UNBOUNDED NOT DIVERGING TRAJECTORIES IN MAPS WITH A VANISHING DENOMINATOR

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To the memory of Professor György Targonski

Abstract. Maps with a denominator which vanishes in a subset of the phase space may generate unbounded trajectories which are not divergent, i.e. trajectories involving arbitrarily large values of the dynamic variables but which are not attracted to infinity. In this paper we propose some simple one-dimensional and two-dimensional recurrences which generate unbounded chaotic sequences, and through these examples we try to explain the basic mechanisms and bifurcations leading to the creation of unbounded sets of attraction.

1. Introduction

The literature on chaotic dynamical systems is mainly concerned with bounded attracting sets, and unbounded trajectories are usually considered as synonymous of diverging trajectories. Nevertheless, unbounded and not diverging chaotic trajectories naturally arise in the iteration of maps with denominator. A first example of "non bounded chaotic solution" was given in [11] (see also [13], p. 38).

In this paper we propose some one-dimensional and two-dimensional recurrences which generate unbounded chaotic sequences, and through these examples we try to explain the basic mechanisms and bifurcations leading to the creation of *Unbounded Sets of Attraction* (USA). We show that unbounded chaotic sets can be easily observed in the iteration of one-dimensional

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maps characterized by the presence of a vertical asymptote in the graph of the map, the abscissa of which cancels the map denominator, and the creation of USA can be explained as the effect of non classical bifurcations, due to a contact between a critical points and a vertical asymptote. We also show an example of USA for a two-dimensional recurrence which is not defined in the whole plane, due to the presence of a vanishing denominator, and we argue that the mechanisms that creates two-dimensional USA is similar to the one observed in one-dimensional maps. In fact, a curve δ_s where a denominator vanishes may be considered as a two-dimensional analogue of a vertical asymptote, and the creation of two-dimensional USA in noninvertible maps may be caused by a contact between δ_s and critical curves.

The study of such peculiar dynamic behaviors of maps with denominator has been motivated by practical reasons, because discrete dynamical systems obtained by the iteration of maps with denominator are often met in applications. For example, many iterative methods to find numerical solutions of equations, based on the well known Newton method, are expressed by recurrences with denominator (see e.g. [8, 3]). Moreover, some discrete-time dynamical systems used to model the evolution of economic and financial systems, which are often expressed by implicit recurrences $F(x_n, x_{n+1}) = 0$, assume the form of recurrences with denominator when they are expressed as $x_{n+1} = f(x_n)$ (see e.g.[10, 2, 6]).

2. One-dimensional examples

We propose two simple one-dimensional rational fractional maps in order to show that the occurrence of USA naturally arise in maps characterized by the presence of a vertical asymptote.

2.1. A recurrence with an unbounded chaotic set. Let us consider the recurrence $x_{n+1} = f(x_n)$, where f is the fractional rational map

$$x' = f(x) = \frac{x^2 - 1}{2x + 1}\,. \tag{1}$$

This map is not defined in the set $\delta_s = \{-1/2\}$ where the denominator vanishes. So the recurrence is well defined provided that the initial condition x_0 belongs to the set $E = \mathbb{R} \setminus \Lambda$,where Λ is the union of the preimages of any rank of δ_s

$$\Lambda = \bigcup_{k=0}^{\infty} f^{-k}(\delta_s)\,.$$

In fact, only the points belonging to the set E generate non interrupted trajectories by the iteration of the map $f : E \to E$. We notice that, being

δ_s an isolated point of $\mathbb{R}$, the set Λ of points excluded from the phase space of the recurrence has zero Lebesgue measure in $\mathbb{R}$.

At $x = -1/2$ the graph of (1) has a vertical asymptote, and for large values of $|x|$ it approaches an asymptote of equation $x' = \frac{1}{2}x - \frac{1}{4}$ (see fig. 1). The shape of the graph of (1) gives us an intuitive understanding of the mechanism which is at the basis of the creation of an USA. In fact, points arbitrarily close to $x = -1/2$ have images arbitrarily large (close to infinity) and the iterated images of points very far from the origin have images of smaller and smaller modulus. Indeed, for large values of $|x|$ the map (1) is approximated by the linear contraction $x' = \frac{1}{2}x - \frac{1}{4}$, until x approaches the value $x = -1/2$, so that large values are obtained again and so on.

Indeed, it is easy to prove that the iteration of the map (1) generates chaotic trajectories. In fact, (1) is conjugate to the map

$$z' = s(z) = \begin{cases} 2z & \text{for} \quad 0 \leq z \leq 1/2 \\ 2z - 1 & \text{for} \quad 1/2 < z \leq 1 \end{cases} \tag{2}$$

by the conjugacy transformation (see [4])

$$z = h(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{2x+1}{\sqrt{3}} \tag{3}$$

The dynamics of (3) are well known, from both a topological and a measure theoretical point of view: it has chaotic dynamics in the interval $[0, 1]$ with an absolutely continuous invariant ergodic measure associated with it. Hence the fractional map (3) has chaotic dynamics in the unbounded interval $(-\infty, +\infty)$ with an absolutely continuous invariant measure on it.

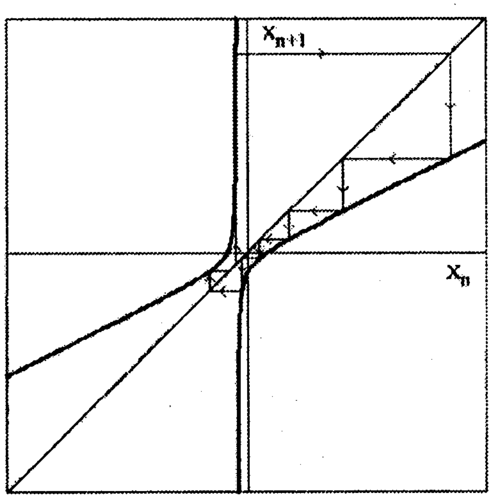


Fig. 1

To sum up, the chaotic trajectories generated by the map (3) include arbitrarily large values, due to the presence of the vertical asymptote, but such trajectories do not diverge, because the infinity is repelling. This is the basic mechanism for the existence of an USA.

2.2. Contact bifurcations which create and destroy an unbounded set of attraction. Let us consider the one-dimensional noninvertible map

$$x' = f(x) = x + \frac{b}{x^2} - 2 \quad , \quad b > 0 \tag{4}$$

whose set of non definition, due to the vanishing of the denominator, is $\delta_s = \{0\}$. Following the terminology of [13] and [1], this is a $Z_1 - Z_3$ map, because the critical point (image of $x = c_{-1}$, local minimum of $f(x)$) $c = \sqrt[3]{2b} + \sqrt[3]{b/4} - 2$ separates the range of the map into the intervals $Z_1 = (-\infty, c)$ and $Z_3 = (c, +\infty)$ whose points have one or three preimages respectively (see fig. 2). The critical point c is characterized by two merging preimages, located in $c_{-1} = \sqrt[3]{2b}$. The map (4) has two fixed points at finite distance: $p^* = \sqrt{b/2}$ and $q^* = -\sqrt{b/2}$. It is immediate to see that q^* is repelling for any $b > 0$, whereas p^* is stable for $b > 8$, and at $b = 8$ it loses stability via a flip bifurcation, which creates a stable cycle of period 2, followed, as b is further decreased, by a sequence of period-doubling bifurcations leading to the creation of chaotic attractors. The graph of (4) is characterized by the presence of a vertical asymptote at $x = 0$ and an inclined asymptote of equation $x' = x - 2$.

As far as $c > 0$, any trajectory $\{x_n = f^n(x_0)\}$ with $x_0 > q^*$ enters the interval $I = [c, c_1]$, with $c_1 = f(c)$, and then never escape, i.e. such trajectories are ultimately bounded inside I. The interval I is called absorbing interval (see [13]), and inside I the asymptotic dynamics may converge to the fixed point p^* (for $b > 8$) or to an attracting cycle or to a bounded chaotic attractor. In fig. 2a, obtained for $b = 2.5$, two trajectories are represented by the Koenig-Lemeray staircase diagram: one, starting from $x > q^*$, shows an apparently chaotic behavior inside I, the other one is a diverging trajectory starting from $x = -1.2 < q^*$.

As b decreases, the critical point c also decreases, until it reaches the value $c = 0$ at $b = b_c = 32/27$. This contact between the critical point c and the point $x = 0$, at which the vertical asymptote is located, marks the occurrence of a bifurcation at which the bounded absorbing interval I is transformed into an interval which is not bounded above. In fact, $c_1 \to +\infty$ as $b \to b_c^+$, and for $b < b_c$ we have $c < 0$, so that the point $\delta_s = 0$ is included into $[c, c_1]$. This implies that at the bifurcation value b_c the interval $[c, c_1] = [0, +\infty)$ is an unbounded chaotic attractor. Just after the bifurcation, for

$b < b_c$, other sequences of "box-within-a-box" bifurcations occur, however a trajectory may have an unbounded transient, i.e. it may involve arbitrarily large values. In fact, even if a bounded attractor exists, for example a cycle, its basin is unbounded, with a boundary formed by an unbounded invariant set on which the restriction of f is chaotic. In fig. 2b the early points of a typical trajectory obtained with $b < b_c$, and starting from $x_0 = 0.3 > q^*$, are shown. In the same figure also a typical diverging trajectory starting from $x = -1 < q^*$ is represented, so that the difference between a diverging trajectory and an unbounded not diverging trajectory can be easily seen.

If we compare the dynamical properties of the map (4) just before and just after the contact bifurcation occurring at $b = b_c$ we can notice that the closure of the unstable set $W^u(p^*)$ of the fixed point p^* is bounded inside $I = [c, c_1]$ for $b > b_c$ whereas it is not bounded above for $b < b_c$ since $W^u(p^*)$ contains points which are arbitrarily close to the point $x = 0$. In fact, at the contact δ_s enters the region Z_3, hence infinitely many new preimages of it are created inside the interval $(c, +\infty)$ for $b < b_c$.

The basin of the USA shown in fig. 2b is $\mathcal{B} = (q^*, +\infty)$, whereas $\mathcal{B}(-\infty) = (-\infty, q^*)$. The boundary which separates the two basins is the repelling fixed point q^*. Another important bifurcation occurs at $b = b_f = 1/2$, when the critical point c has a contact with the basin boundary, i.e. $c = q^*$. This is the *final bifurcation* (see [1,13]) which causes the destruction of the USA. For $b < b_f$ the generic trajectory diverges (see fig. 1c).

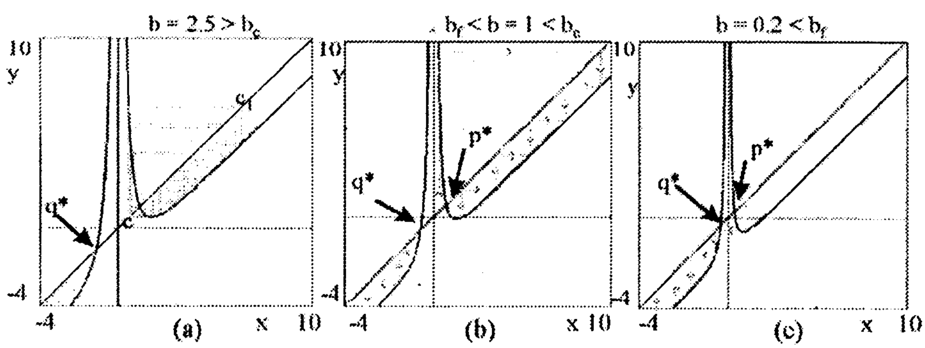


Fig. 2

The bifurcations described above can also be seen in the bifurcation diagram of fig. 3. This bifurcation diagram is obtained by the usual procedure: for each value of the parameter b a trajectory starting from an initial condition close to p^* is numerically generated and, after a transient of the early 1000 iterates has been discarded, 3000 points are plotted along the vertical line through b. For $b > 8$ the asymptotic dynamics is characterized by convergence to a fixed point (this is not visible in the range of b considered in fig. 3), then for decreasing values of b the usual sequence of period doubling bifurcations is obtained leading to chaotic behavior. At $b = b_c$, the contact bifurcation, at which the absorbing interval $[c, c_1]$ opens and becomes unbounded, is revealed by a sudden decrease of the density of the iterated points, due to the fact that they are distributed over an unbounded interval.

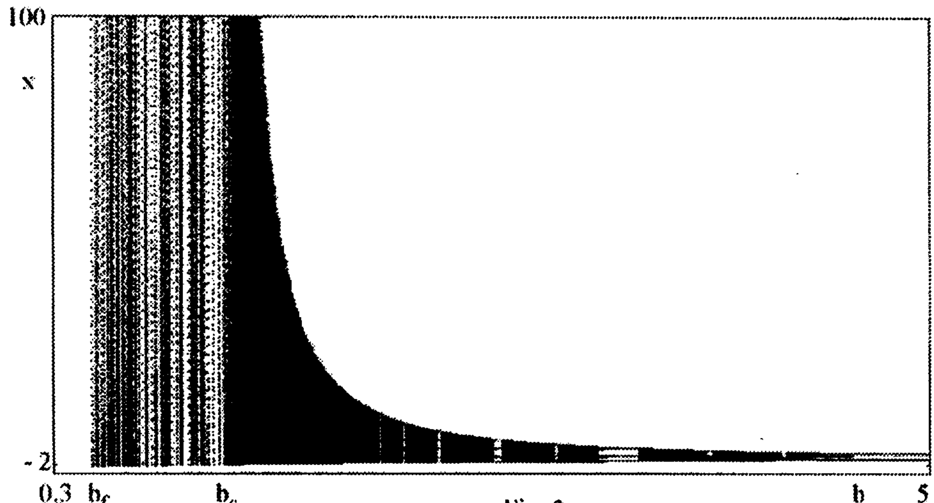


Fig. 3

3. A two-dimensional example

Chaotic USA are easily observed in two-dimensional recurrences obtained by the iteration of a map $T : (x, y) \to (x', y')$ which is not defined in the whole plane, due to the presence of one or more curves at which a denominator vanishes. For noninvertible two-dimensional maps, unbounded sets of attraction may be created by contact bifurcations similar to the one described in the one-dimensional examples of the previous section. In fact, let δ_s be a curve where a denominator vanishes. We may consider such a curve as a two-dimensional analogue of a vertical asymptote (see [7] for a more rigorous approach). As shown in [7], generally the image $T(\gamma)$ of a bounded arc γ crossing δ_s is made up of two disjoint unbounded arcs. If A is an invariant attracting set for the recurrence $(x_{n+1}, y_{n+1}) = T(x_n, y_n)$, then for any $x \in A$ we have $W^u(x) \subseteq A$, since an attracting set includes the unstable sets of all its points. So, if some $W^u(x)$ is unbounded, due to a crossing with δ_s, then also the attracting set A must be unbounded.

Moreover, for a two-dimensional noninvertible map, a chaotic area A is often bounded by segments of critical curves $LC_i = T^i(LC)$, where LC is the critical curve of rank-1 (the two-dimensional analogue of critical points of one-dimensional maps, see e.g. [13, 1]). Hence, the first crossing of a $W^u(x) \subseteq A$ with δ_s occurs just after the first contact between a critical curve, on the boundary of A, and δ_s. This represents the two-dimensional analogue of the contact bifurcation occurring at $b = b_c$ for the one-dimensional map (4).

In order to illustrate these arguments by an example, let us consider the two-dimensional recurrence $(x_{n+1}, y_{n+1}) = T(x_n, y_n)$ where T is defined by

$$(5) \qquad T: \begin{cases} x' = \dfrac{y^2 + 2xy - x^2 - 2x - 2y + b}{x^2 + 6xy + y^2 - 6x - 6y + 3} \\ y' = \dfrac{x^2 + 2xy - y^2 - 2x - 2y + b}{x^2 + 6xy + y^2 - 6x - 6y + 3} \end{cases}.$$

The set δ_s, where the denominator vanishes, is given by the curve

$$\delta_s = \left\{(x, y) \in \mathbb{R}^2 | x^2 + 6xy + y^2 - 6x - 6y + 3 = 0\right\}$$

which is an hyperbola with centre in $(3/4, 3/4)$ and symmetry lines of equation $y = x$ and $y = -x+3/2$. The map (5) is a noninvertible map of $Z_2 - Z_4$ type, where Z_2 and Z_4 represent the regions of the phase plane whose points have two or four distinct rank-1 preimages respectively. These regions are separated by the critical curves $LC = T(LC_{-1})$, where LC_{-1} are the curves of merging preimages (or critical curves of rank-0) which in this case are given by coordinate axes (see fig. 4, obtained with $b = 1$).

The map (5) with $b = 1$ has the very peculiar property that for any initial condition $(x_0, y_0) \in A$, where $A = A_1 \cup A_2$ with

$$A_1 = \left\{(x, y) \in \mathbb{R}^2 | \quad x > 1 \text{ and } y > 1\right\},$$

$$A_2 = \left\{(x, y) \in \mathbb{R}^2 | \quad x < 1 \text{ and } \ y < 1 \text{ and } 2x + y < 1 \text{ and } x + 2y < 1\right\}$$

the recurrence obtained by the iteration of T has a closed form solution $\{x_n, y_n\}$ expressed in terms of elementary functions

$$(6) \qquad x_n = \frac{\cos(2^n C_1)}{\cos(2^n C_1) + \cos(2^n C_2)}, \qquad y_n = \frac{\cos(2^n C_2)}{\cos(2^n C_1) + \cos(2^n C_2)}$$

where the constants C_1 and C_2 are determined by the initial condition $(x_0, y_0) \in A$

$$C_1 = \arccos \frac{x_0}{x_0 + y_0 - 1}, \qquad C_2 = \arccos \frac{y_0}{x_0 + y_0 - 1}.$$

This solution is determined by the method described in [13] ch.1, based on the Schröder functional equation. As explained in [13], the sequence (6) defines a typical chaotic trajectory, characterized by sensitive dependence on initial conditions, which spans the whole chaotic area A. Hence A is an unbounded chaotic area. In fig. 5a the early points of a trajectory starting inside A are represented, and in fig. 5b we show that the boundary of the chaotic area A can also be obtained by segments of critical curves

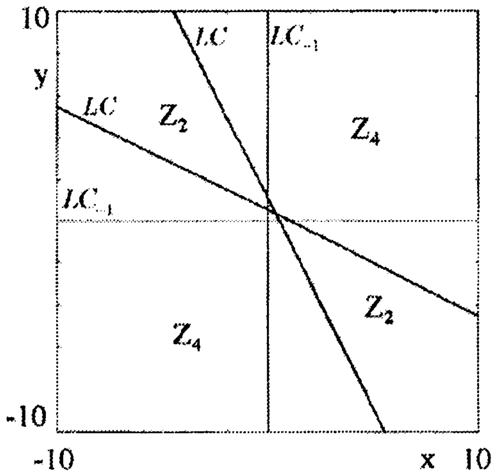
$$LC = \{2x + y = 0\} \cup \{x + 2y = 0\} \text{ and } LC_1 = \{x = 1\} \cup \{y = 1\}.$$

The initial conditions taken out of A generate trajectories converging to one of the stable fixed points $P = (1, 0)$ $Q = (0, 1)$, indicated in fig. 5. In this particular case the lines $x = 1$ and $y = 1$ are invariant, and belong to the boundary of the basins of P and Q. We observe that in this case the usual contact bifurcation between critical curves and basin boundaries (see [13]) does not occur at isolated points, but involves a merging of whole portions of critical segments with segments of basin boundaries.

In fig. 5 it is evident that some portions of the hyperbola δ_s are included inside A, and this is consistent with the fact that the chaotic area A is unbounded, as already remarked in the one-dimensional examples discussed in the previous section.

The property that the boundaries of the chaotic area are formed by segments of critical curves also holds for $b \neq 1$, when the closed form solution is not known in terms of elementary functions. In order to show the kind of bifurcation that leads to the creation of a USA for the map (6) we consider a value of the parameter b for which the attractor does not contain portions of δ_s, so that it is included inside a bounded absorbing area $A(b)$, and then we gradually change the value of the parameter b until a contact between the boundary of $A(b)$, formed by segments of critical curves, and δ_s occurs.

Our starting value is $b = 0.93$. At this stage a numerically generated trajectory fills up the chaotic area $A(b)$ shown in fig. 6. As b is increased the chaotic area enlarges until it has a contact with the lower branch of the hyperbola δ_s. This occurs for $b = 0.933...$ (see the enlargement in fig. 7a) and it can be described as a contact between LC and δ_s (see fig. 7b). After this contact an unbounded attracting set appears (see fig. 8, obtained with $b = 0.934$).


10
y
-10
-10
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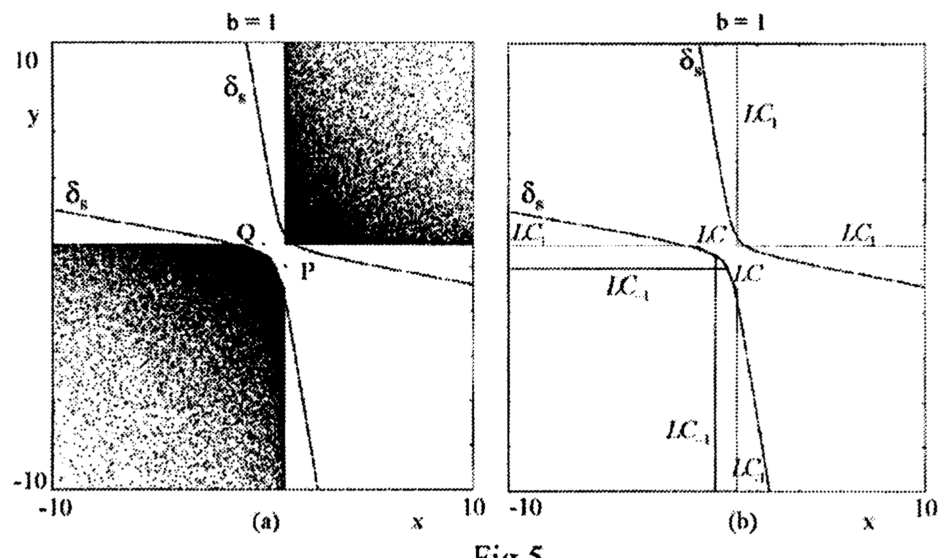
LC
LC
LC_{-1}
LC_{-1}
Z_2
Z_4
Z_4
Z_2

Fig. 4


b = 1
10
y
-10
-10
10
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(a)
δ_s
δ_s
Q
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b = 1
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LC
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Fig.5

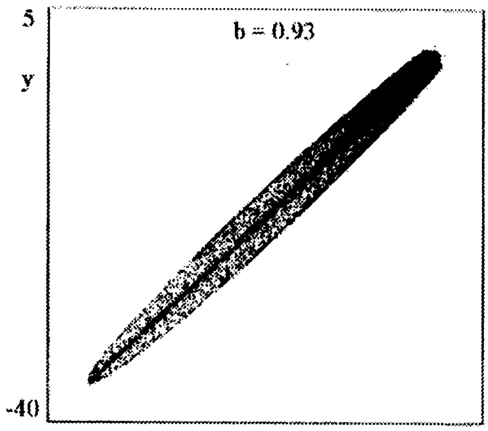

Fig.6

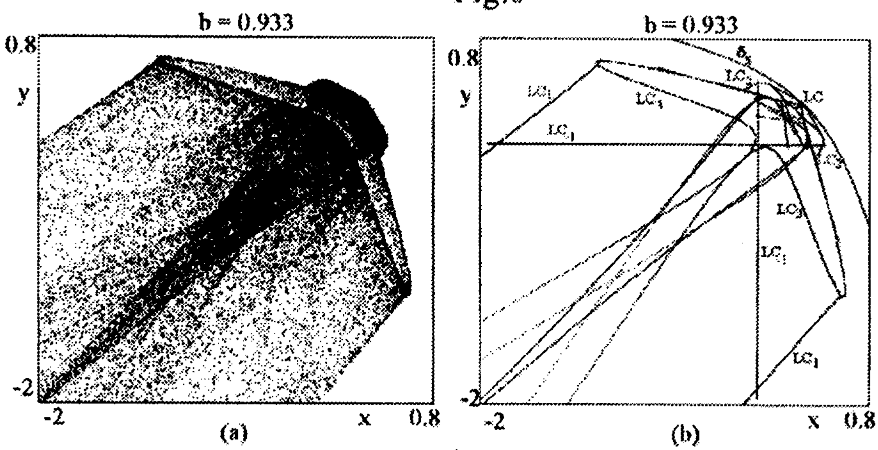

Fig.7

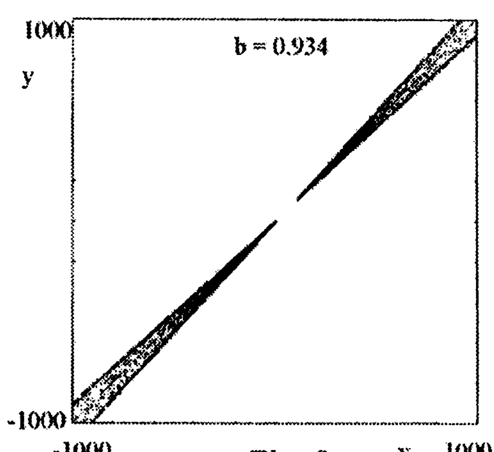

Fig. 8

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