Annales Mathematicae Silesianae 16 (2003)

Prace Naukowe Uniwersytetu Śląskiego nr 2142, Katowice

ON BLOCH HYPERHARMONIC FUNCTIONS

STEVO STEVIĆ

Abstract. In this note we give necessary and sufficient conditions for a hyperharmonic function to be a Bloch function.

1. Introduction

Throughout this paper n is an integer greater than 1, D is a domain in the Euclidean space $\mathbb{R}^n$, $B(a,r) = \{x \in \mathbb{R}^n \mid |x-a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \mathbb{R}^n$ and B is the open unit ball in $\mathbb{R}^n$. $S = \partial B = \{\, x \in \mathbb{R}^n \mid |x| = 1\}$ is the boundary of B.

Let dV denote the Lebesgue measure on $\mathbb{R}^n$, $d\sigma$ the surface measure on S, σ_n the surface area of a S.

We say that a real valued locally integrable function f on D possesses $HL-$property, with a constant c if

$$f(a) \leqslant \frac{c}{r^n} \int_{B(a,r)} f(x)dV(x) \quad \text{whenever } B(a,r) \subset D$$

for some $c > 0$ depending only on n.

For example, a subharmonic function possesses $HL-$property with $c = 1$. In [8] Hardy and Littlewood essentially proved that $|u|^p$, $p > 0$, $n = 2$ also possesses $HL-$property, whenever u is a harmonic function in D. In

Received: 06.03.2001. Revised: 02.01.2002.

AMS (1991) subject classification: Primary 31C05.

Key words and phrases: Hyperharmonic function, $HL-$property, Möbius automorphism, Laplace-Beltrami operator.

the case $n \geqslant 3$ a generalization was made by Fefferman and Stein [7,p.172] and Kuran [9]. An elementary proof of this can be found in [13]. Some other classes of functions which possess $HL-$property can be found in [13],[14],[16] and [17].

A function $f \in C^1(B)$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup_{x\in B}(1-|x|)|\nabla f(x)| < +\infty,$$

where $|\nabla f(x)| = \left(\sum_1^n \left|\frac{\partial f(x)}{\partial x_i}\right|^2\right)^{1/2}$. The space of Bloch functions is denoted by $\mathcal{B}(B)$.

Bloch functions have been extensively studied, especially equivalent conditions for f being a Bloch function in B. Basic results, of this type, for functions harmonic in the unit ball, can be found in [4],[10],[15]. In [15] author considers also a class of $C^1(D)$ Bloch functions. Definition of the Bloch functions and basic results for functions analytic on the unit disc can be found in [2],[3],[5] and for analytic functions in several complex variables in [6],[11],[12] and [18].

For general definition of the Laplace–Beltrami operator Δ_2 see Ahlfors [1]. In the case of a conformal metric $ds = \rho|dx|$, we have

$$\Delta_2 f = \rho^n \frac{\partial}{\partial x_i}\left(\rho^{2-n}\frac{\partial f}{\partial x_i}\right).$$

In the special case $\rho = \frac{1-|x|^2}{2}$ (the Poincaré metric), we obtain the Laplace–Beltrami operator, $\Delta_2 = \Delta_h$, for the hyperbolic metric, which is invariant under Möbius transformations, i.e. for $f \in C^2(B)$ and for every $\mathcal{M} \in M(B)$ we have

$$\Delta_2(f\circ\mathcal{M}) = (\Delta_2 f)\circ\mathcal{M}, \tag{1}$$

which are isometries of the space.

We say that a real valued function $f \in C^2(B)$ is hyperharmonic if it satisfies the Laplace–Beltrami equation $\Delta_h f = 0$. For more on these functions see, for example [1].

By ∇_h we denote the hyperbolic gradient, i.e.

$$\nabla_h f = \frac{(1-|x|^2)}{2}\nabla f.$$

For $\zeta \in S$ and $\delta \in (0, \pi]$, let

$$D_\delta(\zeta) = \{ x \in B \mid \cos\delta \leqslant \langle x, \zeta\rangle/|x|,\ 1-\delta \leqslant |x| < 1 \}.$$

A positive measure μ on B is called a Bergman–Carleson measure if and only if $\mu(D_\delta(\zeta)) = \mathcal{O}(\delta^n)$.

In this note we give necessary and sufficient conditions for a hyperharmonic function to be a Bloch function. A characterization of Bloch functions is connected with a Bergman–Carleson measure. We were motivated by Theorem 3 in [11].

2. Main results

We are now in a position to formulate and prove the main result.

THEOREM 1. *Let u be a hyperharmonic function. Then the following conditions are equivalent:*

(a) $$M = \sup_{a \in B} \int_B |\nabla_h u(x)|^p \left(\frac{1-|a|^2}{1-2\langle a, x\rangle + |a|^2|x|^2}\right)^n dV(x) < +\infty,$$

for some $p > 0$;
(b) $u \in \mathcal{B}(B)$;
(c) $|\nabla_h u(x)|^p dV(x)$ *is a Bergman-Carleson measure.*

PROOF. By Theorem 2 in [14] and Theorem 3 in [13], $|\nabla u|^p$, $p > 0$ possesses $HL-$property, i.e. there is a $C = C(n,p)$ such that

$$|\nabla u(a)|^p \leqslant \frac{C}{r^n} \int_{B(a,r)} |\nabla u(x)|^p dV(x), \qquad \text{whenever} \quad B(a,r) \subset B.$$

Let $T_a(x)$ be Möbius automorphism of the ball B which maps point a in the origin. It is known ([1]) that

$$T_a(x) = \frac{(1-|a|^2)(x-a) - |x-a|^2 a}{[x,a]^2},$$

where $[x,a]^2 = 1 - 2\langle x, a\rangle + |x|^2|a|^2$.

Since u is hyperharmonic, by (1) we obtain that $u \circ T_a$ is hyperharmonic. Thus $u \circ T_a$, also possesses $HL-$property. Especially, we have

$$|\nabla(u \circ T_{-a})(0)|^p \leqslant \frac{C}{r^n} \int_{rB} |\nabla(u \circ T_{-a})(x)|^p dV(x),$$

where r is a fixed number between 0 and 1.
By some simple calculation we obtain

$$|\nabla(u\circ T_{-a}(0))|^p=|\nabla(u(T_{-a}(0))\cdot T'_{-a}(0)|^p=\left(|\nabla u(a)|(1-|a|^2)\right)^p.$$

Applying the change $T_{-a}(x)=y$, i.e. $x=T_a(y)$, we obtain

$$\begin{aligned}\int_{rB}|\nabla(u\circ T_{-a})(x)|^p dV(x)&=\int_{rB}|\nabla u(T_{-a}(x))\cdot T'_{-a}(x)|^p dV(x)\\ &\leqslant\int_{rB}|\nabla u(T_{-a}(x))|^p|T'_{-a}(x)|^p dV(x)\\ &=\int_{T_{-a}(rB)}|\nabla u(y)|^p\left(\frac{1-|a|^2}{[-a,T_a(y)]^2}\right)^p\\ &\quad\times\left(\frac{1-|a|^2}{[a,y]^2}\right)^n dV(y)=I.\end{aligned}$$

The next identity is well-known

$$1-|T_a(y)|^2=(1-|a|^2)(1-|y|^2)/[a,y]^2.$$

Since $1-|x|^2\geqslant 1-r^2$, we have

$$(1-r^2)(1-2\langle a,y\rangle+|a|^2|y|^2)\leqslant(1-|a|^2)(1-|y|^2).$$

If we use the following inequality

$$1-2\langle a,y\rangle+|a|^2|y|^2>(1-|a||y|)^2>(1-|a|)^2,$$

we obtain

$$(1-|a|)(1-r^2)<(1+|a|)(1-|y|^2),$$

and consequently,

$$(1-|a|^2)(1-r^2)<4(1-|y|^2). \tag{2}$$

By (2) we have

$$I\leqslant\left(\frac{4}{1-r^2}\right)^p\int_{T_{-a}(rB)}|\nabla u(y)|^p\left(\frac{1-|y|^2}{[-a,T_a(y)]^2}\right)^p\left(\frac{1-|a|^2}{[a,y]^2}\right)^n dV(y). \tag{3}$$

Since $|T_a(y)|<r$, we have

$$\begin{aligned}[-a,T_a(y)]^2&=1+2\langle a,T_a(y)\rangle+|a|^2|T_a(y)|^2\\ &\geqslant(1-|a|\,|T_a(y)|)^2\geqslant(1-r)^2.\end{aligned} \tag{4}$$

Applying (4) in (3) we obtain

$$I \leqslant \left(\frac{4}{1-r^2}\right)^p \frac{1}{(1-r)^{2p}} \int_{T_{-a}(rB)} |\nabla u(y)|^p (1-|y|^2)^p \left(\frac{1-|a|^2}{[a,y]^2}\right)^n dV(y)$$
$$\leqslant \frac{4^p}{(1-r^2)^p} \frac{2^p}{(1-r)^{2p}} \int_B |\nabla_h u(y)|^p \left(\frac{1-|a|^2}{[a,y]^2}\right)^n dV(y)$$

From all of the above we obtain

$$(5) \qquad [(1-|a|^2)|\nabla u(a)|]^p \leqslant \frac{C\,2^{3p}}{r^n\,(1-r)^{3p}(1+r)^p} \times \int_B |\nabla_h u(x)|^p \left(\frac{1-|a|^2}{1-2\langle a,x\rangle + |x|^2|a|^2}\right)^n dV(x).$$

If we take supremum in (5) over $a \in B$, we obtain that (a) implies (b).

Let us denote

$$I(p,a) = \int_B |\nabla_h u(x)|^p \left(\frac{1-|a|^2}{1-2\langle a,x\rangle + |a|^2|x|^2}\right)^n dV(x).$$

Then by definition of Bloch function and using polar coordinates we obtain

(6)

$$I(p,a) \leqslant \|u\|_{\mathcal{B}} \sup_{a\in B} \int_B \frac{(1-|a|^2)^n}{(1-2\langle a,x\rangle + |a|^2|x|^2)^n} dV(x)$$
$$\leqslant \|u\|_{\mathcal{B}} \sup_{a\in B} (1-|a|^2)^n \int_0^1 \rho^{n-1} \int_S \frac{d\sigma(\zeta)}{(1-2|a|\rho\langle \eta,\zeta\rangle + |a|^2\rho^2)^n} d\rho,$$

where $\eta = a/|a|$. Let

$$J(a) = \int_S \frac{d\sigma(\zeta)}{(1-2|a|\rho\langle\eta,\zeta\rangle + |a|^2\rho^2)^n} = \int_S \frac{d\sigma(\zeta)}{(1-2r\cos\gamma + r^2)^n},$$

where $\gamma = \angle(\eta,\zeta)$ and $r = \rho|a|$. By some calculation we can obtain

$$(7) \qquad J(a) \leqslant C(1-r)^{-n-1} = C(1-\rho|a|)^{-n-1},$$

for some $C > 0$ independent of r. From (6) and (7) we obtain

$$I(p,a) \leqslant C\|u\|_{\mathcal{B}} \sup_{a\in B} (1-|a|^2)^n \int_0^1 (1-\rho|a|)^{-n-1} d\rho \leqslant C\,\|u\|_{\mathcal{B}},$$

as desired. Hence (b) implies (a).

Let (a) holds and $D_\delta(\zeta_0)$ be an arbitrary set defined as in the introduction. We may assume that $\delta < 1/4$. Choose $a = (1-\delta/2)\zeta_0$. For $x \in D_\delta(\zeta_0)$ we have that there is a constant $C > 0$ such that

$$\frac{1-|a|^2}{1-2\langle a,x\rangle + |a|^2|x|^2} \geqslant \frac{C}{1-|a|^2}.$$

Hence

$$\begin{aligned}\int_{D_\delta(\zeta_0)} |\nabla_h u(x)|^p dV(x) \leqslant C(1-|a|)^n \times \\ \int_{D_\delta(\zeta_0)} |\nabla_h u(x)|^p \left(\frac{1-|a|^2}{1-2\langle a,x\rangle + |a|^2|x|^2}\right)^n dV(x) \\ \leqslant CM(1-|a|)^n = 2^n CM\delta^n,\end{aligned}$$

from which (a) implies (c).

Let $|\nabla_h u(x)|^p dV(x)$ be a Bergman–Carleson measure (we denote this measure by μ), and $a \in B$. If $|a| < 3/4$, then there is a $C > 0$ such that

(8)
$$\begin{aligned}\int_B |\nabla_h u(x)|^p \left(\frac{1-|a|^2}{1-2\langle a,x\rangle + |a|^2|x|^2}\right)^n dV(x) \leqslant C\int_B |\nabla_h u(x)|^p dV(x) \\ = C\mu(B) < \infty.\end{aligned}$$

Let $|a| > 3/4$ and

$$F_k = \{\, x \in B \mid |x - a/|a|| < 2^k(1-|a|)\,\}, \quad k \in \mathbb{N}.$$

By definition of the Bergman–Carleson measure there is a constant $N(\mu)$ such that $\mu(F_k) \leqslant N(\mu)(\,2^k(1-|a|)\,)^n$ for all $k \in \mathbb{N}$. On the other hand, it is easy to prove that there is a $C > 0$ such that

$$\frac{1-|a|^2}{1-2\langle a,x\rangle + |a|^2|x|^2} \leqslant \frac{C}{1-|a|} \qquad \text{for} \quad x \in F_1,$$

$$\frac{1-|a|^2}{1-2\langle a,x\rangle + |a|^2|x|^2} \leqslant \frac{C}{2^{2k}(1-|a|)} \qquad \text{for} \quad x \in F_k \setminus F_{k-1}, k \geqslant 2.$$

Thus

(9)
$$\begin{aligned}\int_B |\nabla_h u(x)|^p \left(\frac{1-|a|^2}{1-2\langle a,x\rangle + |a|^2|x|^2}\right)^n dV(x) \leqslant \int_{F_1} + \sum_{k=2}^{\infty} \int_{F_k\setminus F_{k-1}} \\ \leqslant \sum_{k=1}^{\infty} \frac{C\mu(F_k)}{(\,2^{2k}(1-|a|)\,)^n} \leqslant CN(\mu)\sum_{k=1}^{\infty}\frac{1}{2^{kn}}.\end{aligned}$$

From (8) and (9) we obtain that (c) implies (a), finishing the proof.

REMARK 1. Throughout the above proof C denotes a positive constant that may change from one step to the next.

From the proof of Theorem 1 we see that the following statement is true:

THEOREM 2. *Let u be a hyperharmonic function such that*

$$\sup_{a\in B}\int_B |\nabla_h u(x)|^p \left(\frac{1-|x|^2}{1-2\langle a,x\rangle+|a|^2|x|^2}\right)^n dV(x) < +\infty, \tag{10}$$

for some $p > 0$, then $u \in \mathcal{B}(B)$.

Namely, we replace $(1-|a|)^n$ in (3) by $\left(4\frac{(1-|y|^2)}{1-r^2}\right)^n$ (see inequality (2)) and the rest of the proof is the same.

REMARK 2. It is an open question: is the condition (10) equivalent to the conditions in Theorem 1?

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Matematički Fakultet
Studentski Trg 16
11000 Beograd, Serbia
Yugoslavia

e-mail: sstevic@ptt.yu; sstevo@matf.bg.ac.yu