Report of Meeting

The Third Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities January 29 – February 1, 2003 Będlewo, Poland

The Third Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held from January 29 to February 1, 2002, at the Mathematical Research and Conference Center of Polish Academy of Sciences, Będlewo, Poland.

24 participants came from the Silesian University of Katowice (Poland) and the University of Debrecen (Hungary) at 12 from each of both cities.

Professor Roman Ger opened the Seminar and welcomed the participants to Będlewo.

The scientific talks presented at the Seminar focused on the following topics: equations in a single and several variables, iteration theory, equations on algebraic structures, conditional equations, differential functional equations, Hyers-Ulam stability, functional inequalities and mean values. Interesting discussions were generated by the talks.

There was a very profitable Problem Session.

The social program consisted of visiting the palace in Kórnik, and an excursion to the city of Poznań where the participants of the meeting among others visited a museum of old instruments and took part in a festive dinner.

The closing address was given by Professor Zsolt Páles. His invitation to hold the Fourth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in February 2004 in Hungary was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in section 1, problems and remarks in approximate chronological order in section 2, and the list of participants in the final section.

1. Abstracts of talks

ROMAN BADORA: *Set ideals and invariant means*

In the first part of this talk we prove the existence of some generalized invariant mean on the space of all real functions defined on an amenable semigroup S which are essentially bounded with respect to an axiomatically given family of subsets of S. In the second part we present an application of our results to the study of the problem of separation of two functions by an additive function.

KAROL BARON: *A direct proof of a theorem of van der Corput for functionals*

The theorem mentioned in the title concerns the Cauchy functional congruence

$$f(x+y) - f(x) - f(y) \in \mathbb{Z} \tag{1}$$

and reads as follows (see [3; p. 64]).

If $f : \mathbb{R} \to \mathbb{R}$ satisfies (1) *for all $x, y \in \mathbb{R}$ and there exist nonempty and open subsets U, W of $\mathbb{R}$ such that*

$$f(U) \cap (W + \mathbb{Z}) = \emptyset, \tag{2}$$

then there exists a $c \in \mathbb{R}$ with

$$f(x) - cx \in \mathbb{Z} \quad \textit{for every} \quad x \in \mathbb{R}.$$

Using this theorem it is possible to extend it to real functionals. In this manner the following was proved in [2].

Suppose E is a real topological vector space. If $f : E \to \mathbb{R}$ satisfies (1) *for all $x, y \in E$ and there exist nonempty and open subsets U of E and W of $\mathbb{R}$ such that* (2) *holds, then there exists an $x^* \in E^*$ with*

$$f(x) - x^*x \in \mathbb{Z} \quad \textit{for every} \quad x \in E.$$

Another proof of Theorem 1 is presented in [5; Remark 4] by M. Sablik. J.A. Baker [1; Theorem 1 and the next Remark] proved it in the case where E is the additive group of any rational metric linear space. More exactly J.A. Baker proved that if χ is a character of such a group E, then either χ is continuous, or $\chi(U)$ is dense in the unit circle for every nonempty open subset

U of E. P. Flor [4] showed that this theorem of J.A. Baker does not hold for an arbitrary topological group. We prove that in the case of an arbitrary topological group we have the Baker's alternative with the continuity of the character replaced by the continuity of a power of it.

References

[1] J.A. Baker, *On some mathematical characters*, Glasnik Mat. **25** (45) (1990), 319–328.

[2] K. Baron, P. Volkmann, *On a theorem of van der Corput*, Abh. Math. Sem. Univ. Hamburg **61** (1991), 189–195.

[3] J.G. van der Corput, *Goniometrische functies gekarakteriseerd door een functionaalbetrekking*, Eudides **17** (1940), 55–75.

[4] P. Flor, *Remark on Baker's problem*, Report of meeting, The twenty-eight international symposium on functional equations (August 23 – September 1, 1990, Graz-Mariatrost, Austria), Aequationes Math. **41** (1991), 298.

[5] M. Sablik, *A functional congruence revisited*, Selected topics in functional equations and iteration theory, Proceedings of the Austrian-Polish seminar (Graz 1991), Grazer Math. Ber. **316** (1992), 181–200.

LECH BARTLOMIEJCZYK: *Irregular solutions of iterative functional equation of the second order*

(Joint work with Janusz Morawiec)

We describe the structure of orbits generated by two commuting bijections $f, g : X \to X$. Using this description we construct irregular solutions of general functional equation of the second order:

$$h(x, \varphi(x), \varphi(f(x)), \varphi(g(x))) = 0.$$

Graph of such a solution is connected and almost covers the plane $X \times X$ in the sense of measure and topology

MIHÁLY BESSENYEI: *Generalized Hadamard inequalities*
(Joint work with Zsolt Páles)

Let $I \subset \mathbb{R}$ a nonempty interval, $\omega_1, \ldots, \omega_n : I \to \mathbb{R}$ be given functions. A function $\omega_0 : I \to \mathbb{R}$ is said to be $(\omega_1, \ldots, \omega_n)$-*convex* if

$$(-1)^n \begin{vmatrix} \omega_0(x_0) & \ldots & \omega_0(x_n) \\ \omega_1(x_0) & \ldots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_0) & \ldots & \omega_n(x_n) \end{vmatrix} \geqslant 0$$

whenever $x_0 < \ldots < x_n$ and $x_0, \ldots, x_n \in I$. This notion is a common generalization of higher-order monotonicity and (ω_1, ω_2)-convexity, in particular, the classical convexity, too.

In the talk we present Hadamard-type inequalities for $(\omega_1, \ldots, \omega_n)$-convex functions.

ZOLTÁN BOROS: *$\mathbb{Q}$-subgradient of Jensen-convex functions*

(Joint work with Zsolt Páles)

Throughout this presentation D is a convex open subset of $\mathbb{R}^N$ and $\mathbb{Q}^+$ denotes the set of positive rationals. Moreover, let

$$\mathcal{A} = \{\, A : \mathbb{R}^N \to \mathbb{R} \mid A \text{ is additive} \,\}.$$

DEFINITION. Let $f : D \to \mathbb{R}$, $x_0 \in D$, and $u \in \mathbb{R}^N$. The set

$$\partial_{\mathbb{Q}} f(x_0) = \{\, A \in \mathcal{A} \mid f(x_0) + A(x - x_0) \leqslant f(x) \text{ for every } x \in D \,\}$$

is called the $\mathbb{Q}$-subgradient of f at x_0. If the finite limit

$$d_{\mathbb{Q}} f(x_0\,,\, u) = \lim_{\mathbb{Q}^+ \ni r \to 0} \frac{f(x_0 + ru) - f(x_0)}{r}$$

exists, it is called the radial $\mathbb{Q}$-derivative of f at x_0 in the direction u. We shall say that f is radially $\mathbb{Q}$-differentiable at x_0 if $d_{\mathbb{Q}} f(x_0\,,\, v) \in \mathbb{R}$ exists for every $v \in \mathbb{R}^N$. We shall say that f is radially $\mathbb{Q}$-differentiable if f is radially $\mathbb{Q}$-differentiable at x for every $x \in D$. If f is radially $\mathbb{Q}$-differentiable at x_0, the set

$$\delta_{\mathbb{Q}} f(x_0) = \left\{\, A \in \mathcal{A} \mid A(v) \leqslant d_{\mathbb{Q}} f(x_0\,,\, v) \text{ for every } v \in \mathbb{R}^N \,\right\}$$

is called the weak $\mathbb{Q}$-subderivative of f at x_0.

PROPOSITION. *Suppose that $f : D \to \mathbb{R}$ is radially $\mathbb{Q}$-differentiable at $x \in D$, $u \in \mathbb{R}^N$, and $q \in \mathbb{Q}^+$. Then $d_{\mathbb{Q}} f(x, qu) = q d_{\mathbb{Q}} f(x, u)$.*

THEOREM. *Suppose that $f : D \to \mathbb{R}$ is Jensen-convex. Then f is radially $\mathbb{Q}$-differentiable. Moreover, for every $x \in D$, the mapping $\psi(u) = d_{\mathbb{Q}} f(x, u)$ $(u \in \mathbb{R}^N)$ is subadditive and $\partial_{\mathbb{Q}} f(x) = \delta_{\mathbb{Q}} f(x) \neq \emptyset$.*

PROPOSITION. *If $f : D \to \mathbb{R}$ such that $\partial_{\mathbb{Q}} f(x) \neq \emptyset$ for every $x \in D$, then f is Jensen-convex.*

Péter Czinder: *Inequalities for two parameter homogeneous means*

(Joint work with Zsolt Páles)

We investigate some inequalities concerning the two variable Gini and Stolarsky means, defined (in the most general case) by the formulae

$$G_{a,b}(x,y) = \left(\frac{x^a + y^a}{x^b + y^b}\right)^{\frac{1}{a-b}} \quad \text{and} \quad S_{a,b}(x,y) = \left(\frac{x^a - y^a}{a}\frac{b}{x^b - y^b}\right)^{\frac{1}{a-b}}.$$

After giving the summary of preliminary results (comparison theorems and Minkowski/ reversed Minkowski-type theorems), we present some generalizations of them, obtained together by the authors. Finally, we show our new – partial – results regarding the comparison of Gini and Stolarsky means.

Borbála Fazekas: *Decision functions and their properties*

Our main aim is to characterize the relation between the properties of the so called decision functions and the properties of the decision generating functions. A function $D\colon \bigcup_{i=1}^{\infty} I^i \to I$ is called a decision function, if it is symmetric, reflexive, regular and internal. We can generate a decision function D_d with a generalization of the least squares method using a decision generating function $d\colon I \times I \to \mathbb{R}$. The reverse statement is also true, for every decision function D there exists a decision generating function d, that generates it. The main result, that characterizes the monotonicity property, is the following

Theorem. *A decision function $D_d\colon \bigcup_{i=1}^{\infty} I^i \to I$, generated by the decision generating function $d\colon I \times I \to \mathbb{R}$, is monotonic if and only if*

$$d(x_1, y_1) + d(x_2, y_2) \leqslant d(x_1, y_2) + d(x_2, y_1)$$

holds for every $x_1, x_2, y_1, y_2 \in I$, $x_1 \leqslant x_2$, $y_1 \leqslant y_2$.

Roman Ger: *Residual sets in amenable groups and functional equations*

We are presenting some results that are complementary to those obtained recently by F. C. Sánchez (*Stability of additive mappings on large subsets*, Proceedings of the American Mathematical Society **128** (2000), 1071-1077). A subset B of a group $(G,+)$ is termed *m-residual* provided that $m(1_B) = 1$ for some invariant mean m on $(G,+)$. Then, it seems natural to call a set $E \subset G$ to be *m-null* whenever $G \setminus E$ is $m-$residual. We shall say that a subset E of an amenable group is *universally null* if and only if it happens to be $m-$null with respect to every invariant mean m on $(G,+)$.

We prove, among others, that

Every subset E of an amenable group $(G, +)$ such that $E - E$ belongs to a proper linearly invariant set ideal of subsets of G is universally null

and the following

THEOREM. *Let B be an m–residual subset of a group $(G, +)$. Given an Abelian group $(H, +)$ admitting sufficiently many real characters, if a map $a : B \longrightarrow H$ satisfies the condition*

$$x, y, x + y \in B \Rightarrow a(x + y) = a(x) + a(y),$$

then there exists exactly one homomorphism $A : G \longrightarrow H$ such that

$$A(x) = a(x) \ \ \textit{for all} \ \ x \in B.$$

ATTILA GILÁNYI: *On convex functions of higher order*
(Joint work with Zsolt Páles)

In this talk connections between symmetrically convex functions of higher order and Wright-convex functions of higher order are investigated. During the talk I denotes a nonempty interval, $f : I \to \mathbb{R}$ is a function, $n \geqslant 2$ stands for an integer, and $t_1, \ldots, t_n$ are positive real numbers. The function f is said to be symmetrically $(t_1, \ldots, t_n)$-convex on I, if

$$[x, x + t_{i_1}h, \ldots, x + t_{i_1}h + \cdots + t_{i_n}h; f] \geqslant 0$$

for all $h > 0$, $x, x + t_1h + \cdots + t_nh \in I$ and for all permutations $(i_1, \ldots, i_n)$ of the integers $\{1, \ldots, n\}$. We call it $(t_1, \ldots, t_n)$– Wright-convex on I, if

$$\Delta_{t_1h} \ldots \Delta_{t_nh} f(x) \geqslant 0$$

for $h > 0$, $x, x + t_1h + \cdots + t_nh \in I$.

We prove that a symmetrically $(t_1, \ldots, t_n)$-convex function is also $(t_1, \ldots, t_n)$–Wright-convex on I. Concerning the opposite direction, we show that there exist positive numbers $t_1, \ldots, t_n$ for which $(t_1, \ldots, t_n)$-Wright-convexity does not imply symmetrical $(t_1, \ldots, t_n)$-convexity.

ATTILA HÁZY: *On approximately t-convex functions*
(Joint work with Zsolt Páles)

A real valued function f defined on an open convex set D is called (ε, δ, p) t-convex if it satisfies

$$f\,(tx + (1 - t)y) \leqslant tf(x) + (1 - t)f(y) + \varepsilon|x - y|^p + \delta \qquad \text{for} \quad x, y \in D.$$

Our main result shows that if $0 < p < 1$, and f is locally bounded from above at a point of D and is (ε, δ, p) t-convex then it satisfies the convexity-type inequality

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y) + \max\left\{\frac{1}{t}, \frac{1}{1-t}\right\}\delta + \varepsilon\varphi(\lambda)|x-y|^p$$

for $x, y \in D$, $\lambda \in [0,1]$, where $\varphi : [0,1] \to \mathbb{R}$ is a continuous function satisfying

$$\varphi(\lambda) \leqslant c(\lambda(1-\lambda))^p$$

with

$$c < \max\left\{\frac{1}{(1-t)^p - (1-t)}; \frac{1}{t^p - t}\right\}.$$

The particular case $\varepsilon = 0$ of this result is due to Páles [2]. The case $p = 1$ and $t = \frac{1}{2}$ was investigated in Házy and Páles [1].

References

[1] A. Házy, Zs. Páles, *Approximately midconvex functions.*

[2] Zs. Páles, *Bernstein-Doetsch type results for general functional inequalities*, Roczn. Nauk.-Dydakt. Prace Mat. 17 (2000), 197–206 (Dedicated to Professor Zenon Moszner on his 70th birthday).

WITOLD JARCZYK: *Invariant sets and operations*

Let $I = [0,1]$ and denote by $\mathcal{B}$ $[\mathcal{B}_+;\ \mathcal{B}_-]$ the set of all bijections [increasing bijections; decreasing bijections] of I. Given a set $\Phi \subset \mathcal{B}$ we say that $A \subset I^n$ is *Φ–invariant* if the condition

$$(x_1, \dots, x_n) \in A \Rightarrow (\varphi(x_1), \dots, \varphi(x_n)) \in A$$

holds for every $\varphi \in \Phi$. A set $A \subset I^n$ is called *minimal Φ–invariant* if it contains no non–void proper Φ–invariant set.

Denote by $\mathcal{S}_n$ the family of subsets of I^n defined as follows:

$A \in \mathcal{S}_n$ iff there are a permutation σ of $\{1, \dots, n\}$ and symbols $\dashv_0, \dots, \dashv_n \in \{<, =\}$ such that

$$A = \{(x_1, \dots, x_n) \in I^n :\ 0 \dashv_0 x_{\sigma(1)} \dashv_1 \dots \dashv_{n-1} x_{\sigma(n)} \dashv_n 1\}.$$

Given a set $\Phi \subset \mathcal{B}$ we say that a set $\mathcal{F} \subset \mathcal{B}$ *generates* Φ if every element of Φ is a composition of a finite number of functions from $\mathcal{F}$.

A set $\Phi \subset \mathcal{B}_+$ $[\Phi \subset \mathcal{B}_-]$ is called an *interpolation family* if for every $n \in \mathbb{N}$ and numbers $x_1, \dots, x_n, y_1, \dots, y_n \in (0,1)$ satisfying

$$x_1 < \dots < x_n \quad \text{and} \quad y_1 < \dots < y_n \qquad [y_1 > \dots > y_n]$$

there is a $\varphi \in \Phi$ with

$$\varphi(x_i) = y_i \quad \text{for} \quad i \in \{1, \dots, n\}.$$

THEOREM. *Let $\Phi \subset \mathcal{B}_+$ be an interpolation family and let $\mathcal{F} \subset \Phi$ generate Φ. A set $A \subset I^n$ is minimal Φ–invariant [$\mathcal{F}$–invariant] if and only if $A \in \mathcal{S}_n$.*

If $A \in \mathcal{S}_n$ and

$$A = \{(x_1, \dots, x_n) \in I^n : \ 0 \dashv_0 x_{\sigma(1)} \dashv_1 \dots \dashv_{n-1} x_{\sigma(n)} \dashv_n 1\}$$

for a permutation σ of $\{1, \dots, n\}$ and some $\dashv_0, \dots, \dashv_n \in \{<, =\}$ then put

$$R(A) = \{(x_1, \dots, x_n) \in I^n : \ 0 \dashv_n x_{\sigma(n)} \dashv_{n-1} \dots \dashv_1 x_{\sigma(1)} \dashv_0 1\}.$$

THEOREM. *Let $\Phi \subset \mathcal{B}_-$ be an interpolation family and let $\mathcal{F} \subset \Phi$ generate Φ. A set $B \subset I^n$ is minimal Φ–invariant [$\mathcal{F}$–invariant] if and only if $B = A \cup R(A)$ for an $A \in \mathcal{S}_n$.*

Given a set $\Phi \subset \mathcal{B}$ we say that $F : I^n \to I$ is *Φ–invariant* if the condition

$$\varphi(F(x_1, \dots, x_n)) = F(\varphi(x_1), \dots, \varphi(x_n)) \quad \textit{for} \quad (x_1, \dots, x_n) \in I^n$$

holds for every $\varphi \in \Phi$.

The second part of the talk deals with characterizations of invariant functions. A number of examples is presented.

ANTAL JÁRAI: *A remark on the translation equation*

A somewhat more general version of the theorem of Guzik with a new proof is given stating that — under certain conditions — solutions f of the functional equation

$$f(u + v, x) = \sum_{i=1}^{n} h_i(a_i(v, x), f_i(u, b_i(v, x))), \ \ 0 < u, v < \infty, x \in X$$

are continuous.

ZOLTÁN KAISER: *On stability of the monomial functional equation in normed spaces over fields with valuation*

A generalized Hyers–Ulam type stability result is proved for the monomial functional equation in Banach spaces over fields of characteristic zero with arbitrary valuation.

Zygfryd Kominek: *On a problem of Wu Wei Chao*

Answering a question posed by Wu Wei Chao (American Mathematical Monthly, Vol. 108, No 10, December 2001), we show that the identity function is the only solution of the equation

$$f(x^2+y+f(y)) = f(x)^2 + 2y, \qquad x, y \in \mathbb{R}$$

László Losonczi: *Homogeneous Cauchy means*

Let $I \subset \mathbb{R}^+$ be an open interval containing the point 1, $f, g : I \to \mathbb{R}$ be differentiable functions, $g' \neq 0$ on I and suppose that $h := f'/g'$ is strictly monotonic. The Cauchy mean of $x, y \in I$ is defined by

$$D_{f,g}(x,y) := h^{-1}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right) \quad \text{if } x \neq y$$

and $D_{f,g}(x,x) := x$.

We determine the homogeneous two variable Cauchy means i.e. the means satisfying

$$D_{f,g}(tx,ty) = tD_{f,g}(x,y) \quad (x,y \in I, t \in I_{x,y}),$$

where $I_{x,y} = \{\, t \in \mathbb{R} \;:\; tx, ty \in I \,\}$ assuming that f, g are seven times continuously differentiable, $h' \neq 0$ and certain functions built up from f, g are either identically zero or nonzero on I.

Gyula Maksa: *Wright convexity of higher order*
(Joint work with Zsolt Páles)

Let p be a positive integer and $\emptyset \neq I \subset \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is $p-$Wright convex if

$$\Delta_{h_1} \dots \Delta_{h_{p+1}} f(x) \geqslant 0$$

for all $h_1, \dots h_{p+1} \in]0, \infty[$, $x, x + h_1 + \cdots + h_{p+1} \in I$. In the talk we present the following

Theorem. *If $f : I \to \mathbb{R}$ is $p-$Wright convex, then $f = C + P$ where $C : I \to \mathbb{R}$ is a continuous and $p-$convex function and $P : \mathbb{R} \to \mathbb{R}$ is a polynomial function of order p, that is, $\Delta_h^{p+1} P = 0$ on $\mathbb{R}$ for all $h \in \mathbb{R}$.*

Janusz Matkowski: *Regularity of functions in equality problem for Cauchy mean values*

Let $I \subset \mathbb{R}$ be an open interval and $f, g : I \to \mathbb{R}$ differentiable functions such that $g' \neq 0$ and $\frac{f'}{g'}$ invertible. Then the function $D_{f,g} : I^2 \to I$,

$$D_{f,g}(x,y) := \begin{cases} \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right) & x \neq y \\ x & x = y \end{cases};$$

is correctly defined and it is called a Cauchy mean value. In a recent paper L. Losonczi [1] determined all families of functions $f_1, g_1, f_2, g_2 : I \to \mathbb{R}$ satisfying the equation

$$D_{f_1,g_1} = D_{f_2,g_2} \tag{1}$$

under the assumption that these functions are seven times continuously differentiable. In the present paper we show that this strong regularity can be assumed without any loss of generality. This gives an answer to Problem posed by Zs. Páles [2]. Moreover we reduce the equation (1) to the following functional-differential equation

$$\frac{f'(x) - \frac{f(x)-f(y)}{x-y}}{f'(y) - \frac{f(x)-f(y)}{x-y}} = \frac{g'(x) - h'(x)\frac{g(x)-g(y)}{h(x)-h(y)}}{g'(y) - h'(y)\frac{g(x)-g(y)}{h(x)-h(y)}}.$$

References

[1] L. Losonczi, *Equality of two variable Cauchy mean values*, Aequationes Math. (to appear).

[2] Zs. Páles, *Problems in regularity theory in functional equations*, Aequationes Math. **63** (2002), 1–17.

Andrzej Olbryś: *Some conditions implying the continuity of $t-$Wright convex functions*

Let D be a nonempty convex and open subset of a real linear topological space X and $t \in (0,1)$ be a fixed number. A function $f : D \to \mathbb{R}$ is said to be $t-$Wright convex if it follows the following functional inequality:

$$f(tx + (1-t)y) + f((1-t)x + ty) \leqslant f(x) + f(y)\,, \quad \textit{for every } x, y \in D.$$

First result, which was given, was a generalized version of the theorem of Z. Kominek which says that every $t-$Wright convex function $f : (a,b) \to \mathbb{R}$ continuous at least at one point is continuous everywhere. Also it was shown that every $t-$Wright convex function upper semicontinuous everywhere or such that the restriction $f_{|T}$ is lower semicontinuous must be continuous (where T is a set of positive Lebesgue measure of second category with the Baire property). From this fact and from the Lusin theorem it follows that every t–Wright convex function such that the restriction $f_{|T}$ is Lebesgue or Baire measurable is continuous everywhere.

Ágota Orosz: *Sine and cosine equation on discrete polynomial hypergroups*

László Székelyhidi proved, that spectral analysis and spectral synthesis hold for any polynomial hypergroup. Actually, any translation invariant linear subspace of the complex valued functions on the hypergroup is finite dimensional and it is generated by exponential monomials. By applying these theorems the solutions of the sine equation can be described:

Theorem. *Let $(\mathbb{N},*)$ is the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$, $f,g:\mathbb{N}\to\mathbb{C}$ and f is not identically zero. Then*

$$f(n*m)=f(n)g(m)+f(m)g(n)$$

holds for all $m,n\in\mathbb{N}$ if and only if f and g can be written in one of the following forms:

1. $f(n)=aP_n(\lambda)$
   $g(n)=\frac{1}{2}P_n(\lambda)$
2. $f(n)=a(P_n(\lambda_1)-P_n(\lambda_2))\quad \lambda_1\neq\lambda_2$
   $g(n)=\frac{1}{2}(P_n(\lambda_1)+P_n(\lambda_2))$
3. $f(n)=bP_n'(\lambda)$
   $g(n)=P_n(\lambda)$

where $a,b,\lambda,\lambda_1,\lambda_2$ are arbitrary complex numbers.

A similar statement is true for the functional equation

$$f(n*m)=f(n)f(m)-g(n)g(m).$$

Zsolt Páles: *Solution of two variable functional inequalities*

Motivated by the Jensen-convexity and Wright-convexity properties of real functions, the following functional inequalities are investigated

$$f(M(x,y))\leqslant F\Big(x,y,f|_{[x,y]}\Big)\qquad (x,y\in I)$$

and

$$G\Big(x,y,f|_{]x,y[}\Big)\leqslant H(f(x),f(y))\qquad (x,y\in I),$$

where M is a strict mean on the interval I, the functionals F and G enjoy certain monotonicity properties, and the function $f:I\to\mathbb{R}$ is considered as an unknown function. The first inequality generalizes Jensen-convexity, while the second functional inequality contains Wright-convexity as a particular case.

Assuming that the corresponding functional equations have a rich solution set, we prove that a continuous function f is a solution for one of the above functional inequalities if and only if it is Beckenbach-convex with respect to the solution set of the corresponding functional equation.

MACIEJ SABLIK: *Polynomials and divided differences*
(Joint work with Thomas Riedel and Abe Sklar)

We define recursively a sequence of linear operators on the set of functions from R to R, as follows: For fixed $c > 0$ and any integer n, let:

$$(\delta_c^{(1)} f)(x) = \frac{f(x+c) - f(x-c)}{2c}$$

$$(\delta_c^{(n+1)} f) = \frac{4^n}{4^n - 1}(\delta_c^{(n)} f) - \frac{1}{4^n - 1}(\delta_{2c}^{(n)} f).$$

We ask for the functions that are differentiated by the above defined operators which leads to the equation

$$(\delta_c^n(f))(x) = f'(x)$$

or a more general functional equation

$$(\delta_c^n(f))(x) = g(x). \tag{1}$$

We arrive also at another functional equation connected with the problem

$$2g(x) = \sum_{k=1}^{n} a_k^{(n)} \left(g(x + 2^{k-1}c) + g(x - 2^{k-1}c)\right). \tag{2}$$

We prove that functional equations (1) and (2) characterize polynomials or polynomial functions of degree $2n$ and $2n + 1$, respectively.

JUSTYNA SIKORSKA: *On mapping preserving equilateral triangles*
(Joint work with Tomasz Szostok)

Let E be a euclidean space, $\dim E \geqslant 2$. We say that $f : E \to E$ *preserves equilateral triangles* if for all triples of points $x, y, z \in E$ with $\|x - y\| = \|y - z\| = \|x - z\|$ we have

$$\Big\|f(x) - f(y)\Big\| = \Big\|f(y) - f(z)\Big\| = \Big\|f(x) - f(z)\Big\|.$$

We show that if E is a finite-dimensional euclidean space, $\dim E \geqslant 2$, $f : E \to E$ is continuous at a point and preserves equilateral triangles, then it is a similarity mapping (an isometry multiplied by a constant).

Some generalizations as well as some interesting examples are also presented.

JANUSZ WALORSKI: *On homeomorphic solutions of the Schröder equation in Banach spaces*

Let X be a Banach space, $f: X \to X$ be a homeomorphism and $A: X \to X$ be a continuous linear operator. Following the proof of the Grobman-Hartman theorem presented in [Z. Nitecki, *An introduction to the orbit structure of diffeomorphisms*, The MIT Press, 1971] we establish conditions under which there exists a homeomorphism $\varphi: X \to X$ which solves the Schröder equation.

2. Problems and Remarks

1. Problem. If $f : I \to \mathbb{R}$ is a convex function then it satisfies the functional inequality

$$f\Big(\sum_{i=1}^{n} \lambda_i(x,y)M_i(x,y)\Big) \leqslant \sum_{i=1}^{n} \lambda_i(x,y)f(M_i(x,y)) \qquad (x,y \in I),$$

where $M_1, \dots, M_n : I^2 \to I$ are two variable means on I and $\lambda_1, \dots, \lambda_n : I^2 \to \mathbb{R}$ are nonnegative continuous functions with $\lambda_1 + \dots + \lambda_n = 1$. Conversely, if f is locally bounded from above then the above inequality (and certain further properties of the data) results that f is convex (see [1] for the details). Motivated by known regularity results for convexity (e.g., Sierpiński's theorem) find conditions on the data so that if f is a measurable solution of the above equation then it must be convex. As a particular case of the above problem prove (or disprove) that the measurable solutions of

$$f\Big(\frac{x+y+\sqrt{xy}}{3}\Big) \leqslant \frac{f(x)+f(y)+f(\sqrt{xy})}{3} \qquad (x,y \in \mathbb{R}_+)$$

are convex functions.

Reference

[1] Zs. Páles, *Bernstein–Doetsch-type results for general functional inequalities*, Roczn. Nauk.--Dydakt. Akad. Ped. Kraków **204** (2000), 197–206.

ZS. PÁLES

2. Problem. The following inequality plays an important role in the comparison of Gini and Stolarsky means (see the abstract of my talk):

$$\forall x \in \mathbb{R}_+ \setminus \{1\} : \quad \left(\frac{x^{-1+\frac{2}{\sqrt{5}}}+1}{x^{-1-\frac{2}{\sqrt{5}}}+1} \right)^{\frac{\sqrt{5}}{4}} \leqslant \exp\left(\frac{\log x}{1-x^3} + \frac{1}{3} \right).$$

A proof of this inequality would highly be appreciated.

P. CZINDER

3. Problem. Prove that

$$\min_{|z|=1,\, z\in\mathbb{C}} \left| \sum_{k=1}^{m} k z^{m-k} \right| \geqslant \frac{m}{2} \sec \frac{\pi}{2m+2}$$

if m is an odd natural number. With $z = \cos t + i \sin t$ this inequality can be reformulated as

$$\left[\frac{m}{2} + \frac{1}{2} \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^2 \right]^2 + \left[\frac{m \sin t - \sin mt}{4 \sin^2 \frac{t}{2}} \right]^2 \geqslant \left(\frac{m}{2} \sec \frac{\pi}{2m+2} \right)^2$$

(m odd, $t \in [0,\pi]$).

Using this inequality a theorem of A. Schinzel [2] concerning the location of zeros of some self-inversive polynomials can be sharpened (see [1]).

Remark added on June 11, 2003. A. Schinzel proved the above inequality if m is sufficiently large (private communication dated May 30, 2003).

References

[1] P. Lakatos, L. Losonczi, *On zeros of reciprocal polynomials of odd degree*, (submitted to J. of Inequalities in Pure and Applied Math.).

[2] A. Schinzel, *Self-inversive polynomials with all zeros on the unit circle*, (to appear in Ramanujan Journal).

L. LOSONCZI

4. Problem. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a given function. We consider the following functional inequality

$$(1) \qquad \bigwedge_{a\in\mathbb{R}^+} \bigvee_{\gamma(a)\in(0,\frac{1}{2})} \bigwedge_{x,y\in\mathbb{R}^+,\, x\leqslant ay} f\left(\frac{x+y}{2}\right) \leqslant \gamma(a)[f(x)+f(y)].$$

The following theorem was proved in [1]

THEOREM. *Let $f : (0,\infty) \to (0,\infty)$ be an arbitrary continuous function satisfying inequality () with some increasing function γ such that*

$$\lim_{x\to 0} f(x) = 0.$$

Then

$$\bigwedge_{b>0} \bigvee_{c>0,d>1} \bigwedge_{x\in(b,\infty)} cx^d \leqslant f(x).$$

As we can see the above theorem provides us with some information concerning the solutions of the inequality (1) on the interval (b,∞). It would be of interest to obtain a similar result on the interval $(0,b)$.

Reference

[1] T. Szostok, *On a modified version of Jensen Inequality*, J. of Inequal. & Appl. **3** (1999), 331–347.

T. SZOSTOK

5. Remark. During the conference Professor Zsolt Páles posed the following problem:

Prove that if a Lebesgue measurable function $f : 0,\infty \to \mathbb{R}$ satisfies the functional inequality

$$(1) \qquad f\Big(\frac{x+\sqrt{xy}+y}{3}\Big) \leqslant \frac{f(x)+f(\sqrt{xy})+f(y)}{3} \quad \text{for all } x,y \in 0,\infty$$

then f is a convex function.

Páles has proved that if a function f satisfying (1) is locally bounded above then it is convex. An easy calculation shows that the following general "measurability implies locally boundedness above" type statement can be applied to deduce the local boundedness from above of f from equation (1) and hence to solve the remaining part of the problem:

STATEMENT. *Suppose that*

$$f(g(x,y)) \leqslant h\Big(f_1(g_1(x,y)),\dots,f_n(g_n(x,y))\Big)$$

for all $(x,y) \in D$, where $D \subset \mathbb{R}\times\mathbb{R}$ is an open set, and

(1) f_i, $i = 1,2,\dots,n$ are real valued Lebesgue measurable functions;

(2) $h : \mathbb{R}^n \to \mathbb{R}$ and for each $k > 0$ there exists a $K > 0$ such that if $z_i \leqslant k$ for $i = 1,2,\dots,n$ then $h(z_1,\dots,z_n) \leqslant K$;

(3) the functions $g, g_1, \ldots, g_n$ mapping D into $\mathbb{R}$ are continuously differentiable;

(4) for t_0 there exist x_0 and y_0 such that $(x_0, y_0) \in D$, $t_0 = g(x_0, y_0)$, and for $i = 1, 2, \ldots, n$ we have

$$\det \begin{pmatrix} \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \\ \frac{\partial g_i}{\partial x}(x_0, y_0), & \frac{\partial g_i}{\partial y}(x_0, y_0) \end{pmatrix} \neq 0.$$

Then f is locally bounded above at t_0.

This easily follows if we introduce locally the new variable $t = g(x, y)$ instead of x and apply Theorem 5.1 in my book "Regularity properties of functional equation in several variables" (to appear by Kluwer Publisher). A similar result was proved in my paper "On measurable solutions of functional equations", Publ. Math. Debrecen 26 (1979), 17–35. If "Lebesgue measurable" is replaced by "have Baire property" then the Statement above remains true. About the Baire category case, see my paper "Regularity properties of functional equations", Aequationes Math. 25 (1982), 52–66.

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