\tau) - \Phi(\mu_{\lambda_k}).
\end{aligned}$$

Hence, by taking the limit as $k \to \infty$ we get

$$\Big\langle -\sum_{j=1}^{m} A_j^T x_j^{*}, \tau - \pi^{*} \Big\rangle + \Phi(\tau) - \Phi(\pi^*) \ge 0, \quad \forall \tau \in B_{\mathbb{R}^n}(0,2M) \cap \mathbb{R}^n_+. \tag{18}$$

Substituting $\tau = 0$ easily leads to the conclusion that

$$\Phi(\pi^\star) \le \sum_{j=1}^{m} \phi_j(\pi^\star) + \Phi(0),$$

which by (H_5^0) means that $|\pi^\star| \le M$. Accordingly, since (18) holds for any $\tau \in B_{\mathbb{R}^n}(0,2M) \cap \mathbb{R}^n_+$, we easily deduce (15). The proof is complete. □

4. The case ϕ_j positive homogeneous of degree 1

In the case when ϕ_j are positive homogeneous of degree 1 we construct an approximation scheme using the previous results. This approach provides the existence of the solution of the modified problem (P).

THEOREM 3. *Suppose that for any $j = 1, \dots, m$ the following conditions are satisfied:*

(H_1) $0 \in \mathrm{cl}\,(\mathrm{Dom}\,\partial\overline{V}_j)$, $(\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0,r_j) \subset \mathrm{Int}\,\mathrm{Dom}\,\overline{V}_j^\star$ *for some* $r_j > 0$;

(H_2) $\{x \in \mathbb{R}^n_+ \colon \{\langle x^\star, x\rangle : x^\star \in \partial\overline{V}_j(x)\} \cap \mathbb{R}_- \neq \emptyset\} \subset B_{\mathbb{R}^n}(0,M_j)$, *for some* $M_j > 0$;

(H_4^3) $\phi_j(t\tau) = t\phi_j(\tau)$, $\forall \tau \in \mathbb{R}^n_+$, $\forall t > 0$;

(H_0) $\min\{\phi_j(\tau) \colon \tau \in \mathbb{R}^n_+, |\tau| = 1\} =: \gamma_j$, $\gamma_j > 0$;

(H_6^2) *$\Phi = \sum_{j=1}^m \phi_j$ is convex;*

(H_9^1) *$\sum_{j=1}^m A_j^T x_j \notin \partial\Phi_+(0)$ for any $x_j \in \partial\overline{V}_j^\star(0)$.*

Then there exist a number $r \geq 1$ and a system $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}_+^n \times (\mathbb{R}_+^n)^m \times (\mathbb{R}_+)^m$, $\pi \neq 0$, such that

$$\tag{19} \begin{cases} -\alpha_j A_j \pi \in \partial\overline{V}_j(x_j), \\ \langle A_j\pi, x_j\rangle - r\phi_j(\pi) \in \partial \mathrm{ind}_{\geq 0}(\alpha_j), \\ \Phi(\tau) - \Phi(\pi) \geq \left\langle \tau - \pi, \sum_{j=1}^m A_j^T x_j \right\rangle. \quad \forall \tau \in \mathbb{R}_+^n, \end{cases}$$

PROOF. Let $\psi_j^\varepsilon(\tau) = |\tau|^{-\varepsilon}\phi_j(\tau)$, $\tau \in \mathbb{R}_+^n \setminus \{0\}$, $\psi_j^\varepsilon(0) = 0$, where $0 < \varepsilon < 1$, $j = 1, \dots, m$. We claim that the assumptions of Theorem 2 are satisfied. Indeed, from the (H_0) we get that ψ_j^ε, $j = 1, \dots, m$ are continuous on $\mathbb{R}_+^n$. The functions ψ_j^ε, $j = 1, \dots, m$ are positively homogeneous of degree $1 - \varepsilon < 1$, i.e. (H_4^1) holds. Moreover, in view of (H_4^3) and (H_6^1), from

$$\tag{20} \Phi(\tau) \leq \sum_{j=1}^m \psi_j^\varepsilon(\tau) = |\tau|^{-\varepsilon} \sum_{j=1}^m \phi_j(\tau) = |\tau|^{-\varepsilon}\Phi(\tau)$$

it follows that $|\tau| \leq 1$, i.e. (H_5^0) is satisfied. The remaining assumptions can be easily verified.

Accordingly, from Theorem 2 it follows that for any $\varepsilon < 1$ there exists a system $(\pi^\varepsilon, (x_j^\varepsilon), (\alpha_j^\varepsilon))$ such that

$$\tag{21} \begin{cases} -\alpha_j^\varepsilon A_j \pi^\varepsilon \in \partial\overline{V}_j(x_j^\varepsilon), \\ \langle A_j\pi^\varepsilon, x_j^\varepsilon\rangle - |\pi^\varepsilon|^{-\varepsilon}\phi_j(\pi^\varepsilon) \in \partial \mathrm{ind}_{\geq 0}(\alpha_j^\varepsilon), \\ \Phi(\tau) - \Phi(\pi^\varepsilon) \geq \langle \sum_{j=1}^m A_j^T x_j^\varepsilon, \tau - \pi^\varepsilon\rangle, \quad \forall \tau \in \mathbb{R}_+^n. \end{cases}$$

The (H_9^1) ensures easily that $\pi^\varepsilon \neq 0$. Hence $(\pi^\varepsilon, (x_j^\varepsilon), (\alpha_j^\varepsilon))$ is a solution of (19) with $r = |\pi^\varepsilon|^{-\varepsilon}$.

Moreover, if $|\pi^\varepsilon| \geq c\varepsilon$ for some $c > 0$, then $\lim_{\varepsilon\to 0} |\pi^\varepsilon|^{-\varepsilon} = 1$ and there exists a solution of (19) with $r = 1$. Indeed, since $0 \geq \langle -\alpha_j^\varepsilon A_j \pi^\varepsilon, x_j^\varepsilon\rangle \in \langle \partial\overline{V}_j(x_j^\varepsilon), x_j^\varepsilon\rangle$, then from (H_2) we get the boundedness of the sets $\{x_j^\varepsilon\}_{0<\varepsilon<1}$, $j = 1, \dots, m$. From (20) we have that $0 < |\pi^\varepsilon| \leq 1$, for any $\varepsilon < 1$.

We shall now prove that $\{\alpha_j^\varepsilon |\pi^\varepsilon|\}_{0<\varepsilon<1}$, $j = 1, \dots, m$ are bounded. Indeed, suppose to the contrary that for some $j \in \{1, \dots, m\}$ $\alpha_j^\varepsilon|\pi^\varepsilon| \to +\infty$, as $\varepsilon \to 0$ (choosing a subsequence, if necessary). From the the first two conditions of (21) we get

$$-\alpha_j^\varepsilon|\pi^\varepsilon|^{-\varepsilon}\phi_j(\pi^\varepsilon) = \langle -\alpha_j^\varepsilon A_j\pi^\varepsilon, x_j^\varepsilon\rangle = V_j(x_j^\varepsilon) + \overline{V}_j^\star(-\alpha_j^\varepsilon A_j \pi^\varepsilon).$$

Using the fact $\partial\overline{V}_j^\star(0) \neq \emptyset$ we obtain

$$V_j(y) \geq -c_j, \quad \forall y \in \mathrm{Dom}\, V_j, \text{ for some } c_j \in \mathbb{R}.$$

By the definition of Fenchel's conjugate functions we get the estimate

$$\gamma_j \le \gamma_j|\pi^\varepsilon|^{-\varepsilon} \le \frac{c_j + V_j(y)}{\alpha_j^\varepsilon|\pi^\varepsilon|} + \left\langle A_j \frac{\pi^\varepsilon}{|\pi^\varepsilon|}, y\right\rangle \le \frac{c_j + V_j(y)}{\alpha_j^\varepsilon|\pi^\varepsilon|} + |A_j||y|, \quad \forall y \in \operatorname{Dom} V_j.$$

Letting $\varepsilon \to 0$ in above we get

$$0 < \gamma_j \le |A_j||y|, \quad \forall y \in \operatorname{Dom} V_j,$$

which contradics the assumption $0 \in \operatorname{cl}(\operatorname{Dom} \partial\overline{V}_j)$.

Hence we can assume that there exist $p \in \mathbb{R}^n_+$, $|p| = 1$, $x_j \in \mathbb{R}^n_+$, $\widetilde{\alpha_j} \in \mathbb{R}_+$, $j = 1, \dots, m$ such that $\frac{\pi^\varepsilon}{|\pi^\varepsilon|} \to p$, $x_j^\varepsilon \to x_j$, $\alpha_j^\varepsilon|\pi^\varepsilon| \to \widetilde{\alpha_j}$ (passing to a subsequence, if necessary). From the positive homogeneity of the functions ϕ_j, $j = 1, \dots, m$ and Φ we equivalently rewrite (21) as

$$\begin{cases} -\alpha_j^\varepsilon|\pi^\varepsilon|A_j\frac{\pi^\varepsilon}{|\pi^\varepsilon|} \in \partial\overline{V}_j(x_j^\varepsilon), \\ \left\langle A_j\frac{\pi^\varepsilon}{|\pi^\varepsilon|}, x_j^\varepsilon\right\rangle - |\pi^\varepsilon|^{-\varepsilon}\phi_j(\frac{\pi^\varepsilon}{|\pi^\varepsilon|}) \in \partial \operatorname{ind}_{\ge 0}(\alpha_j^\varepsilon|\pi^\varepsilon|), \\ \Phi(\tau) - \Phi(\frac{\pi^\varepsilon}{|\pi^\varepsilon|}) \ge \left\langle \sum_{j=1}^m A_j^T x_j^\varepsilon, \tau - \frac{\pi^\varepsilon}{|\pi^\varepsilon|}\right\rangle, \quad \forall \tau \in \mathbb{R}^n_+, \end{cases}$$

Here we used the substitution $\tau \mapsto \frac{\tau}{|\pi^\varepsilon|}$. Letting $\varepsilon \to 0$ we now get

$$\begin{cases} -\widetilde{\alpha}_j A_j p \in \partial\overline{V}_j(x_j), \\ \left\langle A_j p, x_j\right\rangle - \phi_j(p) \in \partial \operatorname{ind}_{\ge 0}(\widetilde{\alpha}_j), \\ \Phi(\tau) - \Phi(p) \ge \left\langle \sum_{j=1}^m A_j^T x_j, \tau - p\right\rangle, \quad \forall \tau \in \mathbb{R}^n_+, \end{cases}$$

which means that $(p, (x_j), (\widetilde{\alpha}_j))$ is the solution of (19) with $r = 1$. □

COROLLARY 1. *Under the assumptions of Theorem 3, let* $r \ge 1$ *and let* $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}^n_+ \times (\mathbb{R}^n_+)^m \times (\mathbb{R}_+)^m$, $\pi \ne 0$, *be a solution of the problem* (19). *Then*

i) $\forall j = 1, \dots, m, \quad \left(\frac{|A_j|\widetilde{M}_j}{\gamma_j} < 1 \;\Rightarrow\; \alpha_j = 0\right),$

ii) $\exists j' \in \{1, \dots, m\}, \quad \frac{|A_{j'}|\widetilde{M}_{j'}}{\gamma_{j'}} \ge 1,$

where $\widetilde{M}_j = \inf\{M_j : M_j$ *is a constant fulfilling the condition* $(H_2)\}$.

PROOF.
i) Suppose to the contrary that $\frac{|A_j|\widetilde{M}_j}{\gamma_j} < 1$ and $\alpha_j > 0$ for some $j \in \{1, \dots, m\}$. Then the second condition of $(19)_2$ takes the form $\langle A_j\pi, x_j\rangle = r\phi_j(\pi)$. From the positive homogeneity of the function ϕ_j we get

$$\langle A_j p, x_j\rangle = r\phi_j(p) \ge r\gamma_j, \quad p = \frac{\pi}{|\pi|}.$$

Let M_j be as in (H_2), then from the previous estimate we obtain $r \leq \frac{\langle A_j p, x_j \rangle}{\gamma_j} \leq \frac{|A_j| M_j}{\gamma_j}$. Hence

$$1 \leq r \leq \frac{|A_j|\widetilde{M}_j}{\gamma_j} < 1,$$

which is a contradiction.

ii) Suppose to the contrary that $\frac{|A_j|\widetilde{M}_j}{\gamma_j} < 1$ for each $j = 1, \dots, m$. From *i)* it follows that $\alpha_j = 0$, $j = 1, \dots, m$. The first condition of (19) implies $x_j \in \partial \overline{V}_j^\star(0)$, $j = 1, \dots, m$. Moreover, (H_9^1) leads to $\sum_{j=1}^m A_j^T x_j \notin \partial \Phi_+(0)$. Taking into account that Φ is proper, convex, l.s.c and positive homogeneous of degree 1, we get that there exists a closed and convex set $W \subset \mathbb{R}_+^n$ such that $\Phi_+^\star = \operatorname{ind}_W$. The assumptions (H_0), (H_6^2) imply that $\overline{B}(0, \widetilde{\gamma}) \cap \mathbb{R}_+^n \subset W$, where $\widetilde{\gamma} := \sum_{j=1}^m \gamma_j$.

Since, $|\sum_{j=1}^m A_j^T x_j| \geq \widetilde{\gamma} = \sum_{j=1}^m \gamma_j$, there exists an index $j_0 \in \{1, \dots, m\}$ such that $|A_{j_0}^T x_{j_0}| \geq \gamma_{j_0}$. On the other hand, $|x_{j_0}| \leq \widetilde{M}_{j_0}$ leads to

$$\frac{|A_{j_0}|\widetilde{M}_{j_0}}{\gamma_{j_0}} \geq 1,$$

which is a contradiction. □

The next corollary concerns estimates on r.

COROLLARY 2. *Under the assumptions of Theorem 3, let $r \geq 1$ and let $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}_+^n \times (\mathbb{R}_+^n)^m \times (\mathbb{R}_+)^m$, $\pi \neq 0$ be a solution of the problem* (19). *Then*

$$J := \{j \in \{1, \dots, m\} : \alpha_j > 0\} \neq \emptyset$$

and

$$1 \leq r \leq \min_{j \in J} \left\{ \frac{|A_j|\widetilde{M}_j}{\gamma_j} \right\},$$

where $\widetilde{M}_j = \inf\{M_j : M_j$ is a constant fulfilling the condition $(H_2)\}$.

PROOF. First, we claim that $J \neq \emptyset$. Assume to the contrary that $\alpha_j = 0$, $j = 1, \dots, m$. From the first condition of (19) we get $x_j \in \partial \overline{V}_j^\star(0)$, $j = 1, \dots, m$. Taking into account that Φ is proper, convex, l.s.c and positive homogeneity of degree 1, from the third condition of (19) we obtain $\sum_{j=1}^m A_j^T x_j \in \partial \Phi_+(0)$, which contradicts the assumption (H_9^1). Hence $J \neq \emptyset$.

Let $j \in J$. From Corollary 1 it follows that $\frac{|A_j|\widetilde{M}_j}{\gamma_j} \geq 1$. Moreover, the second condition of (19) can be equivalently reformulated as $\langle A_j \pi, x_j \rangle = r\phi_j(\pi)$. By the positive homogeneity of degree 1 of ϕ_j and (H_0) we arrive at

$$1 \leq r \leq \frac{|A_j| M_j}{\gamma_j},$$

where M_j is a constant from (H_2). Accordingly,

$$1 \leq r \leq \min_{j \in J} \left\{ \frac{|A_j| \widetilde{M}_j}{\gamma_j} \right\}.$$

□

The following Corollary provides the conditions for $r = 1$.

COROLLARY 3. *Suppose that for any $j = 1, \dots, m$ the following conditions hold:*

(H_1) $0 \in \operatorname{cl}(\operatorname{Dom} \partial \overline{V}_j)$, $(\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \operatorname{Int} \operatorname{Dom} \overline{V}_j^\star$ *for some* $r_j > 0$*;*

(H_2) $\{x \in \mathbb{R}^n_+ : \{\langle x^\star, x\rangle : x^\star \in \partial \overline{V}_j(x)\} \cap \mathbb{R}_- \neq \emptyset\} \subset B_{\mathbb{R}^n}(0, M_j)$*, for some* $M_j > 0$*;*

(H_4^3) $\phi_j(t\tau) = t\phi_j(\tau)$, $\forall \tau \in \mathbb{R}^n_+$, $\forall t > 0$*;*

(H_0) $\min\{\phi_j(\tau) : \tau \in \mathbb{R}^n_+,\ |\tau| = 1\} =: \gamma_j$, $\gamma_j > 0$*;*

(H_6^2) $\Phi = \sum_{j=1}^m \phi_j$ *is convex;*

(H_9^2) $A_j^T x_j \notin \partial \Phi_+(0)$ *for any* $x_j \in \partial \overline{V}_j^\star(0)$*, where* $\Phi_+ = \Phi + \operatorname{ind}_{\mathbb{R}^n_+}$.

Then there exists a system $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}^n_+ \times (\mathbb{R}^n_+)^m \times (\mathbb{R}_+)^m$, $\pi \neq 0$*, such that*

$$\begin{cases} -\alpha_j A_j \pi \in \partial \overline{V}_j(x_j), \\ \langle A_j \pi, x_j \rangle - \phi_j(\pi) \in \partial \operatorname{ind}_{\geq 0}(\alpha_j), \\ \Phi(\tau) - \Phi(\pi) \geq \left\langle \tau - \pi, \sum_{j=1}^m A_j^T x_j \right\rangle, \quad \forall \tau \in \mathbb{R}^n_+, \end{cases}$$

which means that the problem (P) has a nontrival solution.

PROOF. Note that (H_9^2) implies the condition (H_9^1) of Theorem 19. From Theorem 19 we get that there exist a number $r \geq 1$ and a system $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}^n_+ \times (\mathbb{R}^n_+)^m \times (\mathbb{R}_+)^m$, $\pi \neq 0$ such that

$$\begin{cases} -\alpha_j A_j \pi \in \partial \overline{V}_j(x_j), \\ \langle A_j \pi, x_j \rangle - r\phi_j(\pi) \in \partial \operatorname{ind}_{\geq 0}(\alpha_j), \\ \Phi(\tau) - \Phi(\pi) \geq \left\langle \tau - \pi, \sum_{j=1}^m A_j^T x_j \right\rangle, \quad \forall \tau \in \mathbb{R}^n_+, \end{cases} \tag{22}$$

Thus (H_9^2) leads to $\alpha_j > 0$, $j = 1, \dots, m$.

Indeed, assume to the contrary that $\alpha_j = 0$ for some $j \in \{1, \dots, m\}$. Then the second condition (22) means that $x_j \in \partial \overline{V}_j^\star(0)$. Since the function Φ is proper, convex, l.s.c. and positive homogeneous of degree 1, by the third condition of (22) we obtain $A_j^T x_j \in \partial \Phi_+(0)$, which contradicts the assumption (H_9^2).

As $\alpha_j > 0$, $j = 1, \ldots, m$, the second condition $(22)_2$ takes the form $\langle A_j\pi, x_j\rangle = r\phi_j(\pi), j = 1, \ldots, m$. Now, by (H_6^2) and the third condition of (22) we get

$$\Phi(\pi) = \left\langle \sum_{j=1}^m A_j^T x_j, \pi \right\rangle = r \sum_{j=1}^m \phi_j(\pi) = \Phi(\pi).$$

Consequently $r = 1$. □

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