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Report of Meeting

The Second Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities January 30 – February 2, 2002 Hajdúszoboszló, Hungary

The Second Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities was held in Hajdúszoboszló, Hungary from January 30 to February 2, 2002, at Hotel Délibáb. 20 participants came from the University of Debrecen (Hungary) and the Silesian University of Katowice (Poland) at 10 from each of both cities.

Professor Zsolt Páles opened the Seminar and welcomed the participants to Hajdúszoboszló.

He used this occasion to introduce the Hungarian participants. Following this initiative, Professor Roman Ger introduced the Polish participants. The scientific talks presented at the Seminar focused on the following topics: equations in a single and several variables, iteration theory, equations on algebraic structures, conditional equations, differential functional equations, Hyers-Ulam stability, functional inequalities and mean values. Interesting discussions were generated by the talks.

There were three very profitable Problem Sessions.

The social program included thermal bath, a well-received organ concert performed by Mihály Bessenyei in the reformed church of Hajdúszoboszló, and a festive dinner.

The closing address was given by Professor Roman Ger. His invitation to the Third Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities in February 2003 in Poland was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in section 1, problems and remarks in approximate chronological order in section 2, and the list of participants in the final section.

1. Abstracts of talks

ROMAN BADORA: *On a generalization of Wilson's functional equation to N summands*

Let $(G,+)$ be a locally compact abelian group and let $K = \{k_0 = id_G, k_1, \dots, k_{N-1}\}$ be a finite group of automorphisms of G. Applying the method of the Fourier transformation on the space of almost periodic functions on G (exploited with much success by L. Székelyhidi) we find the set of bounded, continuous solutions $f,\ g : G \to \mathbb{C}$ of the following version of Wilson's functional equation

$$\sum_{n=0}^{N-1} g(x + k_n y) = N g(x) f(y), \quad x,\ y \in G,$$

where g is an almost periodic function on G.

LECH BARTŁOMIEJCZYK: *Irregular scaling functions with orthogonal translations*

(Joint work with Janusz Morawiec)

Following [1] we consider the problem of the existence of irregular compactly supported solutions $\varphi : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$\varphi(x) = \sum_{i=0}^{m-1} \varphi(mx - ik)$$

satisfying

$$\sum_{i\in\mathbb{Z}} \varphi(x+i) = 1 \qquad a.e.$$

and

$$\varphi(x)\varphi(x+i) = 0 \qquad a.e.$$

for every $i \in \mathbb{Z} \setminus \{0\}$.

REFERENCE

[1] J. Cnops, *A scaling equation with only non-measurable orthogonal solutions*, Proc. Amer. Math. Soc. **128** (2000), 1975–1979.

MIHÁLY BESSENYEI: *On further Hadamard-type inequalities*
(Joint work with Zsolt Páles)

Let $\omega_1, \omega_2 : [a,b] \to \mathbb{R}$ be given functions. We say, that a function $f : [a,b] \to \mathbb{R}$ is (ω_1, ω_2)-convex, if

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \geqslant 0,$$

whenever $a \leqslant x < y < z \leqslant b$. If $\omega_1(x) := 1$ and $\omega_2(x) := x$, this notion is consistent with the notion of convexity. We investigate the question whether there are Hadamard-type inequalities when our function $f : [a,b] \to \mathbb{R}$ is supposed to be (ω_1, ω_2)-convex.

For example, if $\omega_1(x) := \cosh x$ and $\omega_2(x) := \sinh x$, we get the inequality

$$2\sinh\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)dx \leqslant \tanh\left(\frac{b-a}{2}\right)(f(a)+f(b)).$$

Similarly, choosing $\omega_1(x) := \cos x$ and $\omega_2(x) := \sin x$ on $[a,b] \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$, we have that

$$2\sin\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)dx \leqslant \tan\left(\frac{b-a}{2}\right)(f(a)+f(b)).$$

ZOLTÁN BOROS: *Decomposition of real functions with monotonic lower and upper strong* $\mathbf{Q}$*-derivatives*

For a real function f we define the lower and upper strong $\mathbf{Q}$-derivatives at the point x and in the direction h by the lower and upper limits of the ratio $(f(y+rh) - f(y))/r$ as r tends to zero through the positive rationals and y tends to x. These limits are denoted by $\underline{D}_h^{\mathbf{Q}} f(x)$ and $\overline{D}_h^{\mathbf{Q}} f(x)$, respectively. We say that f has increasing lower and upper strong $\mathbf{Q}$-derivatives if

$$-\infty < \underline{D}_h^{\mathbf{Q}} f(x_1) \leqslant \overline{D}_h^{\mathbf{Q}} f(x_1) \leqslant \underline{D}_h^{\mathbf{Q}} f(x_2) \leqslant \overline{D}_h^{\mathbf{Q}} f(x_2) < +\infty$$

holds for every $h > 0$ and $x_1 < x_2$. We prove that every function with increasing lower and upper strong $\mathbf{Q}$-derivatives can be represented as the sum of an additive mapping and a convex function.

Zoltán Daróczy: *Functional equations involving means and their Gauss-composition*

Let $I \subset \mathbb{R}$ be a non-void open interval and let $M_i : I^2 \to I$ be strict means on I $(i = 1, 2, 3)$ with

$$M_3 = M_1 \otimes M_2 ,$$

where $\otimes$ denotes the Gauss-composition of means. This talk deals with the connection of the functional equations

$$f(M_1(x,y)) + f(M_2(x,y)) = f(x) + f(y) \qquad (x, y \in I) \tag{1}$$

and

$$2f(M_3(x,y)) = f(x) + f(y) \qquad (x, y \in I), \tag{2}$$

where $f : I \to \mathbb{R}$ is an unknown function.

The main result is the following: *If M_i $(i = 1, 2, 3)$ are quasi-arithmetic means on I, then* (1) *and* (2) *are equivalent.*

Roman Ger: *An interplay between Jensen and Pexider functional equations* (Joint work with Zygfryd Kominek)

Let $(S, +)$ and $(G, +)$ be two commutative semigroups. Assuming that the latter one is cancellative we deal with functions $f : S \longrightarrow G$ satisfying the Jensen functional equation written in the form

$$2f(x + y) = f(2x) + f(2y) .$$

It turns out that functions $f, g, h : S \longrightarrow G$ satisfying the functional equation of Pexider

$$f(x + y) = g(x) + h(y)$$

must necessarily be Jensen. The validity of the converse implication is also studied with emphasis placed on a very special Pexider equation

$$\varphi(x + y) + \delta = \varphi(x) + \varphi(y) ,$$

where δ is a fixed element of G.

Plainly, the main goal is to express the solutions of both: Jensen and Pexider equations in terms of semigroup homomorphisms.

Attila Gilányi: *Hyers-Ulam stability of the Cauchy functional equation on power-symmetric groupoids*

(Joint work with Zsolt Páles)

In this talk stability theorems are proved for the Cauchy functional equation for functions defined on and mapping into power-symmetric groupoids. The results presented are strictly connected to those in [3] and they also generalize stability theorems obtained in [1], [4], [5], [6], as well as Hyers' classical result [2].

References

[1] G.-L. Forti, *An existence and stability theorem for a class of functional equations*, Stochastica 4 (1980), 23–30.

[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA 27 (1941), 222–224.

[3] Zs. Páles, *Hyers-Ulam stability of the Cauchy functional equation on square-symmetric groupoids*, Publ. Math. Debrecen 58 (2001), 651–666.

[4] Zs. Páles, P. Volkmann, R. D. Luce, *Hyers-Ulam stability of functional equations with a square-symmetric operation*, Proc. Natl. Acad. Sci. USA 95 (1998), 12772–12775.

[5] J. Rätz, *On approximately additive mappings*, General Inequalities 2 (ed. W. Walter), Birkhäuser, Basel, 1980, 233–251.

[6] P. Volkmann, *On the stability of the Cauchy equation*, Proceedings of the International Conference Numbers, Functions, Equations '98 (ed. Zs. Páles), Leaflets in Mathematics, 1998, 150–151.

Attila Házy: *Reduction of differential functional equations to differential equations*

Consider the equation

$$(1)\quad l_n(x,y)f^{(n)}(g(x,y))+\ldots+l_0(x,y)f(g(x,y))=F(x,y),\qquad (x,y)\in\Omega,$$

where $\Omega\subset\mathbf{R}^2$ is an open, connected set and $l_0,l_1,\ldots,l_n,g$ and F are given real valued, analytic functions on Ω (such that $g(\Omega)$ is an open set), furthermore f is an unknown real function on $g(\Omega)$. We prove that there exists a differential functional equation

$$(2)\quad h_m(x,y)f^{(m)}(g(x,y))+\ldots+h_0(x,y)f(g(x,y))=H(x,y),\quad (x,y)\in\Omega,$$

(where $m\leqslant n$) whose $(n+1)$-times differentiable solutions coincide with that of (1) such that $h_0,h_1,\ldots,h_m,H$ satisfy the following system of equations

$$(3)\quad \partial_x g\cdot(h_m\cdot\partial_y h_i-h_i\cdot\partial_y h_m)=\partial_y g\cdot(h_m\cdot\partial_x h_i-h_i\cdot\partial_x h_m),\quad i=1,\ldots,m-1$$

and

$$(4)\qquad \partial_x g\cdot(h_m\cdot\partial_y H-H\cdot\partial_y h_m)=\partial_y g\cdot(h_m\cdot\partial_x H-H\cdot\partial_x h_m).$$

These properties of $h_0, h_1, \ldots, h_m$ and H imply that, locally, these functions are of the form

$$h_i(x,y) = h_m(x,y)K_i(g(x,y)), \quad i = 0, \ldots, m-1,$$

and

$$H(x,y) = h_m(x,y)K(g(x,y)).$$

Hence after simplification and the substitution $t = g(x,y)$, (2) reduces to an ordinary differential equation with respect to f, whose order is usually much smaller than the order of (1).

ZOLTÁN KAISER: *The asymptotic stability of the Cauchy equation in p-adic fields*

It is proved that if $\alpha \neq 1$ is a real number and f is a mapping from a non-archimedean normed space $(X, \|\ \|_1)$ over the p-adic field $\mathbb{Q}_p$ to a non-archimedean Banach space $(Y, \|\ \|_2)$ over $\mathbb{Q}_p$ satisfying

$$\|f(x+y) - f(x) - f(y)\|_2 \leqslant K \max\{\|x\|_1^\alpha, \|y\|_1^\alpha\}$$

for some fixed K and all $x, y \in X$, then there exists an additive mapping $g : X \to Y$ for which

$$\|f(x) - g(x)\|_2 \leqslant CK\|x\|_1^\alpha$$

for all $x \in X$, where the coefficient C may depend on α.

RAFAŁ KAPICA: *Sequences of iterates of random-valued vector functions and continuous solutions of a linear functional equation of infinite order*

Given a probability space $(\Omega, \mathcal{A}, P)$, a separable Banach space X, and measurable functions $L : \Omega \to (0, \infty)$, $M : \Omega \to X$, we obtain some theorems on the existence and on the uniqueness of continuous solutions $\varphi : X \to \mathbb{R}$ of the equation

$$\varphi(x) = \int_\Omega \varphi\left(L(\omega)x + M(\omega)\right) P(d\omega).$$

ZYGFRYD KOMINEK: *On the continuity of t-Wright-convex functions*

Let $t \in (0,1)$ be a fixed number. A function $f : (a,b) \to \mathbb{R}$ is called t-Wright-convex iff it satisfies the following inequality

$$f(tx + (1-t)y) + f((1-t)x + ty) \leqslant f(x) + f(y), \qquad x, y \in (a,b).$$

We prove that every t-Wright-convex function which is continuous at a point is continuous everywhere.

KÁROLY LAJKÓ: *One more functional equation in the theory of conditionally specified distributions*

Let (X,Y) be an absolutely continuous bivariate random variable with support in the positive quadrant. Let us denote the joint, marginal, and conditional densities by $f_{X,Y}$, f_X, f_Y, $f_{X|Y}$, $f_{Y|X}$, respectively.

One can write $f_{X,Y}$ in two ways and obtain the relationship

$$(1)\qquad f_{X,Y}(x,y) = f_{X|Y}(x,y)f_Y(y) = f_{Y|X}(x,y)f_X(x) \quad (x,y \in \mathbb{R}_+).$$

Narumi (1923) inquired, for example, about all joint densities whose conditional densities satisfy

$$(2)\qquad \begin{aligned} f_{X|Y}(x,y) &= g_1\left(\frac{x-a_1y}{\sqrt{y^2+b_1y+c_1}}\right); \\ f_{Y|X}(x,y) &= g_2\left(\frac{y-a_2x}{\sqrt{x^2+b_2x+c_2}}\right) \quad (x,y \in \mathbb{R}_+), \end{aligned}$$

where a_1, $a_2 \in \mathbb{R}$, b_1, b_2, c_1, $c_2 \in \mathbb{R}_+$.

We have, from (1) and (2), the functional equation

(3)

$$g_1\left(\frac{x-a_1y}{\sqrt{y^2+b_1y+c_1}}\right) f_Y(y) = g_2\left(\frac{y-a_2x}{\sqrt{x^2+b_2x+c_2}}\right) f_X(x) \quad (x,y \in \mathbb{R}_+)$$

for functions f_X, $f_Y : \mathbb{R}_+ \to \mathbb{R}$ and g_1, $g_2 : \mathbb{R} \to \mathbb{R}_+$.

The general measurable solutions of (3) are determined here.

LÁSZLÓ LOSONCZI: *Comparison and subhomogeneity of integral means*

If $f : I \to \mathbb{R}$ is continuous and strictly monotonic on I then for every $x,y \in I, x < y$, there is a unique point $s \in]x,y[$ such that $f(s) = \int_x^y f(u)\,du/(y-x)$. Hence

$$s = f^{-1}\left(\frac{1}{y-x}\int_x^y f(u)\,du\right).$$

This number s is called the *integral f-mean of x and y* and denoted by $I_f(x,y)$.

Clearly, (requiring I_f to have the mean property or requiring it to be continuous) we have for equal arguments $I_f(x,x) = x \quad (x \in I)$.

I_f was defined and studied by Elezović and Pečarić, *Differential and integral f-means and applications to digamma function*, Math. Ineq. Appl. **3** (2000), 189–196 (we slightly changed their definition). They gave sufficient conditions for the comparison of integral means and applied these to obtain some other inequalities.

Our aim is to give *necessary and sufficient conditions* for the comparison of differential and integral means and discuss the subhomogeneity and homogeneity of these means. We also study the general comparison (involving three integral means).

GYULA MAKSA: *A remark on two variable means*
(Joint work with Zoltán Daróczy)

In this talk we present the following regularity theorem.

THEOREM. *Let $J \subset \mathbb{R}$ be an open interval of positive length, $\varphi : J \to \mathbb{R}$ be a strictly monotonic and continuous function, and $f : J \to]0, +\infty[$. Suppose that*

$$\frac{\varphi(x)f(x) + \varphi(y)f(y)}{f(x) + f(y)} = \varphi\left(\frac{x+y}{2}\right) \tag{1}$$

holds for all $x, y \in J$. Then φ and f are infinitely many times differentiable, furthermore $\varphi'(x) \neq 0$ if $x \in J$.

The equality problem of the weighted arithmetic means with weight functions, when one of the weight functions is constant, leads to equation (1) in the two variable case.

JANUSZ MATKOWSKI: *Means and some functional equations*

Under some regularity assumptions we establish all Lagrangean mean-type mappings for which the arithmetic mean is invariant.

The same problem for means of the form

$$M_f(x,y) = f^{-1}\left(\frac{xf(x) + yf(y)}{x+y}\right)$$

is also considered.

Zsolt Páles: *A regularity theorem for composite functional equations*

We deal with regularity properties of functions f and g satisfying a functional inequality of the following type

$$|f(a(x,y)) - f(a(x,z))| \leqslant |g(b(x,y)) - g(b(x,z))|, \qquad (x,y),(x,z) \in D,$$

where the real valued functions a and b defined on an open set $D \subset \mathbb{R}^2$ enjoy certain sufficiently strong regularity properties. One of the main results states that if g is pointwise Lipschitz on a dense subset of $b(D)$ (for instance if g is differentiable on a dense subset) then f is locally Lipschitz on $a(D)$. Another result states that if f admits a strict inverse pointwise Lipschitz condition on a dense subset of $a(D)$ (for instance, if f is differentiable on a dense subset with nonzero derivative), then g is locally invertible with a locally Lipschitz inverse.

The results so obtained have applications in the regularity theory of composite functional equations, see, for instance, [1], [2].

References

[1] J. Aczél, Gy. Maksa, and Zs. Páles, *Solution to a functional equation arising from different ways of measuring utility*, J. Math. Anal. Appl. **233** (1999), 740–748.

[2] Z. Daróczy and Zs. Páles, *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen, **61** (2002), 157–218.

Iwona Pawlikowska: *A method used in characterizing polynomial functions*

Let X, Y be two linear spaces over a field $\mathbb{K} \subset \mathbb{R}$ and let K be a convex balanced set with $0 \in alg\,int\,K$. Fix $N,\ M \in \mathbb{N} \cup \{0\}$ and $a,\ b \in \mathbb{Q}$, $b \neq 0$. We denote by $I = \{(\alpha,\beta) \in \mathbb{Q} \times \mathbb{Q} : |\alpha| + |\beta| \leqslant 1\}$ and $I^+ = \{(\alpha,\beta) \in I : \beta \neq 0\}$. Assume that $I_0, \dots, I_M$ are finite subsets of I^+. We prove the following lemma: if functions $\varphi_i : K \to SA^i(X;Y)$, $i \in \{0,\dots,N\}$ and $\psi_{j,(\alpha,\beta)} : K \to SA^j(X;Y)$, $(\alpha,\beta) \in I_j$, $j \in \{0,\dots,M\}$ satisfy the equation

$$\sum_{i=0}^{N} \varphi_i(x)((ax+by)^i) = \sum_{i=0}^{M} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha x + \beta y)((ax+by)^i)$$

for every $x,\ y \in K$, then there exists a $p \in \mathbb{N}$ such that φ_N is a local polynomial function of order at most equal to

$$\sum_{i=0}^{M} card\left(\bigcup_{k=i}^{M} I_k\right) - 1$$

on $\frac{1}{p}K$. This outcome in case of $N = 0$ is an extension of a theorem of Z. Daróczy and Gy. Maksa from [1].

We also generalize some results of W. H. Wilson [4], L. Székelyhidi [3] and M. Sablik [2]. We use this lemma to solve functional equations characterizing polynomial functions.

References

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[4] W. H. Wilson, *On a Certain General Class of Functional Equations*, Amer. J. Math. 40 (1918), 263–282.

Tomasz Powierża: *On the smallest set-valued iterative roots of bijections*

We deal with the notion of the set-valued iterative root. We sketch the construction of a certain class of such roots. Necessary conditions of the existence of the smallest set-valued iterative root of a given bijection are also given.

Maciej Sablik: *A functional equation stemming from a variant of Flett's Mean Value Theorem*

(Joint work with Thomas Riedel)

Thomas Riedel presented at the 8th ICFEI in Złockie, Poland, September 2001, the following variant of Flett's Mean Value Theorem:

Theorem. *Let f be differentiable on $[a,b]$, then there is a point c in (a,b) such that*

$$\frac{1}{c-a}\left(f'(c)-\frac{f(c)-f(a)}{c-a}\right)+\frac{1}{c-b}\left(f'(c)-\frac{f(c)-f(b)}{c-b}\right)=\frac{f'(b)-f'(a)}{b-a}.$$

A question arises as natural as the analogous one asked in the case of Lagrange MVT: which are the functions satisfying equality in Riedel's theorem with $c = \frac{a+b}{2}$ for all $a, b \in \mathbb{R}$? This leads, after a pexiderization procedure, to the following functional equation

$$(1)\qquad -8\,f\left(\frac{a+b}{2}\right)+4\,f(a)+4\,f(b)=(g(b)-g(a))\,(b-a).$$

Our task in the present talk is to solve (1) completely with no regularity assumption on f or g.

Tomasz Szostok: *A generalization of the sine function and a characterization of inner product spaces*

The function
$$s(x,y) = \inf_{\lambda\in\mathbb{R}} \frac{\|x+\lambda y\|}{\|x\|}$$
is considered. This function was originally used to provide unconditional equations in place of orthogonal equations in the sense of Birkhoff-James. Further properties of the function s are determined. Since in an Euclidean space the value $s(x,y)$ is equal to the absolute value of the sinus of the angle between vectors x and y, s may be viewed as a generalization of sinus. Moreover, we deal with another generalization of the sine function which, in particular, leads to a new characterization of inner product spaces.

2. Problems and Remarks

1. Remark (a generalization of a quasi-arithmetic mean). Let $I \subset \mathbb{R}$ be an interval. If $f, g : I \to \mathbb{R}$ are continuous, both increasing or both decreasing, and $f+g$ is strictly monotonic, then the function $M_{f,g} : I^2 \to \mathbb{R}$ given by
$$M_{f,g}(x,y) := (f+g)^{-1}(f(x)+g(y)), \qquad x,y \in I,$$
is a mean; if, moreover, f and g are strictly monotonic, then $M_{f,g}$ is a strict mean.

Note that
1) if $g = f$ then $M_{f,g}$ is a quasi-arithmetic mean;
2) if $\varphi : I \to \mathbb{R}$ is continuous, strictly monotonic, and $p \in (0,1)$ is fixed, then, setting $f := p\varphi$, $g := (1-p)\varphi$ gives
$$M_{f,g}(x,y) := \varphi^{-1}(p\varphi(x) + (1-p)\varphi(y)), \qquad x,y \in I$$
(thus $M_{f,g}$ is a generalization of the weighted quasi-arithmetic means).

Without any regularity assumptions one can
a) characterize the functions $f, g, F, G : I \to \mathbb{R}$ such that $M_{F,G} = M_{f,g}$;
b) determine all f and g for which $M_{f,g}$ is homogeneous;
c) determine all f and g for which $M_{f,g}$ is translative.

J. Matkowski

2. Remark of Problem. As it was discussed in the talk of Z. Daróczy, the functional equations

$$(1) \qquad f(px+(1-p)y) + f((1-p)x+py) = f(x)+f(y) \qquad (x,y \in I)$$

and

$$(2) \qquad f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \qquad (x, y \in I)$$

may and also may not be equivalent to each other, depending on the value of the parameter $p \in]0,1[$ (where I is an open real interval and the unknown function f maps I to $\mathbb{R}$). The reason is that the general solutions of (2) are of the form

$$f(x) = A_0 + A_1(x) \qquad (x \in I),$$

where A_0 is an arbitrary constant and $A_1 : \mathbb{R} \to \mathbb{R}$ is an arbitrary additive function, while, by a result of K. Lajkó, the general solutions of (1) are of the form

$$f(x) = A_0 + A_1(x) + A_2(x,x) \qquad (x \in I),$$

where, in addition, $A_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric biadditive function such that

$$(3) \qquad A_2(px, (1-p)x) = 0 \qquad (x \in \mathbb{R}).$$

Therefore, (1) and (2) are equivalent functional equations if and only if there is no non-identically-zero symmetric biadditive function A_2 that satisfies (3). For instance, if p is rational, then (3) yields that $A_2(x,x) = 0$ for all x, hence A_2 is identically zero.

Gy. Maksa tried to find A_2 in the form

$$A_2(x,y) = a(x)y + a(y)x,$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive function with the homogeneity property

$$(4) \qquad a(px) = qa(x) \qquad (x \in \mathbb{R}).$$

Clearly, then

$$A_2(px,(1-p)x) = a(px)(1-p)y + a((1-p)x)px = [q(1-p)+(1-q)p]a(x)x.$$

If A_2 (and therefore a) is non identically zero, then (3) holds if and only if

$$(5) \qquad p + q = 2pq.$$

By a basic result of Z. Daróczy, nontrivial additive functions satisfying (4) exist if an only if either p and q are transcendental, or p is algebraic and q is one of the algebraic conjugates of p.

Z. Daróczy observed that if p is transcendental and $q = p/(2p-1)$, then q is also transcendental, therefore there is a nontrivial additive function a satisfying (4). On the other hand, (5) holds, thus A_2 is a nontrivial symmetric biadditive function that satisfies (3). Hence, the functional equations (1) and (2) are not equivalent for transcendental p.

Gy. Maksa observed that if p is a second-order algebraic number whose generating polynomial is of the form

$$p^2 - 2rp + r = 0, \tag{6}$$

where r is a rational number, then the algebraic conjugate q of p trivially satisfies (5). Therefore, for such algebraic numbers, (1) and (2) are again not equivalent to each other.

My observation was that if p is a second-order algebraic number, then a nontrivial A_2 satisfying (3) (but not necessarily of the form $A(x,y) = a(x)y + a(y)x$) can exist if and only if the generating polynomial of p is exactly of the form (6). Furthermore, I was also able to prove that, for all third-order algebraic numbers p, (3) holds if and only if A_2 is identically zero.

Therefore, I pose the following problem: *Prove or disprove that, for all third or higher-order algebraic numbers p,* (3) *holds if and only if A_2 is identically zero (and hence, then* (1) *and* (2) *are equivalent functional equations).*

Zs. Páles

3. Problem. Let $I \subset \mathbb{R}_+$ be a non-void open interval and let

$$M_1(x,y) := \sqrt{\frac{x+y}{2}x}\,, \quad M_2(x,y) := \sqrt{\frac{x+y}{2}y} \qquad (x,y \in I).$$

M_1 and M_2 are strict means on I. It is known (Carlson, 1971) that

$$M_1 \otimes M_2\,(x,y) = \sqrt{\frac{x^2-y^2}{2\log\frac{x}{y}}} \quad \text{if} \quad x \neq y\,.$$

We consider the functional equations

$$f\left(\sqrt{\frac{x+y}{2}x}\right) + f\left(\sqrt{\frac{x+y}{2}y}\right) = f(x) + f(y) \qquad (x,y \in I) \tag{1}$$

and

$$2f\left(\sqrt{\frac{x^2-y^2}{2\log\frac{x}{y}}}\right) = f(x) + f(y) \qquad (x \neq y\,;\; x,y \in I). \tag{2}$$

The implication (1) $\Longrightarrow$ (2) is open.

REMARK. If $f : I \to \mathbb{R}$ is a *continuous solution* of (1), then there exists a real number c such that $f(x) = c$ for all $x \in I$. My conjecture is the following: (1) has only constant solutions, therefore (1) $\Longrightarrow$ (2), i.e., (1) $\Longleftrightarrow$ (2).

Z. DARÓCZY

4. Remark (in connection with the talk of Zoltán Daróczy). Assume that $g : (0, \infty) \to \mathbb{R}$ satisfies the equation

$$(1) \qquad g\big(x(x+y)\big) + g\big(y(x+y)\big) = g\big(2x^2\big) + g\big(2y^2\big), \quad x, y \in (0, \infty).$$

As we know, any continuous solution of (1) is constant. Zsolt Páles remarked that with the aid of A. Járai's method one can show that any Lebesgue measurable solution of (1) is continuous and hence constant. But even non-measurable real functions on $(0, \infty)$ admit the existence of finite limits at 0 and ∞. Therefore, the following statement provides an additional information on the behaviour of solutions of (1).

THEOREM. *Let $g : (0, \infty) \to \mathbb{R}$ be a solution to* (1) *such that either*
(i) $d := \lim_{t \to \infty} g(t)$ *exists and* $d \in \mathbb{R}$
or
(ii) *the limits* $c := \lim_{t \to 0} g(t)$ *and* $d := \lim_{t \to \infty} g(t)$ *do exist and* $c \in \mathbb{R}$.
Then g is constant.

PROOF. Assuming (i) fix arbitrarily an $x \in (0, \infty)$ and pass to the limit as $y \to \infty$ in (1) getting

$$d + d = g(2x^2) + d.$$

Due to the finiteness of d and the unrestricted choice of x we infer that g is constant, as claimed.

Assuming (ii) observe first that equation (1) may equivalently be written in the form

$$(2) \qquad g(s) + g(t) = g\left(\frac{2s^2}{s+t}\right) + g\left(\frac{2t^2}{s+t}\right), \quad s, t \in (0, \infty),$$

whence, by induction,

$$\begin{aligned} &g(s) + g(t) \\ &\quad = g\left(\frac{2^n s^{2^n}}{(s+t)(s^2+t^2)\cdots(s^{2^{n-1}}+t^{2^{n-1}})}\right) \\ &\quad\quad + g\left(\frac{2^n t^{2^n}}{(s+t)(s^2+t^2)\cdots(s^{2^{n-1}}+t^{2^{n-1}})}\right) \end{aligned}$$

valid for all $s, t \in (0, \infty)$ and all $n \in \mathbb{N}$.

Fix arbitrarily an $s \in (0, \infty)$ and $\alpha \in (0, 1)$. Setting here $t = \alpha s$ we get

$$(3) \qquad g(s) + g(\alpha s) = g\left(\frac{2^n s}{\varphi_n(\alpha)}\right) + g\left(\frac{2^n \alpha^{2^n} s}{\varphi_n(\alpha)}\right),$$

where

$$\varphi_n(\alpha) := \prod_{k=0}^{n-1} \left(1 + \alpha^{2^k}\right), \quad n \in \mathbb{N}.$$

Plainly, this product converges to a number from $[1, \infty)$, whence, passing to the limit as $n \longrightarrow \infty$ in (3), we obtain

$$g(s) + g(\alpha s) = d + c,$$

showing that d has to be finite, i.e. that (i) is satisfied. Thus the proof has been completed.

R. Ger

5. Remark (to Matkowski's 1. Remark). In his remark, J. Matkowski introduced the following class of two variable means

$$M_{f,g}(x + y) := (f + g)^{-1}(f(x) + g(y)) \qquad (x, y \in I),$$

where I is a real interval and f, g are strictly increasing real valued continuous functions on I. Matkowski also announced the solution of the equality and homogeneity problem. In the following result, assuming differentiability conditions, we give necessary and sufficient conditions for the comparison problem of such means.

THEOREM. *Let $f, g, F, G : I \to \mathbb{R}$ be twice continuously differentiable functions with everywhere positive derivatives. Then the comparison inequality*

$$(1) \qquad M_{f,g}(x, y) \leqslant M_{F,G}(x, y)$$

holds for all $x, y \in I$ if and only if

$$(2) \qquad \frac{F'}{f'} = \frac{G'}{g'}$$

and this function is nondecreasing on I.

PROOF. *Necessity.* Let u be an arbitrarily fixed point in I. Consider the function Φ defined by

$$\Phi(x,y) = M_{F,G}(x,y) - M_{f,g}(x,y) \qquad (x,y \in I).$$

By (1), we have that Φ is nonnegative and also $\Phi(u,u) = 0$. In other words, Φ has a minimum at the point (u,u). Therefore,

$$\partial_1\Phi(u,u) = 0, \qquad \partial_1^2\Phi(u,u) \geqslant 0. \tag{3}$$

The first equality yields that

$$\frac{F'(u)}{F'(u)+G'(u)} - \frac{f'(u)}{f'(u)+g'(u)} = 0,$$

which simplifies to (2). The second relation of (3) results that

$$\frac{F''(u)(F'(u)+G'(u))^2 - F'^2(u)(F''(u)+G''(u))}{(F'(u)+G'(u))^3} \\ - \frac{f''(u)(f'(u)+g'(u))^2 - f'^2(u)(f''(u)+g''(u))}{(f'(u)+g'(u))^3} \geqslant 0.$$

By (2), we have that

$$g' = \frac{f'G'}{F'}, \qquad g'' = \frac{f''F'G' + f'F'G'' - f'F''G'}{F'^2}.$$

Replacing g' and g'' by the above expressions and simplifying the last inequality, we get

$$F''(u)f'(u) - F'(u)f''(u) \geqslant 0$$

for all u, which yields that F'/f' is nondecreasing. Thus, the necessity of the conditions is proved.

Sufficiency. Applying Cauchy's Mean Value Theorem, the monotonicity of the functions F'/f' and G'/g' yields that

$$f(x) - f(u) \leqslant \frac{f'(u)}{F'(u)}(F(x) - F(u)) \qquad (x,u \in I)$$

and

$$g(y) - g(u) \leqslant \frac{g'(u)}{G'(u)}(G(y) - G(u)) \qquad (y,u \in I).$$

Let $x, y \in I$ be arbitrary and denote by u the value of $M_{F,G}(x,y)$. Then we have that

$$F(u) + G(u) = F(x) + G(y).$$

Putting these values into the above inequalities and summing them up, we can see that the right hand side is zero due to the choice of u and the identity (2). Thus

$$f(x) + g(y) \leqslant f(u) + g(u),$$

i.e., $M_{f,g}(x,y) \leqslant u = M_{F,G}(x,y)$. The proof is complete.

The second-order regularity conditions are not needed to express the necessary and sufficient conditions of the Theorem, they are only used in the proof of the necessity. It seems to be natural to ask if they can be eliminated at all. As it turned out after the Seminar, the statement of the Theorem can be proved without second-order differentiability assumptions as well. Moreover, necessary and sufficient conditions not involving even first-order differentiability assumptions can also be obtained and proved.

ZS. PÁLES

6. Problem. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a strictly increasing function, c be a positive constant, and assume that, for each fixed $y > 0$,

$$x \mapsto f(x+y) - cf(x)$$

is monotonic.

In the case $c = 1$ this leads to the Jensen-convexity or Jensen concavity of f. Due to the Bernstein–Doetsch theorem, it follows that f is convex or concave. Therefore, f is locally Lipschitz, moreover, it is differentiable everywhere form the left and from the right, and the two-sided derivative exists everywhere but at countably many points.

The problem is *if these differentiability properties can be derived also in the case* $c \neq 1$. By the results presented in my talk, it follows that f and its inverse are locally Lipschitz functions. f.

ZS. PÁLES

7. Problem (concerning Matkowski's talk). Find all triples of Lagrangean means (M_1, M_2, M_3) such that $M_3 = M_1 \otimes M_2$. Note that the problem considered by J. Matkowski (cf. the abstract of his talk in this report) is the particular case when M_3 is the arithmetic mean.

Z. DARÓCZY

8. Problem. A function $f:(0,\infty)\to(0,\infty)$ is called convex with respect to the logarithmic mean

$$L(x,y) := \begin{cases} \dfrac{x-y}{\ln x - \ln y}, & \text{if } x,y\in(0,\infty),\ x\neq y, \\ x, & \text{if } x=y,\ x\in(0,\infty) \end{cases}$$

if f fulfils the inequality

$$\text{(1)} \qquad f\left(\frac{x-y}{\ln x-\ln y}\right) \leqslant \frac{f(x)-f(y)}{\ln f(x)-\ln f(y)}, \quad x,y\in(0,\infty),\ x\neq y.$$

It is known that

(i) every continuous f fulfilling (1) is quasi-convex;
(ii) every decreasing f fulfilling (1) is convex in the usual sense;
(iii) every measurable in the Lebesgue sense f fulfilling (1) is continuous.

The problems are:

(A) Does there exist an increasing function f fulfilling (1) which is not convex?
(B) Does there exist a non-measurable function f fulfilling (1)?

Z. Kominek

(compiled by Zoltán Boros)

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