ON THE EXISTENCE OF TWO SOLUTIONS OF FUNCTIONAL BOUNDARY VALUE PROBLEMS

SVATOSLAV STANĚK

Abstract. The functional differential equation $(g(x'(t)))' = (Fx)(t)$ is considered. Here g is an increasing homomorphism on $\mathbb{R}, g(0) = 0$ and $F : C^1(J) \to L_1(J)$ is a continuous operator satisfying a growth condition with respect to x. A class of nonlinear functional boundary conditions is considered and sufficient conditions for the existence at least one positive and one negative solutions of the boundary value problems are given. Results are proved by the homotopy theory, the Leray–Schauder degree and the Borsuk theorem.

1. Introduction

Let $J = [a, b] \subset \mathbb{R}$ be a compact interval. In this paper $L_1(J)$,$\mathbf{X}$, $\mathbf{Y}$ and $AC(J)$ denote the following Banach spaces:

$$L_1(J) = \{x : J \to \mathbb{R} \text{ measurable and } \int_a^b |x(t)|dt < \infty\} \text{ with norm}$$

$$\|x\|_{L_1} = \int_a^b |x(t)|dt;$$

$$\mathbf{X} = C^0(J) \text{ with norm } \|x\|_0 = \max\{|x(t)| : t \in J\};$$
$$\mathbf{Y} = C^1(J) \text{ with norm } \|x\|_1 = \|x\|_0| + \|x'\|_0;$$

AMS (1991) subject classification: Primary 34K10

Key words and phrases: Multiplicity, functional differential equation, functional boundary conditions, homotopy, Leray–Schauder degree, Borsuk theorem, p-Laplacian, Emden–Fowler equation.

$$AC(J) = \{x : J \to \mathbb{R} \text{ absolutely continuous}\} \text{with norm}$$
$$\|x\|_{AC} = \|x\|_0 + \|x'\|_{L_1}.$$

For each functional $\varphi : \mathbf{X} \to \mathbb{R}$, $Im(\varphi)$ denotes the range of φ.

By $\mathcal{A}$ we understand the set of all functionals $\varphi : \mathbf{X} \to \mathbb{R}$ that are
(i) continuous,
(ii) $\varphi(x) = \varphi(|x|)$ for $x \in \mathbf{X}$,
(iii) $x, y \in \mathbf{X}$, $|x(t)| < |y(t)|$ for $t \in J \Rightarrow \varphi(x) < \varphi(y)$,
(iv) $\lim_{u \in \mathbb{R}, u \to \infty} \varphi(u) = \infty$;[1)]
and set $\mathcal{A}_0 = \{\varphi : \varphi \in \mathcal{A}, \varphi(0) = 0\}$.

REMARK 1. *The set $\mathcal{A}$ was introduced in [8] the first time.*

EXAMPLE 1. Let $p : [0, \infty) \to \mathbb{R}$ be continuous increasing and $\lim_{u \to \infty} p(u) = \infty$. Set $\varphi(x) = \int_a^b p(|x(t)|)dt$ for $x \in \mathbf{X}$ (see [1]). Then $\varphi \in \mathcal{A}$. Next functionals belonging to the set $\mathcal{A}$ are given below:

$$\max\{|x(t)| : t \in J_1\}, \quad \min\{|x(t)| : t \in J_1\}, \quad \sum_{i=1}^{n} a_i |x(t_i)|,$$

where $J_1 \subset J$ is a compact interval, $a \le t_1 < t_2 < \ldots < t_n \le b$ and $a_i \in (0, \infty)$ (see [8]).

Let $\mathcal{B}$ be the set of all functionals $\varphi : \mathbf{X} \to \mathbb{R}$ that are
(j) continuous, $\varphi(0) = 0$, and
(jj) $x, y \in \mathbf{X}$, $x(t) < y(t)$ for $t \in J \Rightarrow \varphi(x) < \varphi(y)$.

EXAMPLE 2. Let $J_1 \subset J$ be a compact interval, $a \le a_1 < b_1 \le b$ and $n \in \mathbb{N}$. Then the functionals

$$\max\{x(t) : t \in J_1\}, \quad \min\{x(t) : t \in J_1\}, \quad \int_{a_1}^{b_1} x^{2n+1}(t)dt,$$

belong to $\mathcal{B}$ (see [8]).

Consider the functional differential equation

$$(1) \qquad (g(x'(t)))' = (Fx)(t),$$

where g and F satisfy the following assumptions:

---

1) We identificate the subspace of $\mathbf{X}$ of constant functions with $\mathbb{R}$.

(H_1) $g : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with inverse g^{-1} and such that $g(0) = 0$;

(H_2) $F : \mathbf{Y} \to L_1(J)$ is a continuous operator.

Together with (1) we concern in the boundary conditions

$$\omega(x) = A, \tag{2}$$

$$\gamma(x') = 0, \tag{3}$$

where $\omega \in \mathcal{A}, \gamma \in \mathcal{B}$ and $A \in \mathbb{R}$.

A function $x \in \mathbf{Y}$ is said to be a *solution of boundary value problem* (BVP for short) (1)–(3) if $g(x') \in AC(J)$, x satisfies boundary conditions (2), (3) and (1) is satisfied for a.e. $t \in J$.

REMARK 2. *The special case of g in (1) is the p-Laplacian $g_p : \mathbb{R} \to \mathbb{R}$ defined by*

$$g_p(u) = |u|^{p-2}u$$

for $p > 1$ (see e.g. [3], [5] and references therein).

The special case of (1) (with $g(u) \equiv u$) is the functional differential equation

$$x''(t) = (Fx)(t) \tag{4}$$

Note that multiplicity results for (4) with nonlinear functional boundary conditions were given by Brykalov ([1], [2]) and the author ([7]–[9]). In [1], [2] results are proved under the assumption that F is bounded and in [7]–[9] under the assumption that F satisfies a growth condition of the type

$$|(Fx)(t)| \le f(|x'(t)|)$$

for a.e. $t \in J$ and each $x \in \mathbf{Y}$, where $f : [0, \infty) \to (0, \infty)$ is nondecreasing and $\int_0^\infty \frac{t\,dt}{f(t)} = \infty$ ([9]) resp. $\int_0^\infty \frac{dt}{f(t)} = \infty$ ([7], [8]). In [7]–[9] the results were proved by the Bihari lemma, the theory of homotopy, the Leray–Schauder degree and the Borsuk theorem.

In the present paper we assume that F satisfies a growth condition depending only on x and give sufficient conditions for any solution of BVP (1)–(3) does not vanish on J and there exist at least one positive and at least one negative solutions. In contradiction to [7]–[9] we can't now apply the Bihari lemma. Results are proved by the theory of homotopy, the Leray–Schauder degree and the Borsuk theorem (see, e.g., [4] and [10]).

2. Lemmas

LEMMA 1. [8]. *Let* $\varphi \in \mathcal{A}, A \in Im(\varphi)$. *Then*
(a) $\varphi(0) \le \varphi(x)$ *for each* $x \in \mathbf{X}$,
(b) $\varphi(r) = A$ *for a unique nonnegative constant* r,
(c) $x, y \in \mathbf{X}$, $|x(t)| \le |y(t)|$ *for* $t \in J \Rightarrow \varphi(x) \le \varphi(y)$.

LEMMA 2. [8]. *Let* $\varphi \in \mathcal{A}$ *and* $\varphi(x) = \varphi(y)$ *for some* $x, y \in \mathbf{X}$. *Then there exists a* $\tau \in J$ *such that*

$$|x(\tau)| = |y(\tau)|.$$

LEMMA 3. [7]. *Let* $\varphi \in \mathcal{A}$ *and* $\varphi(x) \le \varphi(y)$ *for some* $x, y \in \mathbf{X}$. *Then there exists a* $\xi \in J$ *such that*

$$|x(\xi)| \le |y(\xi)|.$$

LEMMA 4. [8]. *Let* $\varphi \in \mathcal{B}, A \in Im(\varphi)$. *Then* $\varphi(d) = A$ *for a unique* $d \in \mathbb{R}$.

LEMMA 5. [8]. *Let* $\varphi \in \mathcal{B}$ *and* $c \in [0,1]$. *Let the equality*

$$\varphi(x) - c\varphi(-x) = 0$$

be satisfied for an $x \in \mathbf{X}$. *Then exists a* $\tau \in J$ *such that*

$$x(\tau) = 0.$$

LEMMA 6. *Let assumption* (H_1) *be satisfied and let* $\{y_n\} \subset \mathbf{Y}$ *be a bounded sequence such that*

$$|g(y_n'(t_1)) - g(y_n'(t_2))| \le \left| \int_{t_1}^{t_2} \psi(t)dt \right| \tag{5}$$

for each $t_1, t_2 \in J$ *and* $n \in \mathbb{N}$, *where* $\psi \in L_1(J)$ *is a nonnegative function. Then* $\{y_n\}$ *is compact (in* $\mathbf{Y}$*).*

PROOF. By assumption, $\{y_n\}$ is bounded in $\mathbf{Y}$. To prove our lemma it is sufficient to show that there exists a convergent subsequence of $\{y_n'\}$ in $\mathbf{X}$. We see that $\{g(y_n')\}$ is bounded in $\mathbf{X}$, and consequently (5) and the Arzelà–Ascoli theorem imply the existence of subsequence $\{g(y_{k_n}')\}$ converging in $\mathbf{X}$. Set

$z_{k_n} = g(y'_{k_n})$. Then $y'_{k_n} = g^{-1}(z_{k_n})$ and since (cf. (H_1)) g^{-1} is increasing on $\mathbb{R}$ and $\{z_{k_n}\}$ is convergent in $\mathbf{X}$, $\{y'_{k_n}\}$ is convergent in $\mathbf{X}$ as well.

For each g satisfying (H_1), define the function $G^{-1} : [0,\infty) \to [0,\infty)$ by the formula

$$G^{-1}(v) = \max\{g^{-1}(v), -g^{-1}(-v)\}.$$

Then G is continuous and increasing on $[0,\infty)$.

REMARK 3. *If g satisfying (H_1) is an odd function, then $G^{-1}(v) = g^{-1}(v)$ for $v \in [0,\infty)$. In particular $G^{-1}(v) = v$ and $G^{-1}(v) = \sqrt[p-1]{v}$ on $[0,\infty)$ for $g(u) = u$ and $g(u) = |u|^{p-2}u$ $(p > 1)$ on $\mathbb{R}$, respectively.*

We assume throughout this paper that the operator F and the function g satisfy assumptions (H_1)–(H_4), where

(H_3) There exist a continuous nondecreasing function $f : [0,\infty) \to [0,\infty)$ and nonnegative functions $\alpha, \beta \in L_1(J)$ such that

$$|(Fx)(t)| \le \alpha(t) f(\|x\|_0) + \beta(t) \qquad \text{for a.e. } t \in J \text{ and each } x \in \mathbf{Y}; \tag{6}$$

(H_4) The function $S : [0,\infty) \to \mathbb{R}$,

$$S(v) = v - h(v) \tag{7}$$

is increasing on $[0,\infty)$ and

$$\lim_{v\to\infty} S(v) = \infty, \tag{8}$$

where $h : [0,\infty) \to \mathbb{R}$,

$$h(v) = \int_a^b G^{-1}(f(v)\Lambda(t) + \Psi(t))dt \tag{9}$$

and

$$\Lambda(t) = \max\left(\int_a^t \alpha(s)ds, \int_t^b \alpha(s)ds\right),$$

$$\Psi(t) = \max\left(\int_a^t \beta(s)ds, \int_t^b \beta(s)ds\right).$$

The fact that F and g satisfy assumptions (H_1)–(H_4) will be stated explicitly only in assumptions of our theorems.

REMARK 4. *Clearly (cf. (H_3)),* $S \in C^0([0,\infty)), S(0) \le 0, S^{-1}(0) \ge 0, h(v) = \|G^{-1}(b(v)\Lambda(t) + \Psi(t))\|_{L_1}$, *and there exists the inverse function* $S^{-1} : [S(0), \infty) \to [0, \infty)$ *to* S. *Moreover,* $\Lambda(t) \le \|\alpha\|_{L_1}, \Psi(t) \le \|\beta\|_{L_1}$ *for* $t \in J$. *If* $g(u) \equiv u$, *then* $S(v) = v - f(v)\|\Lambda\|_{L_1} - \|\Psi\|_{L_1}$. *For* $g(u) = |u|^{p-2}u$ ($p > 1$) *we have (cf. Remark 3)* $S(v) = v - \int_a^b \sqrt[p-1]{f(v)\Lambda(t) + \Psi(t)}dt$.

LEMMA 7. *Let* $u(t)$ *be a solution of (1) such that* $u(\xi) = 0, u'(\tau) = 0$ *for some* $\xi, \tau \in J$. *Then*

$$\|u\|_0 \le S^{-1}(0). \tag{10}$$

PROOF. Integrating the equality

$$(g(u'(t)))' = (Fu)(t) \qquad \text{for a.e.} \quad t \in J$$

from τ to t we obtain $g(u'(t)) = \int_\tau^t (Fu)(s)ds$ since $g(u'(\tau)) = 0$, and therefore (for $t \in J$)

$$u'(t) = g^{-1}\left(\int_\tau^t (Fu)(s)ds\right),$$

$$u(t) = \int_\xi^t g^{-1}\left(\int_\tau^s (Fu)(v)dv\right)ds.$$

Then

$$|u(t)| \le \left|\int_\xi^t \left|g^{-1}\left(\int_\tau^s (Fu)(v)dv\right)\right|ds\right| \tag{11}$$

for $t \in J$. Since (cf. (H_3))

$$\left|\int_\tau^s (Fu)(v)dv\right| \le \left|\int_\tau^s |(Fu)(v)|dv\right| \le f(\|u\|_0)\left|\int_\tau^s \alpha(v)dv\right| + \left|\int_\tau^s \beta(v)dv\right|$$

we see that

$$\left|g^{-1}\left(\int_\tau^s (Fu)(v)dv\right)\right| \le G^{-1}\left(f(\|u\|_0)\left|\int_\tau^s \alpha(v)dv\right| + \left|\int_\tau^s \beta(v)dv\right|\right)$$

for $s \in J$. Hence

$$\begin{aligned}
&\left|\int_{\xi}^{t} g^{-1}\left(\int_{\tau}^{s}(Fu)(v)dv\right) ds\right| \\
(12)\qquad &\leq \left|\int_{\xi}^{t} G^{-1}\left(f(\|u\|_0)\left|\int_{\tau}^{s}\alpha(v)dv\right| + \left|\int_{\tau}^{s}\beta(v)dv\right|\right) ds\right| \\
&\leq \int_{a}^{b} G^{-1}\left(f(\|u\|_0)\Lambda(t) + \Psi(t)\right) dt
\end{aligned}$$

for $t \in J$, and consequently (cf. (9) and (11)) $\|u\|_0 \leq h(\|u\|_0)$. This gives $S(\|u\|_0) \leq 0$ which implies (10).

COROLLARY 1. *Let $u(t)$ be a solution of (1), $u'(\tau) = 0$ for a $\tau \in J$ and $\|u\|_0 > S^{-1}(0)$. Then*

$$|u(t)| > 0 \qquad \text{for} \quad t \in J.$$

PROOF. If not, there exists a $\xi \in J$ such that $u(\xi) = 0$. Then, by Lemma 7, $\|u\|_0 \leq S^{-1}(0)$, a contradiction.

Consider the functional differential equation

$$(13_\lambda)\qquad (g(x'(t)))' = \lambda(Fx)(t), \quad \lambda \in [0,1]$$

depending on the parametr λ.

LEMMA 8. *Let $m > 0$ a constant and $u(t)$ be a solution of (13_λ) for a $\lambda \in [0,1]$ such that $|u(\xi)| = m$, $u'(\tau) = 0$ for some $\xi, \tau \in J$. Then*

$$(14)\qquad \|u\|_0 \leq S^{-1}(m)$$

and

$$(15)\qquad \|u'\|_0 \leq G^{-1}(f(S^{-1}(m))\|\alpha\|_{L_1} + \|\beta\|_{L_1}).$$

PROOF. Integrating the equality (for a.e. $t \in J$)

$$(g(u'(t)))' = \lambda(Fu)(t)$$

from τ to t we obtain

$$g(u'(t)) = \lambda \int_{\tau}^{t} (Fu)(s)ds$$

and therefore

$$u'(t) = g^{-1}\left(\lambda \int_{\tau}^{t} (Fu)(s)ds\right) \tag{16}$$

$$u(t) = u(\xi) + \int_{\xi}^{t} g^{-1}\left(\lambda \int_{\tau}^{s} (Fu)(v)dv\right) ds$$

for $t \in J$. Using the inequalities

$$|u(t)| \leq m + \left|\int_{\xi}^{t} g^{-1}\left(\lambda \int_{\tau}^{s} (Fu)(v)dv\right) ds\right|,$$

$$\left|\lambda \int_{\tau}^{t} (Fu)(s)ds\right| \leq f(\|u\|_0)\Lambda(t) + \Psi(t)$$

we see that (cf. proof of Lemma 7) $|u(t)| \leq m + h(\|u\|_0)$ for $t \in J$. Consequently, $S(\|u\|_0) \leq m$ and inequality (14) holds. Then

$$\left|\lambda \int_{\tau}^{t} (Fu)(s)ds\right| \leq f(\|u\|_0)\left|\int_{\tau}^{t} \alpha(s)ds\right| + \left|\int_{\tau}^{t} \beta(s)ds\right| \leq f(S^{-1}(m))\|\alpha\|_{L_1} + \|\beta\|_{L_1}$$

which and (16) together imply (15).

For each $x \in \mathbf{X}$ define $x_+, x_- \in \mathbf{X}$ by the formulas

$$x_+(t) = \begin{cases} x(t) & \text{for } x(t) \geq 0 \\ 0 & \text{for } x(t) < 0, \end{cases} \qquad x_-(t) = \begin{cases} 0 & \text{for } x(t) \geq 0 \\ -x(t) & \text{for } x(t) < 0. \end{cases}$$

Then $x_+(t) \geq 0, x_-(t) \geq 0$ for $t \in J$ and $x = x_+ - x_-$.

LEMMA 9. *Let* $\omega \in \mathcal{A}_0, \gamma \in \mathcal{B}, r, k, l, l_1$ *be positive constants,* $k > r$ *and*

$$\Omega = \{(x, \alpha, \beta) : (x, \alpha, \beta) \in \mathbf{Y} \times \mathbb{R}^2, \|x\|_0 < k, \|x'\|_0 < l, |\alpha| < k, |\beta| < l_1\}.$$

Let

$$\Gamma_i : \overline{\Omega} \to \mathbf{Y} \times \mathbb{R}^2 \quad (i = 1, 2),$$

$$\Gamma_1(x, \alpha, \beta) = (\alpha + g^{-1}(\beta)(t - a), \alpha + \omega(x_+) - \omega(\mu), \beta + \gamma(x')),$$

$$\Gamma_2(x, \alpha, \beta) = (\alpha + g^{-1}(\beta)(t - a), \alpha + \omega(x_-) - \omega(\mu), \beta + \gamma(x')).$$

Then

$$D(I - \Gamma_i, \Omega, 0) \neq 0 \qquad \text{for} \quad i = 1, 2. \tag{17}$$

Here "D" denotes the Leray-Schauder degree and I is the identity operator on $\mathbf{Y}$.

PROOF. First of all, we see that Ω is an open bounded and symmetric subset of the Banach space $\mathbf{Y} \times \mathbb{R}^2$ with usual norm and $\omega(\tau) > 0$ since $\omega \in \mathcal{A}_0$ and $\tau > 0$. Define (for $i = 1, 2$)

$$H_i : [0, 1] \times \overline{\Omega} \to \mathbf{Y} \times \mathbb{R}^2$$

by

$$H_1(\lambda, x, \alpha, \beta) = \big(\alpha + (g^{-1}(\beta) - (1 - \lambda)g^{-1}(-\beta))(t - a),$$

$$\alpha + \omega(x_+) - \omega((1 - \lambda)x_-) - \lambda\omega(\mu), \beta + \gamma(x') - (1 - \lambda)\gamma(-x')\big).$$

$$H_2(\lambda, x, \alpha, \beta) = \big(\alpha + (g^{-1}(\beta) - (1 - \lambda)g^{-1}(-\beta))(t - a),$$

$$\alpha + \omega(x_-) - \omega((1 - \lambda)x_+) - \lambda\omega(\mu), \beta + \gamma(x') - (1 - \lambda)\gamma(-x')\big).$$

Clearly,

$$H_i(1, x, \alpha, \beta) = \Gamma_i(x, \alpha, \beta)$$

for $(x, \alpha, \beta) \in \overline{\Omega}$ and $i = 1, 2$. Hence, to prove (17) it is sufficient to verify, by the homotopy theory and the Borsuk theorem, that (for $i = 1, 2$)

(a) $H_i(0, \cdot, \cdot, \cdot)$ is an odd operator, that is, $H_i(0, -x, -\alpha, -\beta) = -H_i(0, x, \alpha, \beta)$ for $(x, \alpha, \beta) \in \overline{\Omega}$,

(b) H_i a compact operator, and

(c) $H_i(\lambda, x, \alpha, \beta) \neq (x, \alpha, \beta)$ for $(\lambda, x, \alpha, \beta) \in [0, 1] \times \partial\Omega$.

We prove, for instance, (17) for $i = 1$. The proof of (17) with $i = 2$ is similar. Fix $(x, \alpha, \beta) \in \overline{\Omega}$. Then

$$H_1(0, -x, -\alpha, -\beta) = \big(-\alpha + (g^{-1}(-\beta) - g^{-1}(\beta))(t - a), -\alpha + \omega(x_-) - \omega(x_+),$$

$$-\beta + \gamma(-x') - \gamma(x')\big) = -\big(\alpha + (g^{-1}(\beta) - g^{-1}(-\beta))(t - a),$$

$$\alpha + \omega(x_+) - \omega(x_-), \beta + \gamma(x') - \gamma(-x')\big) = -H_1(0, x, \alpha, \beta)$$

since $(-u)_+ = u_-$ and $(-u)_- = u_+$ for any $u \in \mathbf{X}$. Hence $H_1(0,\cdot,\cdot,\cdot)$ satisfies (a) (with $i = 1$).

We proceed to show that H_1 is a compact operator. Let $\{(\lambda_n, x_n, \alpha_n, \beta_n)\} \subset [0,1] \times \overline{\Omega}$ be a sequence, Then $0 \le \lambda_n \le 1, \|x_n\|_0 \le k, \|x_n'\|_0 \le l, |\alpha_n| \le k, |\beta_n| \le l_1$ for each $n \in \mathbb{N}$. Consequently, $\{\omega((x_n)_+)\}, \{\omega((1-\lambda_n)(x_n)_-)\}$, $\{\gamma(x_n')\}$ and $\{\gamma(-x_n')\}$ are bounded sequences (in $\mathbb{R}$) and, by the Bolzano–Weierstrass theorem, without restiction of generality, we can assume that $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}, \{\omega((x_n)_+)\}, \{\omega((1-\lambda_n)(x_n)_-)\}, \{\gamma(x_n')\}$ and $\{\gamma(-x_n')\}$ are convergent. Hence there exists limits

$$\lim_{n\to\infty} (\alpha_n + (g^{-1}(\beta_n) - (1-\lambda_n)g^{-1}(-\beta_n))(t-a))$$

in $\mathbf{Y}$ and

$$\lim_{n\to\infty} (\alpha_n + (\omega((x_n)_+) - \omega((1-\lambda_n)(x_n)_-) - \lambda_n\omega(\tau)),$$

$$\lim_{n\to\infty} (\beta_n + \gamma(x_n') - (1-\lambda_n)\gamma(-x_n'))$$

in $\mathbb{R}$, and consequently there exists $\lim_{n\to\infty} H_1(\lambda_n, x_n, \alpha_n, \beta_n)$ in $\mathbf{Y} \times \mathbb{R}^2$. Moreover, from the continuity od ω and γ we deduce that H_1 is a continuous operator. It follows that H_1 is a compact operator.

It remains to prove (c) (for $i = 1$). Assume, on the contrary, that

$$H_1(\lambda_0, x_0, \alpha_0, \beta_0) = (x_0, \alpha_0, \beta_0)$$

for a $(\lambda_0, x_0, \alpha_0, \beta_0) \in [0,1] \times \partial\Omega$. Then

(18) $$x_0(t) = \alpha_0 + \left(g^{-1}(\beta_0) - (1-\lambda_0)g^{-1}(-\beta_0)\right)(t-a) \qquad \text{for} \quad t \in J,$$

(19) $$\omega((x_0)_+) - \omega((1-\lambda_0)(x_0)_-) = \lambda_0\omega(\tau)$$

and

(20) $$\gamma(x_0') - (1-\lambda_0)\gamma(-x_0') = 0.$$

From (20) and Lemma 5 (with $\varphi = \gamma, c = 1 - \lambda_0, x = x_0'$) it follows that $x_0'(\tau) = 0$ for a $\tau \in J$. Then (cf. (18))

$$g^{-1}(\beta_0) - (1-\lambda_0)g^{-1}(-\beta_0) = 0$$

which is satisfied if and only if $\beta_0 = 0$ since $ug^{-1}(u) > 0$ for all $u \in \mathbb{R}\setminus\{0\}$; hence

$$x_0(t) = \alpha_0 \qquad \text{for} \quad t \in J.$$

By our assumption $(\alpha_0, \alpha_0, 0) = (x_0, \alpha_0, \beta_0) \in \partial\Omega$ and therefore $|\alpha_0| = k$ $(> r)$. Assume $\alpha_0 = k$. By (19), $\omega(k) = \lambda_0\omega(\mu)$, which contradicts $\lambda_0\omega(\mu) \leq \omega(\mu) < \omega(k)$. If $\alpha_0 = -k$, then (cf. (19))

$$-\omega((1-\lambda_0)k) = \lambda_0\omega(\mu).$$

Since $\omega((1-\lambda_0)k) \geq 0$ and $\omega(\mu) > 0$, the last equality is satisfied if and only if $\lambda_0 = 0$ and $\omega(k) = 0$, which is impossible. This completes the proof.

3. Existence results

Existence results for BVP (1)–(3) are given in two theorem. In Theorem 1 we assume that $\omega \in \mathcal{A}_0$. By Theorem 1, existence results for each $\omega \in \mathcal{A}$ are proved in Theorem 2.

THEOREM 1. *Let assumptions (H_1)–(H_4) be satisfied and let $\omega \in \mathcal{A}_0, \gamma \in \mathcal{B}$. Then for each $A \in \mathbb{R}$ such that*

$$A > \omega(S^{-1}(0))$$

any solution of BVP (1)–(3) does not vanish on J, and there exist at least one negative and at least one positive solutions.

PROOF. Fix $A > \omega(S^{-1}(0))$. Then (cf. Remark 4) $A > 0$. By Lemma 1, there exists a unique positive constant μ such that $\omega(\mu) = A$. Hence $\omega(\mu) > \omega(S^{-1}(0))$, and consequently

$$r > S^{-1}(0). \tag{21}$$

Let $u(t)$ be a solution of BVP (1)–(3). Then $\omega(u) = A(= \omega(\mu)), \gamma(u') = 0$, and so (cf. Lemma 2 and Lemma 5 with $c = 0$) $|u(\xi)| = r, u'(\tau) = 0$ for some $\xi, \tau \in J$. Thus (cf. (21)) $\|u\|_0 > S^{-1}(0)$, which yields $|u(t)| > 0$ for $t \in J$ by Corollary 1. We have proved that any solution of BVP (1)–(3) (provided that one exists) does not vanish on J.

We proceed to show that then exists at least one positive solution of BVP (1)–(3). Set

$$k = S^{-1}(r) + r, \qquad l = G^{-1}(l_1),$$

$$\Omega = \{(x, \alpha, \beta) : (x, \alpha, \beta) \in \mathbf{Y} \times \mathbb{R}^2, \|x\|_0 < k, \|x'\|_0 < l, |\alpha| < k, |\beta| < l_1\},$$

where

$$l_1 = f(S^{-1}(r))\|\alpha\|_{L_1} + \|\beta\|_{L_1} + 1.$$

Let the operator $S_1 : [0,1] \times \overline{\Omega} \to \mathbf{Y} \times \mathbb{R}^2$ be given by the formula

$$S_1(\lambda, x, \alpha, \beta)$$
$$= \left(\alpha + \int_a^b g^{-1} \left(\beta + \lambda \int_a^s (Fx)(v)dv \right) ds, \alpha + \omega(x_+) - \omega(\mu), \beta + \gamma(x') \right).$$

Obviously, $S_1(0, x, \alpha, \beta) = \Gamma_1(x, \alpha, \beta)$ for $(x, \alpha, \beta) \in \overline{\Omega}$, where Γ_1 is defined in Lemma 9. Consider the operator equation

$$(22_\lambda) \qquad S_1(\lambda, x, \alpha, \beta) = (x, \alpha, \beta), \qquad \lambda \in [0,1]$$

depending on the parametr λ. We next prove that (22_1) has a solution. As $D(I - \Gamma_1, \Omega, 0) \neq 0$ by Lemma 9, it is sufficient to check that (cf [4], [10])
(a) S_1 is a compact operator, and
(b) $S_1(\lambda, x, \alpha, \beta) \neq (x, \alpha, \beta)$ for each $(\lambda, x, \alpha, \beta) \in [0,1] \times \partial\Omega$.

From the continuity of g^{-1}, F, ω and γ we deduce that S_1 is a continuous operator. Let $\{(\lambda_n, x_n, \alpha_n, \beta_n)\} \subset [0,1] \times \overline{\Omega}$ be a sequence and set

$$(y_n, a_n, b_n) = S_1(\lambda_n, x_n, \alpha_n, \beta_n)$$

for $n \in \mathbb{N}$. Then

$$(23) \qquad y_n(t) = \alpha_n + \int_a^t g^{-1} \left(\beta_n + \lambda_n \int_a^s (Fx_n)(v)dv \right) ds,$$

$$(24) \qquad a_n = \alpha_n + \omega((x_n)_+) - \omega(\mu)$$

and

$$(25) \qquad b_n = \beta_n + \gamma(x_n')$$

for $n \in \mathbb{N}$. We will prove that the sequence $\{y_n\}$ is compact in $\mathbf{Y}$. Since $0 \leq \lambda_n \leq 1, \|x_n\|_0 \leq k, \|x_n'\|_0 \leq l, |\alpha_n| \leq k$ and $|\beta_n| \leq l_1$, we conclude that (cf. (23) and the definition of G^{-1})

$$\begin{aligned} \|y\|_0 &\leq k + (b-a)G^{-1}\left(l_1 + f(\|x_n\|_0)\|\alpha\|_{L_1} + \|\beta\|_{L_1}\right) \\ &\leq k + (b-a)G^{-1}\left(l_1 + f(k)\|\alpha\|_{L_1} + \|\beta\|_{L_1}\right), \end{aligned}$$

$$\|y_n'\|_0 \leq G^{-1}\left(l_1 + f(k)\alpha\|_{L_1} + \|\beta\|_{L_1}\right)$$

and, moreover,

$$|g(y_n'(t_1)) - g(y_n'(t_2))| \le \lambda_n \left| \int_{t_1}^{t_2} (Fx_n)(t)dt \right| \le \left| \int_{t_1}^{t_2} \psi(t)dt \right|$$

for each $n \in \mathbb{N}, t_1, t_2 \in J$, where $\psi(t) = f(k)\alpha(t) + \beta(t)$ $(\in L_1(J))$. By Lemma 6, $\{y_n\}$ is compact in $\mathbf{Y}$. From this and from the inequalities (cf. (24) and (25))

$$|a_n| \le k + \omega(k) + \omega(r), \qquad |b_n| \le l_1 + \max\{\gamma(l), -\gamma(-l)\}$$

for $n \in \mathbb{N}$, we deduce that $\{(y_n, a_n, b_n)\}$ is compact in $\mathbf{Y} \times \mathbb{R}^2$. Hence S_1 is a compact operator.

To prove property (b) of S_1 we assume, on the contrary, that

$$S_1(\lambda_0, x_0, \alpha_0, \beta_0) = (x_0, \alpha_0, \beta_0) \tag{26}$$

for a $(\lambda_0, x_0, \alpha_0, \beta_0) \in [0,1] \times \partial\Omega$. Then

$$x_0(t) = \alpha_0 + \int_a^t g^{-1}\left(\beta_0 + \lambda_0 \int_a^s (Fx_0)(v)dv\right) ds, \qquad t \in J, \tag{27}$$

$$\omega((x_0)_+) = \omega(\mu) \tag{28}$$

and

$$\gamma(x_0') = 0. \tag{29}$$

By (28) and Lemma 2,

$$(x_0)_+(\xi) = r \tag{30}$$

for a $\xi \in J$, and

$$x_0'(\tau) = 0 \tag{31}$$

for a $\tau \in J$ by (29) and Lemma 5 (with $c = 0$). From (27) we see that

$$(g(x_0'(t)))' = \lambda_0 (Fx_0)(t) \qquad \text{for a.e.} \quad t \in J$$

and then Lemma 8 (with $m = r$ and $\lambda = \lambda_0$) implies

$$\|x_0\|_0 \le S^{-1}(r) < k, \tag{32}$$

$$\|x_0'\|_0 \le G^{-1}\left(f(S^{-1}(r))\|\alpha\|_{L_1} + \|\beta\|_{L_1}\right) < G^{-1}(l_1). \tag{33}$$

Since (cf. (27))

$$\alpha_0 = x_0(a), \qquad \beta_0 = g(x_0'(a)),$$

we have (cf. (27), (31) and (32))

$$|\alpha_0| < k, \tag{34}$$

$$\begin{aligned}|\beta_0| = |g(x_0'(a))| = \left|\int_a^T (Fx_0)(t)dt\right| &\le \int_a^b |(Fx_0)(t)|dt \\ &\le f(\|x_0\|_0)\|\alpha\|_{L_1} + \|\beta\|_{L_1} \le f(k)\|\alpha\|_{L_1} + \|\beta\|_{L_1} < l_1.\end{aligned} \tag{35}$$

Hence $(x_0, \alpha_0, \beta_0) \notin \partial\Omega$ which follows from (32)–(35), a contradiction.

We have verified that (22_1) has a solution (in Ω), say (u, α_0, β_0). Then u is a solution of (1) satisfying boundary conditions

$$\omega(u_+) = A \quad (= \omega(r)), \qquad \gamma(u') = 0.$$

Since $|u_+(\xi)| = r$ for a $\xi \in J$ by Lemma 2, we see that $u_+(\xi) = r$, and consequently $u(t) > 0$ on J by Corollary 1. Hence $\omega(u_+) = \omega(u)$ and u is a positive solution of BVP (1)–(3).

If the operator $S_2 : [0,1] \times \partial\Omega \to \mathbf{Y} \times \mathbb{R}^2$,

$$S_2(\lambda, x, \alpha, \beta)$$
$$= \left(\alpha + g^{-1}\int_a^t \left(\beta + \lambda\int_a^s (Fx)(v)dv\right) ds, \alpha + \omega(x_-) - \omega(r), \beta + \gamma(x')\right)$$

is considered instead of S_1, one can prove, in the same manner as above, the existence at least one negative solution of BVP (1)–(3).

THEOREM 2. *Let assumptions (H_1)–(H_4) be satisfied and let $\omega \in \mathcal{A}, \gamma \in \mathcal{B}$. Then for each $A \in \mathbb{R}$ such that*

$$A > \omega(S^{-1}(0))$$

any solution of BVP (1)–(3) does not vanish on J, and there exist at least two solutions, one negative and one positive.

PROOF. Fix $A > \omega(S^{-1}(0))$. Set $\overline{\omega}(x) = \omega(x) - \omega(0)$ for $x \in \mathbf{X}$. Then $\overline{\omega} \in \mathcal{A}_0$. Consider equation (1) subject to the boundary conditions

$$\overline{\omega}(x) = A - \omega(0), \qquad \gamma(x') = 0. \tag{36}$$

Of course, $A - \omega(0) > \overline{\omega}(S^{-1}(0))$ and applying Theorem 1 to BVP (1), (36), any solution of this problem does not vanish on J and there exist at least two solutions, one positive and one negative. Since $u(t)$ is a solution of BVP (1)–(3) if and only if that is a solution of BVP (1), (36), our theorem is proved.

EXAMPLE 3. Let $p > 1, \lambda > 0$ and $K > 0$ be constants such that

$$\lambda \le p - 1, \qquad \sqrt[p-1]{K}(b - a) < 1. \tag{37}$$

Consider the functional differential equation

$$(|x'(t)|^{p-2}x'(t))' = (F_1x)(t)|x(t)|^{\lambda} \operatorname{sign} x(t), \tag{38}$$

where $F_1 : \mathbf{Y} \to L_1(J)$ is continuous and $|(F_1x)(t)| \le K$ for a.e. $t \in J$ and each $x \in \mathbf{Y}$. We see (cf. Remark 2) that the left side of (38) is equal to $(g_p(x'(t)))'$, where g_p is the p–Laplacian, and the right side of (38) has the Emden–Fowler form (see, e.g., [6] and references therein). Set

$$f(v) = \begin{cases} K & \text{for } v \in [0, 1] \\ Kv^{\lambda} & \text{for } v \in (1, \infty). \end{cases} \tag{39}$$

Then f is a nondecreasing on $[0, \infty)$ and since

$$|(F_1(x)(t)|x(t)|^{\lambda}\operatorname{sign} x(t)| \le K \max\{1, \|x\|_0^{\lambda}\},$$

we have

$$|(Fax)(t)|x(t)|^{\lambda}\operatorname{sign} x(t)| \le f(\|x\|_0)$$

for a.e. $t \in J$ and each $x \in \mathbf{Y}$. Consequently (cf. Remark 4),

$$S(v) = v - \sqrt[p-1]{f(v)}(b - a), \qquad v \in [0, \infty).$$

Hence (cf. (39))

$$S(v) = \begin{cases} v - \sqrt[p-1]{K}(b - a) & \text{for } v \in [0, 1] \\ v - \sqrt[p-1]{K}(b - a)v^{\frac{\lambda}{p-1}} & \text{for } v \in (1, \infty). \end{cases}$$

Since (cf. (37)) $(v - \sqrt[p-1]{K}(b-a)v^{\frac{\lambda}{p-1}})' = 1 - \frac{\lambda}{p-1}\sqrt[p-1]{K}(b-a)v^{\frac{\lambda}{p-1}-1} > 0$ for each $v \geq 1$, S is increasing on $[0,\infty)$. Moreover, $\lim_{v\to\infty} S(v) = \infty$ and $S(v_0) = 0$ if and only of $v_0 = \sqrt[p-1]{K}(b-a)$ $(= S^{-1}(0))$. So, equation (38) satisfies assumptions (H_1)–(H_4). Consider (38) subject to the boundary conditions

$$\|x\|_0 = A, \quad \min\{x'(t) : t \in J\} = 0, \tag{40}$$

$$\|x\|_{L_1} = A, \quad \max\{x'(t) : t \in J\} = 0 \tag{41}$$

and

$$\int_a^b \sqrt{1+(x(t))^2}dt = A, \qquad x(a) = x(b) \tag{42}$$

which are the special cases of (2), (3) with $\omega(x) = \|x\|_0, \gamma(x) = \min\{x(t) : t \in J\}$ for (40), $\omega(x) = \|x\|_{L_1}, \gamma(x) = \max\{x(t) : t \in J\}$ for (41) and $\omega(x) = \int_a^b \sqrt{1+(x(t))^2}dt, \gamma(x) = \int_a^b x(t)dt$ for (42). By Theorem 2, for all $A \in \mathbb{R}$ such that

$$A > \sqrt[p-1]{K}(b-a)(\text{resp. } A > \sqrt[p-1]{K}(b-a)^2; A > (b-a)\sqrt{1+\sqrt[p-1]{K^2}(b-a)^2})$$

any solution of BVP (38), (40) (resp. (38), (41); (38), (42)) does not vanish on J and there exist at least two solutions, one positive and one negative.

References

[1] S. A. Brykalov, *A second-order nonlinear problem with two-point and integral boundary conditions*, Proceedings of the Georgian Academy of Science. Math. **1**(1993), 273–279.

[2] S. A. Brykalov, *Solutions with given maximum and minimum*, Diff. Urav. **29**(1993), 938–942 (in Russian).

[3] C. De Coster, *Pairs of positive solutions for one-dimensional p-Laplacian*, Nonlin. Anal. **23**(1994), 669–681.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin–Heidelberg 1985.

[5] A. El Hachimi and J.-P. Gossez, *A note on a nonresonance condition for a quasilinear elliptic problem*, Nonlin. Anal. **22**(1994), 229–236.

[6] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomus Ordinary Differential Equations*, Nauka, Moscow 1990 (in Russian).

[7] S. Staněk, *Multiple solutions for some functional boundary value problem*, Nonlin. Anal. **32** (1998), 427–438.

[8] S. Staněk, *Multiplicity results for functional boundary value problems*, Nonlin. Anal. **30** (1997), 2617–2628.

[9] S. Staněk, *Multiplicity results for second order nonlinear problems with maximum and minimum*, Math. Nachr. **192** (1998), 225–237.

[10] E. Zeidler, *Vorlesungen über nichtlineare Funktionalanalysis I–Fixpunktsätze*, Teubner-Verlag, Leipzig 1976.

Department of Mathematical Analysis
Faculty of Science
Palacký University
Tomkova 40
779 00 Olomouc
Czech Republic