ON THE PERIODIC STRUCTURE OF THE ANTITRIANGULAR MAPS ON THE UNIT SQUARE

FRANCISCO BALIBREA GALLEGO AND ANTONIO LINERO BAS

To the memory of Professor György Targonski

Abstract. In this paper we study the periodic structure of the antitriangular maps on $I^2 = [0,1] \times [0,1]$ and obtain an ordering on the periods of these maps of a Šarkovskii type.

1. Introduction

The periodic structure of discrete dynamical systems (X, f), where X is a phase space of dimension one and f a continuous map on X, has been a subject of great interest in the last thirty years after the appearance in 1964 of the Šarkovskii's Theorem ([S]). When X is $I = [0,1]$, $\mathbb{R}$ or $\mathbb{S}^1$, a tree or a finite graph some results on periodic structure have been obtained in a similar line to those established for the case $X = I$. We call them Šarkovskii´s type results. When we say periodic structure results we mean that there exists some forcing relation among all the periodic points of the map f. Besides these relations can be interpreted in terms of the applications found in Dynamics of Populations.

When $X = I^2 = [0,1] \times [0,1]$ or $\mathbb{R}^2$, it is in general difficult to obtain results similar to those of the unidimensional case except for the case of

---

Received: January 26, 1999.

AMS (1991) subject classification: Primary 58F99.

This paper has been partially supported by D.G.I.C.Y.T. Grant PB95-1004 and the Grant PB/2/FS (Fundación Séneca, Comunidad Autónoma de la Región de Murcia).

triangular maps, that is, maps $F : I^2 \to I^2$ given by

$$F(x, y) = (f(x), g(x, y)), \tag{1}$$

where $f, g \in C(I, I)$ ([Kl]).

Motivated by the cooperative behaviour of some groups of animals in the african savannah or some models of duopoly games (see [D]) we have considered maps $F : I^2 \to I^2$ given by

$$F(x, y) = (g(y), f(x)), \tag{2}$$

where $f, g \in C(I, I)$. We have called them *antitriangular maps* since the effect in the coordinates of the map (2) is the opposite to that on the maps (1).

In this paper we have developed the periodic structure of antitriangular maps and found that we have some type of Šarkovskii´s ordering to describe the set of periodic points. In fact, under this point of view, the antitriangular maps behave like some particular mixture of the behaviours of the maps $f \circ g$ and $g \circ f$.

2. Preliminaries: definitions and notation

In a general setting, let X be a metric or compact metric space and $\varphi : X \to X$ a continuous map. We call *n-iterate of* φ the map $\varphi^n = \varphi^{n-1} \circ \varphi$ for $n \geq 1$ and φ^0 will represent the identity map on X.

The *orbit of a point* $x \in X$ will be the sequence $\mathrm{Orb}_{\varphi}(x) = (\varphi^n(x))_{n=0}^{\infty}$. A point $x \in X$ is said to be *periodic* if $\varphi^n(x) = x$ for some $n > 0$. When $n = 1$ the periodic point is a *fixed point.* The set of fixed points of φ will be denoted by $\mathrm{Fix}(\varphi)$. The *order* or *period* of a periodic point $x \in X$ is the least $m > 0$ such that $\varphi^m(x) = x$ and it will be denoted by $\mathrm{ord}\,\varphi(x)$ or simply $\mathrm{ord}\,(x)$ if it is clear that it is under the map φ. If $x \in X$ is a periodic point then its orbit (finite) will be called a *cycle* or a *periodic orbit.* Finally we will denote by $\mathrm{Per}\,(\varphi)$ the set of periods of the map φ.

For maps F we will use the notation $n \Rightarrow m$ to indicate that the existence of a cycle of order n for F "forces" the existence of another cycle of order m for F. Also the notation $n \Leftrightarrow m$ means that $n \Rightarrow m$ and $m \Rightarrow n$.

3. Main result

The aim of this paper is to describe the periodic structure that the antitriangular maps can have.

First let us recall the well know result given by the Šarkovskii´s theorem on the periodic structure of continuous maps on the interval. If we give an ordering of the elements of the set $\mathbb{N}$ in the following way

$$\begin{array}{ccccccccccc}
3 & >_s & 5 & >_s & 7 & >_s & \dots & >_s & 2n+1 & >_s \dots \\
2\cdot 3 & >_s & 2\cdot 5 & >_s & 2\cdot 7 & >_s & \dots & >_s & 2\cdot(2n+1) & >_s \dots \\
2^2\cdot 3 & >_s & 2^2\cdot 5 & >_s & 2^2\cdot 7 & >_s & \dots & >_s & 2^2\cdot(2n+1) & >_s \dots \\
 & & & & & & \dots & & & \\
2^k\cdot 3 & >_s & 2^k\cdot 5 & >_s & 2^k\cdot 7 & >_s & \dots & >_s & 2^k\cdot(2n+1) & >_s \dots \\
 & & & & & & \dots & & & \\
\dots\dots & >_s & 2^m & >_s & \dots & >_s & 2^3 & >_s & 2^2 \quad >_s & 2 >_s 1,
\end{array}$$

then if a continuous self-map of the interval I has a periodic point of period n, then it has also periodic points of all the orders m such that $n >_s m$. Let $S(n)$ be the set $\{m \in \mathbb{N} : n >_s m\} \cup \{n\}$, where $n \in \mathbb{N}$. In the case of the symbol 2^∞ we have $S(2^\infty) = \{2^k : k = 0, 1, 2, \dots\}$. The Šarkovskii´s result states that given a continuous self-map φ on the unit interval then there is $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{2^\infty\}$ such that

$$\operatorname{Per}(\varphi) = S(n).$$

For general maps $G : I^2 \to I^2$ we can not find similar results to that of Šarkovskii´s theorem, except for those we know as triangular maps

$$F(x,y) = (f(x), g(x,y))$$

(see [Kl]). In fact for those maps the result is the same. In the case of antitriangular maps we obtain not the same but a similar result. The following example proves that the periodic structure of antitriangular maps can not be the same as that given by the Šarkovskii´s result.

EXAMPLE 3.1. Let $F(x,y) = (y, 1-x)$, $0 \le x, y \le 1$. It holds that $\operatorname{Fix}(F) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$. We have not periodic points of period two since the condition $F^2(x,y) = (x,y)$ is only satisfied by the fixed point $\left(\frac{1}{2}, \frac{1}{2}\right)$. For the same reason there are not periodic points of period three but it is true that $F^4(x,y) = (x,y)$ for any point $(x,y) \in I^2$. Therefore $\operatorname{Per}(F) = \{1, 4\}$.

Let A be a subset of natural numbers. By $2(A \setminus \{1\})$ we understand the set of elements of the type $2k$, with $k \in A \setminus \{1\}$.

THEOREM 3.1. *Let $F : I^2 \to I^2$ be an antitriangular map*

$$F(x,y) = (g(y), f(x))$$

where $f, g \in C(I)$.

If $\mathrm{Per}\,(g \circ f) = S(n)$ *for some* $n \in \overline{\mathbb{N}}$, *then the set* $\mathrm{Per}\,(F)$ *is of one of the following types*

(1) $\mathrm{Per}\,(F) = 2\,(S(n) \setminus \{1\}) \cup \{k : \ k \in S(n), k > 1, k \textit{ odd}\} \cup \{1\}$.

(2) $\mathrm{Per}\,(F) = 2\,(S(n) \setminus \{1\}) \cup \{k : \ k \in S(n), k > 1, k \textit{ odd}\} \cup \{1\} \cup \{2\}$.

Both cases can hold and depend on the fixed points of $g \circ f$ since $2 \in$ $\mathrm{Per}\,(F)$ *if and only if $g \circ f$ has at least two fixed points.*

4. Preliminary results

We are starting introducing two results, one concerning the ordering of periodic points in the setting of compact metric spaces and another on the behaviour of the iterates of antitriangular maps.

LEMMA 4.1. *Let $\varphi \in C(X,X)$, where X is a compact metric space. Given $n, m \in \mathbb{N}$ and $x \in X$, it is held that $x \in Fix(\varphi^n)$ if and only if x is a $m-$periodic point of φ, where m is a divisor of n $(m \mid n)$.*

PROOF. It can be seen in [L]. □

REMARK 4.1. The statement of Lemma 4.1 establishes that if $\varphi^n(x) = x$ then it must be $\mathrm{ord}_{\varphi}(x) = m$, with $m \mid n$.

An easy calculation allows us to prove the following result.

LEMMA 4.2. *Let $F : I^2 \to I^2$ be an antitriangular map $F(x,y) = (g(y), f(x))$. For any $k \geq 0$ we have*

(1) $F^{2k+1}(x,y) = \left(g \circ (f \circ g)^k(y), f \circ (g \circ f)^k(x)\right)$.

(2) $F^{2k}(x,y) = \left((g \circ f)^k(x), (f \circ g)^k(y)\right)$.

The following result establishes the behaviour of the periodic points in the composition of interval maps.

LEMMA 4.3. *Given $f, g \in C(I,I)$, we have*

$$\mathrm{Per}\,(f \circ g) = \mathrm{Per}\,(g \circ f).$$

PROOF. Let $x \in I$ be a periodic point of order k for $g \circ f$. Then $f(x)$ is a periodic point of period k for $f \circ g$ since $\mathrm{Orb}_{f \circ g}(f(x)) = f\,(\mathrm{Orb}_{g \circ f}(x))$. Analogously if $y \in I$ is a point of order n for $f \circ g$, then $\mathrm{Orb}_{g \circ f}(g(y)) = g\,(\mathrm{Orb}_{f \circ g}(y))$ and therefore $g(y)$ is a point of order n for $g \circ f$. □

The inclusion of the order 2 in the set $\operatorname{Per}(F)$ depends on the number of fixed points of the maps $g \circ f$ and $f \circ g$ (in Example 3.1, $2 \notin \operatorname{Per}(F)$ since $g \circ f = f \circ g$ has only a fixed point, but if $g \circ f$ had more than one fixed point then $2 \in \operatorname{Per}(F)$).

LEMMA 4.4. *Given an antitriangular map F, the following properties are equivalent:*

(1) $2 \notin \operatorname{Per}(F)$.

(2) $g \circ f$ *has a unique fixed point.*

(3) $f \circ g$ *has a unique fixed point.*

PROOF. Let $x \in I$ be the unique fixed point of $g \circ f$. Then $f(x)$ is the unique fixed point of $f \circ g$ since if $y \neq f(x)$ is another fixed point for $f \circ g$ we have $g(f \circ g(y)) = g(y)$ and so $g(y) = x$ whence $f(x) = (f \circ g)(y) = y$.

Analogously if $z \in I$ is the unique fixed point of $f \circ g$ then $g(z)$ is the unique fixed point of $g \circ f$. Therefore the statements (2) and (3) are equivalent.

Let us suppose that $g \circ f$ would have a unique fixed point $\omega \in I$ (in this case $f \circ g$ would have the unique fixed point $f(\omega)$). If $F^2(x, y) = (x, y)$, then using Lemma 4.2 we would obtain that $f \circ g(y) = y$, but then $g(y) = \omega$, $f(\omega) = y$. Then

$$F(x, y) = F(\omega, f(\omega)) = (g(f(\omega)), f(\omega)) = (\omega, f(\omega)) = (x, y),$$

that is, (x, y) would be a fixed point for F but not of order 2. Therefore $2 \notin \operatorname{Per}(F)$.

Conversely, let us suppose that $2 \notin \operatorname{Per}(F)$. If $g \circ f$ has two distinct fixed points, $x_1 \neq x_2$, then $f(x_1)$, $f(x_2)$ are fixed points of $f \circ g$ which are distinct, since $f(x_1) = f(x_2)$ leads to $x_1 = g \circ f(x_1) = g \circ f(x_2) = x_2$. In that case, the point $(x_1, f(x_2))$ verifies

$$F(x_1, f(x_2)) = (g(f(x_2)), f(x_1)) = (x_2, f(x_1)) \neq (x_1, f(x_2))$$

$$F^2(x_1, f(x_2)) = (g(f(x_1)), f(x_2)) = (x_1, f(x_2)),$$

and it is a point of order 2 for F, in contradiction with $2 \notin \operatorname{Per}(F)$. Therefore $g \circ f$ can have only a fixed point. $\square$

In the following lemma we will prove that $2\,(\mathrm{Per}\,(g \circ f) \setminus \{1\}) \subseteq \mathrm{Per}\,(F)$.

LEMMA 4.5. *If* $k \in \mathrm{Per}\,(g \circ f)$, $k > 1$, *then* $2k \in \mathrm{Per}\,(F)$.

PROOF. Let $x \in I$ be a point of order k for $g \circ f$ with $k > 1$. Then the point $(x, f(x)) \in I^2$ is of order $2k$ for F.

Using Lemma 4.2 we obtain that $F^{2k}(x, f(x)) = (x, f(x))$. Let $p = \mathrm{ord}_F(x, f(x))$. If $p = 2k$ we have finished, but if $p < 2k$ then must be $p \leq k$. In the sequel we analyse all the possibilities:

(a) If $p < k$, then $2p < 2k$ and therefore

$$F^{2p}(x, f(x)) = F^p(F^p(x, f(x))) = F^p(x, f(x)) = (x, f(x)).$$

Using again Lemma 4.2 we obtain that $(g \circ f)^p(x) = x$ with $p < k$, contradicting $\mathrm{ord}_{g \circ f}(x) = k$.

(b) If $p = k$ with k even, again Lemma 4.2 gives us $(g \circ f)^{\frac{k}{2}}(x) = x$. But $\frac{k}{2} < k = \mathrm{ord}\,(x)$, which is a contradiction.

(c) If $p = k$ with k odd, Lemma 4.2 gives us now

$$g \circ (f \circ g)^{\frac{k-1}{2}}(f(x)) = x,$$

that is, $(g \circ f)^{\frac{k+1}{2}}(x) = x$. But $\frac{k+1}{2} < k = \mathrm{ord}\,(x)$ since we are supposing that $k > 1$, getting a new contradiction. □

LEMMA 4.6. *Given an antitriangular map* $F(x, y) = (g(y), f(x))$, *for every* $n \geq 1$ *it is held that* $2n+1 \in \mathrm{Per}\,(F)$ *if and only if* $2(2n+1) \in \mathrm{Per}\,(F)$, *that is,* $2n + 1 \Leftrightarrow 2(2n + 1)$.

PROOF. Let $(x, y) \in I^2$ be a periodic point for F of period $2(2n + 1)$. Then

$$F^{2(2n+1)}(x, y) = \left((g \circ f)^{2n+1}(x), (f \circ g)^{2n+1}(y)\right) = (x, y).$$

From it we have $\mathrm{ord}_{g \circ f}(x) \mid 2n + 1$ and $\mathrm{ord}_{f \circ g}(y) \mid 2n + 1$ (Lemma 4.1), but $\mathrm{ord}_{g \circ f}(x) = \mathrm{ord}_{f \circ g}(y) = 1$ can not happen, since in that case the order of (x, y) for F would be at most $2 < 2(2n + 1)$. Therefore, without loss of generality, we can suppose that $1 < \mathrm{ord}_{g \circ f}(x) \leq 2n + 1$.

If $\mathrm{ord}_{g \circ f}(x) < 2n+1$ then $\mathrm{ord}_{g \circ f}(x)$ would be an odd number less than $2n + 1$ and the Šarkovskii´s theorem would give us

$$\mathrm{Per}\,(g \circ f) = \mathrm{Per}\,(f \circ g) \supseteq S(\mathrm{ord}_{g \circ f}(x)) \supseteq S(2n + 1).$$

In fact there exists $\overline{y} \in I$ of order $2n + 1$ for $f \circ g$. From the orbit

$$\mathrm{Orb}_{g \circ f}(g(\overline{y})) = \left\{g(\overline{y}), g \circ (f \circ g)(\overline{y}), ..., g \circ (f \circ g)^{2n}(\overline{y})\right\}$$

we select the point $\overline{x} = g \circ (f \circ g)^n(\overline{y})$ which is of order $2n+1$ for $g \circ f$ and besides

$$f \circ (g \circ f)^n(\overline{x}) = (f \circ g)^{2n+1}(\overline{y}) = \overline{y}.$$

In this way

$$F^{2n+1}(\overline{x}, \overline{y}) = (g \circ (f \circ g)^n(\overline{y}), f \circ (g \circ f)^n(\overline{x})) = (\overline{x}, \overline{y}).$$

According to the election of $\overline{x}$ and $\overline{y}$, the order of the point $(\overline{x}, \overline{y})$ can not be less than $2n+1$ since in this case we would have an odd order of the type $2k+1$ with $k < n$ giving

$$F^{2k+1}(\overline{x}, \overline{y}) = \left(g \circ (f \circ g)^k(\overline{y}), f \circ (g \circ f)^k(\overline{x})\right) = (\overline{x}, \overline{y})$$

and as a consequence must be $\overline{x} = g \circ (f \circ g)^k(\overline{y})$ in contradiction with the election of $\overline{x}$ within the orbit $\mathrm{Orb}_{g \circ f}(g(\overline{y}))$. Therefore $2n+1 \in \mathrm{Per}(F)$.

Let us suppose now that $(x, y) \in I^2$ be a point of order $2n+1$ for F with $n \geq 1$,

$$F^{2n+1}(x, y) = (g \circ (f \circ g)^n(y), f \circ (g \circ f)^n(x)) = (x, y)$$

and try to obtain a periodic point for F of period $2(2n+1)$. Using the $(2n+1)$–iterate we get

$$x = (g \circ f)^{2n+1}(x), \qquad y = (f \circ g)^{2n+1}(y).$$

Repeating the arguments used to obtain the implication $2(2n+1) \Rightarrow 2n+1$ we find that $\mathrm{Per}(g \circ f) = \mathrm{Per}(f \circ g) \supseteq S(2n+1)$. We can consider that $\mathrm{ord}_{g \circ f}(x) = \mathrm{ord}_{f \circ g}(y) = 2n+1$ (otherwise there is no problem to find another pair $(\overline{x}, \overline{y})$ in such a way that the orders of $\overline{x}$ and $\overline{y}$ are $2n+1$ and $\mathrm{ord}_F(\overline{x}, \overline{y}) = 2n+1$). Then $(x, (f \circ g)(y)) \in I^2$ has order $2(2n+1)$ for F :

(a) $F^{2(2n+1)}(x, (f \circ g)(y)) = \left((g \circ f)^{2n+1}(x), (f \circ g)^{2n+1}(f \circ g)(y)\right) = (x, (f \circ g)(y))$, and therefore $\mathrm{ord}_F(x, (f \circ g)(y)) \mid 2(2n+1)$.

(b) If $\mathrm{ord}_F(x, (f \circ g)(y)) = 1$, then

$$F^2(x, (f \circ g)(y)) = F(x, (f \circ g)(y)) = (x, (f \circ g)(y))$$
$$F^2(x, (f \circ g)(y)) = ((g \circ f)(x), (f \circ g)^2(y)),$$

in contradiction with the fact that $\mathrm{ord}_{g \circ f}(x) = 2n+1 > 1$.

(c) If $\mathrm{ord}_F(x, (f \circ g)(y)) = 2k$, with $k \leq 2n$, the iterate $F^{2k}(x, (f \circ g)(y))$ leads to a contradiction with the fact that x and $(f \circ g)(y)$ are periodic points of order $2n+1$ since $F^{2k} = \left((g \circ f)^k, (f \circ g)^k\right)$, $k \leq 2n$.

(d) If $\operatorname{ord}_F(x, (f \circ g)(y)) = 2m+1$ with $m < n$ and $2m+1 \mid 2n+1$ we have

$$\begin{aligned} F^{2m+1}(x, (f \circ g)(y)) &= (g \circ (f \circ g)^m (f \circ g)(y), f \circ (g \circ f)^m (x)) \\ &= (x, (f \circ g)(y)) \end{aligned}$$

and as consequence must be $(f \circ g)^{2m+1}(x) = x$ which is a contradiction since $\operatorname{ord}(x) = 2n+1 > 2m+1$.

(e) It is $\operatorname{ord}_F(x, (f \circ g)(y)) \neq 2n+1$ since if

$$F^{2n+1}(x, (f \circ g)(y)) = (x, (f \circ g)(y)),$$

then $g \circ (f \circ g)^n (f \circ g)(y) = x$, $f \circ (g \circ f)^n(x) = (f \circ g)(y)$, but

$$\begin{aligned} g \circ (f \circ g)^n (f \circ g)(y) &= g \circ f \circ g \circ (f \circ g)^n (y) = \\ &= (g \circ f) \circ (g \circ (f \circ g)^n)(y) = (g \circ f)(x), \end{aligned}$$

obtaining that $(g \circ f)(x) = x$, which is a contradiction.

(f) Finally, must be $\operatorname{ord}_F(x, (f \circ g)(y)) = 2(2n+1)$, obtaining that $2n+1 \Rightarrow 2(2n+1)$. $\square$

5. Proof of Theorem 3.1

We divide the last section in two parts. In the first we give a picture of forcing relations in $Per(F)$ to illustrate Theorem 3.1 which will be finally proved in the second part.

5.1. Picture of forcings in $\operatorname{Per}(F)$. Using all the results obtained in the previous section we are able to order $\mathbb{N} \setminus \{2\}$ in such a way that the following picture of forcing for all the orders of F is held.

PROPOSITION 5.1. *Given the antitriangular map $F(x, y) = (g(y), f(x))$ in I^2, the following set of forcings for the periods of F is held*

$$\begin{array}{ccccccccccccc}
 & 2\cdot 3 & & 2\cdot 5 & & 2\cdot 7 & & & & 2\cdot(2n+1) & & \\
 & \Updownarrow & & \Updownarrow & & \Updownarrow & & & & \Updownarrow & & \\
 & 3 & \Rightarrow & 5 & \Rightarrow & 7 & \Rightarrow & \ldots & \Rightarrow & 2n+1 & \Rightarrow & \ldots \\
\Rightarrow & 2^2\cdot 3 & \Rightarrow & 2^2\cdot 5 & \Rightarrow & 2^2\cdot 7 & \Rightarrow & \ldots & \Rightarrow & 2^2\cdot(2n+1) & \Rightarrow & \ldots \\
 & & & & & & \ldots & \ldots & \ldots & & & \\
\Rightarrow & 2^r\cdot 3 & \Rightarrow & 2^r\cdot 5 & \Rightarrow & 2^r\cdot 7 & \Rightarrow & \ldots & \Rightarrow & 2^r\cdot(2n+1) & \Rightarrow & \ldots \\
 & & & & & & \ldots & \ldots & \ldots & & & \\
\Rightarrow & \ldots & \Rightarrow & 2^{m+1} & \Rightarrow & 2^m & \Rightarrow & \ldots & & \Rightarrow 2^3 & \Rightarrow 2^2 & \Rightarrow 1.
\end{array}$$

PROOF. The initial chain of double implications is a consequence of Lemma 4.6. Now we are proving that if we consider a particular n of the picture, then there are points of order m for any m placed after n in the ordering.

(a) If n is odd, $n \geq 3$ and $n \in \mathrm{Per}\,(F)$, then $2n \in \mathrm{Per}\,(F)$. From this it is easy to obtain that $n \in \mathrm{Per}\,(f \circ g) = \mathrm{Per}\,(g \circ f)$. Let $(x, y) \in I^2$ be a point of order $2n$ for F. By Lemma 4.2 we get $(g \circ f)^n(x) = x$, $(f \circ g)^n(y) = y$. If $\mathrm{ord}_{g \circ f}(x) = \mathrm{ord}_{f \circ g}(y) = 1$ then we obtain $F^2(x, y) = (x, y)$ but $2n > 2$. In this case some of the orders of x or y is an odd number $m \leq n$ (see Lemma 4.1). According to Šarkovskii´s theorem, $\mathrm{Per}\,(g \circ f) \supseteq S(m) \supseteq S(n)$ and using Lemmas 4.5 and 4.6 we have

$$\mathrm{Per}\,(F) \supseteq 2\,(S(n) \setminus \{1\}) \cup \{k : \quad k \in S(n),\ k \text{ odd}\}\,.$$

Therefore $n \Rightarrow p$ for any p placed after n in the ordering given in the statement of the proposition.

(b) The case $n = 2q$ with q odd and $q \geq 3$ can be brought to the case $n' = q$ using the equivalences given in Lemma 4.6.

(c) If $n = 2^k \cdot m$ with $k \geq 2$, m odd, $m \geq 3$, and $(x, y) \in I^2$ is a point of order n for F, then using Lemmas 4.1 and 4.2 we obtain that $\mathrm{ord}_{g \circ f}(x) \mid 2^{k-1}m$ and $\mathrm{ord}_{f \circ g}(y) \mid 2^{k-1}m$. If $\mathrm{ord}\,(x) = 2^s$ and $\mathrm{ord}\,(y) = 2^t$ with $s, t \leq k - 1$, we obtain that $F^{2^k}(x, y) = (x, y)$ which is a contradiction. In such case some of the two orders given above, for example $\mathrm{ord}\,(x)$, must be of the type $2^s p$ where $s \leq k - 1$, $p > 1$ is odd and $p \mid m$. Using the Šarkovskii´s order we obtain that $\mathrm{Per}\,(g \circ f) \supseteq S(2^s p) \supseteq S(2^{k-1}m)$ and reasoning as in (b) we obtain that $n \Rightarrow q$ for any number q placed after n in the ordering of the statement of the Proposition.

(d) If $n = 2^k$, $k > 2$ and $(x, y) \in I^2$ is a point of order n for F, then it is held that $(g \circ f)^{2^{k-1}}(x) = x$ and $(f \circ g)^{2^{k-1}}(y) = y$. Then $\mathrm{ord}\,(x) \mid 2^{k-1}$ and $\mathrm{ord}\,(y) \mid 2^{k-1}$ (according to Lemmas 4.1 and 4.2). If $q = \max\,\{\mathrm{ord}\,(x), \mathrm{ord}\,(y)\} < 2^{k-1}$ the we would obtain $F^{2q}(x, y) = (x, y)$ which is a contradiction ($2q < 2^k$). Then must be for example $\mathrm{ord}\,(x) = 2^{k-1}$. But this means that $\mathrm{Per}\,(g \circ f) \supseteq S(2^{k-1})$ and therefore $\mathrm{Per}\,(F) \supseteq 2\,(S(2^{k-1} \setminus \{1\})$(Lemma 4.5), that is, $n \Rightarrow 2^s$ for $s = 0, 2, 3, ..., k$. For $n = 2^2$ we have to prove only that $2^2 \Rightarrow 1$, but using the fixed point Brouwer´s theorem that statement is true. □

REMARK 5.1. According to Proposition 5.1 and Lemma 4.4, the kernel of periodicity (see [ALM]) of the antitriangular maps is the set $\{2, 3, 2 \cdot 3\}$ since the period 2 does not force any other period and at the same time it is not forced by any period. If an antitriangular map has a point of period 3 or $2 \cdot 3$ then has points of all periods.

5.2. Proof of the Theorem. The last statement of Theorem 3.1 is a direct consequence of Lemma 4.4. Using Lemmas 4.5 and 4.6 we know that for some $n \in \overline{\mathbb{N}}$ it is held

$$\mathrm{Per}\,(F) \supseteq 2 \cdot (S(n) \setminus \{1\}) \cup \{k : \quad k \in S(n), \ k \ \ \text{odd}\} \cup \{1\} = S_d(n),$$

where $S(n) = \mathrm{Per}\,(g \circ f) = \mathrm{Per}\,(f \circ g)$ is a segment of the Šarkovskii's ordering and $S_d(n)$ can be considered as a segment in the order of the antitriangular maps.

To end the proof it is necessary to prove that there are not other periods in $\mathrm{Per}\,(F)$, except 2, which can appear or not depending on the number of fixed points of $f \circ g$.

Let us suppose that $m \in \mathrm{Per}\,(F)$ and consider the following situations:

(a) If $m = 2^r q$ with $r \geq 2$, $q \geq 1$ and q odd, we can repeat the considerations made in the proof of Proposition 5.1 to obtain that $S(n) = \mathrm{Per}\,(g \circ f) \supseteq S(\frac{m}{2})$. Therefore,

$$m = 2\frac{m}{2} \in 2\left(S(\frac{m}{2}) \setminus \{1\}\right) \subseteq 2\,(S(n) \setminus \{1\}) \subseteq \mathrm{Per}\,(F).$$

(b) If m is odd and $m \geq 3$, it has been proved in Proposition 5.1 that $\mathrm{Per}\,(g \circ f) = S(n) \supseteq S(m)$. Using Lemma 4.6 we have

$$\begin{aligned} m &\in 2\,(S(m) \setminus \{1\}) \cup \{k : \quad k \text{ odd}, \ k \in S(m)\} \subseteq \\ &\subseteq 2\,(S(n) \setminus \{1\}) \cup \{k : \quad k \text{ odd}, \ k \in S(n)\} \subseteq \mathrm{Per}\,(F). \end{aligned}$$

(c) Finally, if $m = 2q$ with q odd and $q \geq 3$, by Lemma 4.6 is $q \in \mathrm{Per}\,(F)$ and similarly to the previous case

$$m = 2q \in 2\,(S(q) \setminus \{1\}) \subseteq 2\,(S(n) \setminus \{1\}) \subseteq \mathrm{Per}\,(F).$$

(d) It only remains the study of the case $m = 2$. The inclusion of this order depends of the number of fixed points in the map $g \circ f$. □

REMARK 5.2. Theorem 3.1 can be easily extended to $\mathbb{R}^2$ but in this case the situation $\mathrm{Per}\,(F) = \emptyset$ can occur (for example, consider the map $F(x, y) = (y + 1, x + 1)$ defined in $\mathbb{R}^2$).

Besides, Theorem 3.1 admits a converse result. Given a segment $S_d(n)$ for some $n \in \overline{\mathbb{N}}$ there are antitriangular maps $F_1, F_2 : I^2 \to I^2$ such that

$$\mathrm{Per}\,(F_1) = S_d(n) \cup \{2\}, \ \mathrm{Per}\,(F_2) = S_d(n).$$

To prove this it is sufficient to consider the antitriangular maps $F_{1,\alpha}(x, y) = (y, \alpha x(1 - x))$, $F_{2,\beta}(x, y) = (y, \cos(\beta\pi(x - \frac{1}{2})))$. When $\alpha \in [2, 4]$, the logistic map $f_\alpha(x) = \alpha x(1 - x)$ has two fixed points, and by Lemma 4.4,

$2 \in \mathrm{Per}\,(F_1, \alpha)$ also, when $\alpha \in [2,4]$, $\mathrm{Per}\,(f_\alpha)$ can be any initial segment $S(n)$ of the Šarkovskiiś ordering, therefore $\mathrm{Per}\,(F_1, \alpha)$ can be any set $S_d(m) \cup \{2\}$ with $m \in \overline{\mathbb{N}}$. If $\beta \in [0,1)$, then $f_\beta(x) = \cos(\beta\pi(x - \frac{1}{2}))$ has a unique fixed point, that is, $2 \notin \mathrm{Per}\,(F_1, \beta)$; like in the above family, we demonstrate that $\mathrm{Per}\,(F_1, \beta)$ can be any segment $S_d(n)$, $n \in \overline{\mathbb{N}}$.

REMARK 5.3. The notion of antitriangular map can be extended to the case of maps on I^n with $n \geq 3$, having a particular structure, and we guess that periodic structures similar to the case $n = 2$ can be found. This must be true even for maps defined on $\mathbb{R}^n$ and $\mathbb{T}^n$ (the n-dimensional torus). But this will be presented in a forthcoming paper.

REFERENCES

[D] R.A.Dana, L.Montrucchio, *Dynamics complexity in duopoly games*, Journal of Economic Theory **40** (1986), 40–56.

[Kl] P. E. Kloeden, *On Sharkovsky's cycle coexistence ordering*, Bull. Austral. Math. Soc. **20** (1979), 171–177.

[L] A. Linero Bas, *Puntos periódicos de aplicaciones continuas en espacios de dimensión uno*, Tesina de Licenciatura, Universidad de Murcia, 1991.

[S] A. N. Šarkovskiĭ, *Coexistence of cycles of a continuous map of the line into itself*, Ukrain. Math. J. **16** (1964), 61–71(in Russian); English version in Internat. Journal of Bifurcation and Chaos **5** (1995).

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE MURCIA
30071 ESPINARDO (MURCIA)
SPAIN

e-mail: balibrea@gaia.fcu.um.es