SOME CHARACTERIZATIONS ABOUT SIEGEL CURVES

MOHAMED HMISSI

To the memory of Professor György Targonski

Abstract. Let $\Psi : \mathbb{R} \times T \mapsto T$ be a continuous dynamical system on the two-dimensional torus T. The aim of this paper is to prove some characterizations about the existence of a Siegel curve, i.e. a simple closed curve which cuts every half-trajectory of the dynamical system (T, Ψ). This result completes and precises some results obtained in our article [7].

0. Introduction

We note by P the plane by T the two-dimensional torus and by $p : P \mapsto T$ the canonical projection. Let $\Psi : \mathbb{R} \times T \mapsto T$ be a continuous dynamical system (DS) on T. It is known that there exists a unique DS $\Phi : \mathbb{R} \times P \mapsto P$ such that p is an SD-morphism. A simple closed curve Λ of T is a Siegel curve of (T, Ψ) if Λ cuts continuousely every half-trajectory of the dynamical system (T, Ψ).

Such curves are constructed first by C. L. Siegel [8] but only if Ψ is defined by a vector field with continuous derivative of the second order. Since this fondamental work, Siegel curves play an important role in ergodic theory (cf. [3]). Recently, the existence of Siegel curves is proved in our paper [7], for (only) continuous dynamical systems.

In fact, it is supposed both in [7] and [8] that the dynamical system Ψ is without periodic trajectories. Moreover, this condition is not necessary to obtain Siegel curves (cf. Example 2.1 above). Hence a natural problem

Received: April 27, 1999.

AMS (1991) subject classification: Primary 58F25.

is to characterize dynamical systems on the torus T which possesses Siegel curves. This is the purpose of this paper.

More precisely, we prove the equivalence of the following properties:

1. Λ is a Siegel curve for (T, Ψ).

2. The mapping $f : \mathbb{R} \times \Lambda \mapsto T$ defined by $f(t, x) = \Psi(t, x)$ is an SD-morphism (whenever $\mathbb{R} \times \Lambda$ is endowed with the parallel DS).

3. The DS (P, Φ) is parallelizable.

This characterizations complete and precise some results obtained in our paper [7].

I want to thank M. Sieveking for many discussions on these and related topics.

1. Preliminaries

In this paragraph, we collect some facts about dynamical system which we need later. The general references are [1] and [2].

1.1. Dynamical systems. A dynamical system (DS) on a topological space X is a continuous mapping $\Phi : \mathbb{R} \times X \mapsto X$ such that

$$\Phi(0, x) = x \quad \text{and} \quad \Phi(s + t, x) = \Phi(s, \Phi(t, x))$$

for all $x \in X$ and $s, t \in \mathbb{R}$. Such a system is always denoted by the pair (X, Φ).

Dynamical systems are natural generalizations of solutions of ordinary differential equations (the associated DS are namely differentiable). But many other sources (for example in iteration theory) give DS which are only continuous.

1.2. Trajectories. Let (X, Φ) be a DS.

1. Let $x \in X$, the trajectory throught x and the trajectory from x, are defined respectively by

$$C(x) := \{\Phi(t, x) : \ t \in \mathbb{R} \}; \quad C^+(x) := \{\Phi(t, x) : \ t \geq 0 \}.$$

2. A subset U of X is said to be *invariant* if $C(x) \subset U$ for every $x \in U$. If U is invariant then the restriction of Φ to $\mathbb{R} \times U$ defines a DS on U which is denoted by (U, Φ).

A trajectory $C(x)$ is *periodical* if there is $a \in]0, \infty[$ such that $\Phi(a, x) = x$.

1.3. Morphism of DS

1. Let (X, Φ) and (Y, Ψ) be two DS; a continuous maping $p : X \mapsto Y$ is called *DS-morphism* from (X, Φ) into (Y, Ψ) if $p(\Phi(t, x)) = \Psi(t, p(x))$ for every $x \in X$ and $t \in \mathbb{R}$.

2. Let $p : (X, \Phi) \mapsto (Y, \Psi)$ be a DS-morphism. If p is a bijection from X into Y and p^{-1} is a DS-morphism then p is said to be a *DS-homeomorphism*. In this case (X, Φ) and (Y, Ψ) are said to be *DS-homeomorphic*.

1.4. Parallelization of DS

1. On a topological space of the form $\mathbb{R} \times Z$, the so called *parallel* DS is denoted by $(\mathbb{R} \times Z, \Theta)$ and defined by $\Theta(t, (s, z)) = (s + t, z)$ for $z \in Z$ and $s, t \in \mathbb{R}$.

2. A DS (X, Φ) is called *parallelizable* iff it is DS-homeomorphic to a parallel DS. In this case, we say that X is parallelizable.

2. Siegel curves on the torus

Let $P := \mathbb{R}^2$ be the plane, let $T := \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus, let S be the unit circle of P and let $p : P \mapsto T$ the canonical projection. If (T, Ψ) is a dynamical system, it is known ([1], p. 222) that there exists a unique DS (P, Φ) such that p is an SD-morphism.

We say that a simple closed curve Λ of T is a *Siegel curve* of the DS (T, Ψ) if the mapping $\tau : T \mapsto]0, \infty[$ defined by

$$\tau(y) := \inf\{t > 0 : \ \Psi(t, y) \in \Lambda\}$$

is continuous.

EXAMPLE 2.1. We endow P with the parallel DS denoted by $(\mathbb{R} \times \mathbb{R}, \Theta)$. Let $A := (\alpha_{m,n})_{1 \leq m,n \leq 2} \in \mathcal{M}_2(\mathbb{R})$ and $q : P \mapsto T$ defined by (using the notation $\mathrm{i}^2 = -1$)

$$\forall\ a, b \in \mathbb{R} : \ q(a, b) := (\exp(\mathrm{i}a\alpha_{11} + b\alpha_{12}), \exp(\mathrm{i}a\alpha_{21} + b\alpha_{22})).$$

Define the DS (T, Ψ) on the torus by

$$\forall\ u, v \in S\ ,\ t \in \mathbb{R} : \ \Psi(t, (u, v)) := (u\exp(\mathrm{i}t\alpha_{11}), v\exp(\mathrm{i}t\alpha_{21})).$$

It is easily proved that $q : (P, \Theta) \mapsto (T, \Psi)$ is an SD-morphism.

REMARK 2.2. In our paper [7], we have proved that (T, Ψ) posseses a Siegel curve whenever it is without periodical trajectories. But the last condition is not necessary; it suffices to choose convenient coefficients α_{ij} in the preceeding example. It is also a natural problem to characterize the existence of Siegel curve for a given DS (T, Ψ).

From now, we fix a DS (T, Ψ) and we denote by (P, Φ) the associated DS. We fix also a simple closed curve Λ in T. Finally, we endow $\mathbb{R} \times \Lambda$ with the parallel dynamical system Θ.

LEMMA 2.3. *Let $h : (X, \Phi) \mapsto (\mathbb{R} \times Z, \Theta)$ a local DS-homemorphism, then (X, Φ) is parallelizable.*

PROOF. It suffices to prove that h is injective: For $x \in X$, let us write $h(x) = (t(x), \alpha(x)) \in \mathbb{R} \times Z$. Let $x, y \in X$ be such that $h(x) = h(y)$. If $y = \Phi(s, x)$ for some $s \in \mathbb{R}$ then

$$\begin{aligned}(t(y), \alpha(y)) &= h(y) = h(\Phi(s, x)) = \Theta(s, h(x)) \\ &= \Theta(s, (t(x), \alpha(x)) = (s + t(x), \alpha(x)).\end{aligned}$$

Hence $\alpha(y) = \alpha(x)$ and $s + t(x) = t(x)$ and so $y = x$.

For the general case, it can be similary proved that $h(x) = h(y)$ implies that $h(C(x)) = h(C(y))$. For $z \in X$, let $V(z)$ a neighborhood of z on which h is injective. Now, if $y \notin C(x)$ then

$$C(y) \cap \bigcup_{z \in C(x)} V(z) = \emptyset.$$

But, this gives a contradiction.

LEMMA 2.4. *Suppose that the mapping $f : \mathbb{R} \times \Lambda \mapsto T$ defined by $f(t, x) := \Psi(t, x)$ is a local DS-homemorphism. Then there exsits a local DS-homeomorphism $q : (P, \Phi) \mapsto (\mathbb{R} \times \Lambda, \Theta)$.*

PROOF. First, we use some well known facts about Riemannian surfaces: Since $p : P \mapsto T$ is an universal covering of T (cf [4], 5.11) and $f : \mathbb{R} \times \Lambda \mapsto T$ is a covering of T, then there exists a covering $q : P \mapsto \mathbb{R} \times \Lambda$ such that $p = f \circ q$. In fact q is a local homeomorphism because p, f are so.

Now we prove that $q : (P, \Phi) \mapsto (\mathbb{R} \times \Lambda, \Theta)$ is a DS-morphism: Let $x \in X$ and

$$I(x) := \{t \in \mathbb{R} : \ q(\Phi(t, x)) = \Theta(t, q(x))\}.$$

It is clear that $I(x)$ is closed in $\mathbb{R}$ and $0 \in I(x)$. It remains to prove that $I(x)$ is open in $\mathbb{R}$. Let $t \in I(x)$, $y := \Phi(t, x)$ and V an open neighborhood of $q(y)$ such that the restriction of p ot V is an homeomorphism. Hence there exist $r \in]0, \infty[$ such that

$$\forall \ |s| < r : \ q(\Phi(s, y)) \quad \text{and} \quad \Theta(s, q(y)) \in V.$$

Let us prove that

$$\forall \ |s| < r : \ q(\Phi(s, y)) = \Theta(s, q(y)).$$

This is equivalent to prove

$$\forall\ |s| < r:\ f \circ q(\Phi(s,y)) = f(\Theta(s, q(y))).$$

Since p and f are SD-morphism and since $f \circ q = p$, then the last relation is verified. Indeed for all $|s| < r$, we have

$$f \circ q(\Phi(s,y)) = p(\Phi(s,y)) = \Psi(s, p(y)) = \Psi(s, f \circ q(y)) = f(\Psi(s, q(y))).$$

Finally $I(x)$ is open in $\mathbb{R}$.

THEOREM 2.5. *The following properties are equivalent:*

(i) *Λ is a Siegel curve of (T, Ψ).*

(ii) *The mapping $f : \mathbb{R} \times \Lambda \mapsto T$ defined by $f(t,x) := \Psi(t,x)$ is a local DS-homemorphism.*

(iii) *(P, Φ) is parallelizable.*

PROOF. (i)$\Rightarrow$ (ii): Since $\Theta(s,(t,x)) = (s+t, x)$ for all $s,t \in \mathbb{R}$ and $x \in \Lambda$, it follows that

$$f(\Theta(s,(t,x))) = f(s+t,x) = \Psi(s+t,x) = \Psi(s, \Psi(t,x)) = \Psi(s, f(t,x))$$

for all $s,t \in \mathbb{R}$ and $x \in \Lambda$. Hence f is a DS-morphism.

Let now $\tau : T \mapsto]0,\infty[$ defined by $\tau(y) := \inf\{t > 0 :\ \Psi(t,y) \in \Lambda\}$. Since $\tau : T \mapsto \Lambda$ is continuous and Λ is compact then $\beta := \inf\{\tau(x) : x \in \Lambda\} > 0$. Let us prove first that $f :]-\beta/3, \beta/3[\times\Lambda \mapsto T$ is injective: Let $x, y \in \Lambda$ and $s,t \in]-\beta/3, \beta/3[$ such that $\Psi(t,x) = \Psi(s,y)$. Then $y = \Psi(t-s, x)$ and $|s-t| < \beta$. In view of the definition of τ, this is possible only if $s - t = 0$ and hence $x = y$. In order to conclude that f is a local homemorphism, it suffices to remark that $\Psi(t_0, \Lambda)$ is also a Siegel curve for all $t_0 \in \mathbb{R}$.

(ii)$\Rightarrow$ (iii): Lemmas 2.3 and 2.4.

(iii)$\Rightarrow$ (i): For the existence of a Siegel curve, let us recall the idea of the proof developped in our paper [7]: Without loss of the generality, it can be supposed that (P, Φ) is a parallel DS, i.e.

$$\forall\ a,b,t \in \mathbb{R}:\ \Phi(t,(a,b)) = (t+a, b).$$

For $r \in]0,\infty[$, let $B_r := \{x \in P :\ \|x\| < r\}$. Since p is a local homeomorphism then $\{p(B_r) :\ r > 0\}$ is a covering of T by open sets. Since Y is compact, then there exists $\eta \in]0,\infty[$ such that $p(B_\eta) = Y$. Let $x = (x_1, x_2) \in P$ such that $|x_2| > \eta$, then there exist $x' \in B_\eta$ for which $p(x) = p(x')$. Since p is a local homeomorphism, x' can be choosen such that the closed curve

$\{p(sx + (1-s)x') : \ s \in [0, 1[\}$ is simple and hence homeomorphic to Λ. For all $y \in T$, let

$$\tau(y) := \inf\{t > 0 : \ \Psi(t, y) \in \Lambda\}.$$

It is also proved in our paper [7] that $\tau : T \mapsto]0, \infty[$. For the continuity of τ, the proof is the same as in classical case where Ψ is defined by a vector field (cf. [3], p.414).

REMARK. If the DS (T, Ψ) possesses a Siegel curve Λ and a periodic trajectory C then C must cut Λ in only one point. On the other hand this condition is sufficient in many examples. Hence we conjecture the following:

Λ is a Siegel curve of (T, Ψ) if and only if every periodic trajectory of (T, Ψ) cuts Λ in exactly one point.

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DÉPARTEMENT DE MATHÉMATIQUES
FACULTÉ DES SCIENCES DE TUNIS
TN-1060 BELVEDERE TUNIS
TUNISIA

e-mail: med.hmissi@fst.rnu.tn