DISTRIBUTIONAL CHAOS FOR CONTINUOUS MAPPINGS OF THE CIRCLE

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To the memory of Professor György Targonski

Abstract. We show that theory of distributional chaos for continuous functions on the unit interval as developed recently by Schweizer and Smítal remains essentially true for continuous mappings of the circle, with natural exceptions.

1. Introduction

Let $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ be the circle and $C(\mathbb{S}, \mathbb{S})$ the set of continuous mappings of the circle into itself. Denote by $\Pi : \mathbb{R} \to \mathbb{S}$ the natural projection defined by $\Pi(x) = x - [x]$, where $[x]$ is the integer part of x. Define a metric on the circle by $\|x_1, x_2\| = \min\{|y_1 - y_2|; x_i = \Pi(y_i)\}$. For $f \in C(\mathbb{S}, \mathbb{S})$, $x, y \in \mathbb{S}$, and $i \in \mathbb{N}$, denote by $\delta_{x,y}(i) = \|f^i(x), f^i(y)\|$ the distance of the iterations. For real t, and any positive integer n denote

$$(1.1) \quad \xi(x, y, t, n) = \sum_{i=0}^{n-1} \chi_{[0,t)}(\delta_{xy}(i)) = \#\{i;\ \ 0 \leq i < n \text{ and } \delta_{xy}(i) < t\},$$

$$(1.2) \qquad F_{xy}^*(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, t, n),$$

Received: January 5, 1999.

AMS (1991) subject classification: Primary 26A18, 58F13.

The research was supported, in part, by the Grant Agency of Czech Republic, grant No. 201/97/0001.

and

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, t, n). \tag{1.3}$$

Clearly both F^*_{xy}, F_{xy} are nondecreasing functions such that $F^*_{xy}(t) = F_{xy}(t) = 0$ for $t < 0$, and $F^*_{xy}(t) = F_{xy}(t) = 1$ for $t > 1$. We identify two functions that coincide everywhere except at a countable set. We take a convention to chose functions F^*_{xy}, F_{xy} as left-continuous. Functions F^*_{xy}, F_{xy} are called, respectively, the *upper* and *lower distribution function* of x and y. A function f *exhibits distributional chaos* if there are points $x, y \in \mathbb{S}$ such that $F^*_{xy}(t) = 1$ for all $t > 0$ and there is a point $s \in (0,1)$ such that $F^*_{xy}(s) > F_{xy}(s)$.

A function $f \in C(\mathbb{S},\mathbb{S})$ has a *horseshoe* if there are disjoint compact intervals U, V such that $f(U) \cap f(V) \supset U \cup V$.

The set $C(\mathbb{S},\mathbb{S})$ can be decomposed into the following disjoint sets (as is shown in [4], [3] and [7]):

$W_0 = \{f \in C(\mathbb{S},\mathbb{S});\ \mathrm{Per}\,(f) = \emptyset\}$,
$W_1 = \{f \in C(\mathbb{S},\mathbb{S});\ \mathrm{P}\,(f^n) = \{1\}$, for some $n \in \mathbb{N}\}$,
$W_2 = \{f \in C(\mathbb{S},\mathbb{S});\ \mathrm{P}\,(f^n) = \{2^i, i = 0, 1, 2, \ldots\}$, for some $n \in \mathbb{N}\}$,
$W_3 = \{f \in C(\mathbb{S},\mathbb{S});\ \mathrm{P}\,(f^n) = \mathbb{N}$, for some $n \in \mathbb{N}\}$.

LEMMA 1.1. *Let $f \in C(\mathbb{S},\mathbb{S})$, let $\tilde\omega = \omega_f(x)$ be an infinite maximal ω-limit set, and let U be an open interval containing a point $a \in \tilde\omega$. Denote $P_U = \bigcap_{n=0}^{\infty}\bigcup_{i=n}^{\infty} f^i(U)$. Then*

$$P_U = J_1^U \cup \ldots \cup J_{n(U)}^U, \tag{1.4}$$

where J_i^U are pairwise disjoint periodic intervals forming a periodic orbit, and $n(U)$ is a positive integer.

PROOF. There are $k < l$ such that $f^k(x), f^l(x) \in U$. Hence, $f^{l-k}(U) \cap U \neq \emptyset$ and consequently, P_U is the union of intervals J_i^U as in (1.4) with $n(U) \le l-k$. Moreover, $f(P_U) = f(\bigcap_{n=0}^{\infty}\bigcup_{i=n}^{\infty} f^i(U)) = \bigcap_{n=0}^{\infty} f(\bigcup_{i=n}^{\infty} f^i(U)) = \bigcap_{n=0}^{\infty}\bigcup_{i=n}^{\infty} f^{i+1}(U) = P_U$. Hence, P_U is invariant. It is a periodic orbit of intervals since trajectory $\{f^i(x)\}_{i=0}^{\infty}$ visits any J_j^U. □

Lemma 1.1 implies the following classification of maximal ω-limit sets. If $\tilde\omega$ is finite then it is a *cycle*. If $\tilde\omega$ is infinite and $\lim_{|U|\to 0} n(U) = \infty$ then $\tilde\omega$ is *solenoid.* If $\tilde\omega$ is infinite, $\lim_{|U|\to 0} n(U) < \infty$ and $\mathrm{Per}\, f \cap \tilde\omega \neq \emptyset$ then $\tilde\omega$ is a *basic set.* If $\tilde\omega$ is infinite, $\lim_{|U|\to 0} n(U) < \infty$ and $\mathrm{Per}\, f \cap \tilde\omega = \emptyset$ we call $\tilde\omega$ a *singular set.* It is well-known that for the continuous maps of the interval only the first three types can occur, i.e., singular sets are impossible (cf., e.g., [8] for references).

2. Main results

THEOREM 2.1. *Let* $f \in C(\mathbb{S},\mathbb{S})$. *If* $\omega_f(u) \subset \tilde{\omega}_u$, $\omega_f(v) \subset \tilde{\omega}_v$, *where* $\tilde{\omega}_u, \tilde{\omega}_v$ *are solenoids or cycles, then*

(i) $F_{uv} = F^*_{uv}$;

(ii) *if, in addition,* $\liminf_{i\to\infty} \delta_{uv}(i) = 0$, *then* $F_{uv} = \chi_{(0,\infty)}$, *i.e.,* $F_{uv}(t) = 1$, *for all* $t > 0$.

PROOF. The proof is analogous to the proof of a similar theorem in [8] for mappings in $C(I,I)$. It is based on the fact that a solenoid has decomposition into periodic portions of arbitrarily high period, and hence, that most of the periodic portions have small diameter. □

THEOREM 2.2. *For* $f \in C(\mathbb{S},\mathbb{S})$, *the following conditions are equivalent.*

(i) *Function* f *has positive topological entropy.*

(ii) *Function* f^n *has a horseshoe, for some* $n \in \mathbb{N}$.

(iii) *Function* f *belongs to* W_3.

(iv) *Function* f *exhibits distributional chaos.*

(v) *Function* f *has a basic set.*

THEOREM 2.3. *Let* $f \in C(\mathbb{S},\mathbb{S})$ *have zero topological entropy. Then* $F_{uv} = F^*_{uv}$, *for all* u, v *in* $\mathbb{S}$. *If in addition,* $\liminf_{i\to\infty} \delta_{uv}(i) = 0$, *then* $F_{uv} = \chi_{(0,\infty)}$, *i.e.,* $F_{uv}(t) = 1$ *for all* $t > 0$.

PROOF. If a function has zero topological entropy then, by Theorem 2.2, it has no basic set and the result follows from Lemma 3.1 and Theorem 2.1. □

Similar result, for mappings on the interval, is proved in [8].

3. Properties of distributively chaotic functions

LEMMA 3.1. *If* $\tilde{\omega} = \omega_f(x)$ *is a singular set then* f *belongs to* W_0.

PROOF. Let $\tilde{\omega}$ be a singular set. If $\overline{P_U}$ contains a periodic interval $J \neq \mathbb{S}$ of period m then the map $g = f^m|_J$ is conjugate to a one-dimensional map on the interval and hence, $\tilde{\omega}$ would not be a singular set. Thus $\overline{P_U} = \mathbb{S}$, for every U. Assume that f has a periodic point q_0 of period n such that $\operatorname{Orb}(q_0) = \{q_0, q_1, \ldots, q_{n-1}\}$ and $f(q_k) = q_{k+1}$, where k is taken modulo n. Let V_i be the maximal open interval disjoint from $\tilde{\omega}$ such that $q_i \in V_i$, for $i = 0, 1, \ldots, n-1$. Denote $V = V_0 \cup V_1 \cup \ldots \cup V_{n-1}$. Then $f(V) \subsetneq V$ is impossible (otherwise $f(\tilde{\omega}) \not\supseteq \tilde{\omega}$) as well as $f(V) = V$ (because then the end points of V, which belong to $\tilde{\omega}$, would be periodic). Thus, $f(V) \subsetneq V$, so $f(V_i) \supset V_{i+1}$, for any i modulo n, and there is a k such that $f(V_k) \setminus V_{k+1}$ contains an interval. Without loss of generality we may assume that

$$f(V_0) \setminus V_1 \text{ contains an interval.} \tag{3.1}$$

Let $\{V_0^k\}_{k=0}^{\infty}$ be the system of maximal intervals in V_0 such that $f(V_0^0) = V_1$ and $f^n(V_0^{k+1}) = V_0^k$. Then $V_0 \supsetneq V_0^0 \supsetneq V_0^1 \supsetneq \dots$. Let $\mathcal{K}$ be the system of component intervals of $V_0^k \setminus V_0^{k+1}$, for $k \geq 0$. If $L \in \mathcal{K}$ is contained in $V_0^k \setminus V_0^{k+1}$ then, by (3.1), $f^{kn+1}(L)$ contains a neigborhood U_k of a point of $\tilde{\omega}$, for any k. Denote $V_0^{\infty} = \bigcap_{k=0}^{\infty} \overline{V_0^k}$. The end points of V_0^{∞} are periodic since $f^n(V_0^{\infty}) = V_0^{\infty}$. One of them, say a point b, is an accumulation point of the system $\mathcal{K}$. Consequently, there are sequences of compact intervals $\{U_k\}_{k=1}^{\infty}$ and $\{K_k\}_{k=1}^{\infty}$ converging to a and b, respectively such that $f^{m_k}(U_k) \supset K_k$ and $f^{n_k}(K_k) \supset V_k$. By the Itinerary lemma (cf. also Lemma 3.4 from [8]), there is an x such that $\omega_f(x) \supset \tilde{\omega} \cup \{b\}$. By the maximality of $\tilde{\omega}$, $b \in \tilde{\omega}$ which is a contradiction since b is periodic. □

LEMMA 3.2. *If every $\tilde{\omega}$ is either a cycle or a solenoid then f is not distributionally chaotic.*

PROOF. It follows from Theorem 2.1. □

LEMMA 3.3. *Let $f \in C(\mathbb{S}, \mathbb{S})$. If, for some $n > 0$, f^n is distributionally chaotic then f is distributionally chaotic.*

PROOF. Let G_{uv}, G_{uv}^* be the lower and upper distribution function for f^n, respectively. Then $G_{uv}(t) < 1$ implies $F_{uv}(t) < 1$, where F_{uv} is the lower distribution function for f. Let F_{uv}^* be the upper distribution function for f. Since f is continuous, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|u, v\| < \delta$ implies $\delta_{uv}(i) < \varepsilon$, for $i = 0, 1, \dots, n-1$. Consequently, $G_{uv}^*(\delta) = 1$ implies $F_{uv}^*(\varepsilon) = 1$, and hence, $F_{uv}^* \equiv 1$. □

REMARK. Actually the two conditions in Lemma 3.3 are equivalent, but we do not need the other implication.

PROPOSITION 3.4. *If $f \in C(\mathbb{S}, \mathbb{S})$ has a basic set then f^k has a horseshoe, for some $k \in \mathbb{N}$.*

PROOF. Let $\tilde{\omega} = \omega_f(x)$ be a basic set, and $q \in \tilde{\omega}$ a periodic point of period m. By the definition there is $n \in \mathbb{N}$ such that $n(U) = n$, for any sufficiently small U. Let U be such a small interval with $q \in U$, and let

$g = f^{nm}$. Then P_U^g is an interval and q is a fixed point of g. If $P_U^g \neq \mathbb{S}$, for some U, then $\mathbb{S} \setminus P_U^g$ is an interval or a point. In the first case, $\overline{P_U^g} \neq \mathbb{S}$ and g restricted to P_U^g is conjugate to a one-dimensional map on the interval. Hence, g^k and consequently f^{kmn} has a horseshoe, for some k.

If $\mathbb{S} \setminus P_U^g$ is a singleton $\{a\}$ then a is a fixed point of f. Again, g is conjugate to a one-dimensional map $[a, a+1] \to [a, a+1]$ with a basic set and consequently, g and hence, f^{mn} has a horseshoe.

Finally, let $P_U^g = \mathbb{S}$, for every U. Then, for any U with $U \cap \tilde{\omega} \neq \emptyset$, $q \in \operatorname{Orb}(U)$. Hence, there are an $r \in \mathbb{N}$, and compact disjoint intervals V, H, containing x and q, respectively, such that $g^r(V) = g^r(H) =: W$. Since W contains $g^r(x)$ and q there is a $p \in \mathbb{N}$ such that $g^p(W) \supset W \cup V \cup H$. So g^k and consequently f^{mnk} has horseshoe, for some k. □

PROPOSITION 3.5. *If $f \in C(\mathbb{S}, \mathbb{S})$ has no basic set then f exhibits no distributional chaos.*

PROOF. If f has no basic set then every maximal ω-limit set is cycle, solenoid or singular set. If $\tilde{\omega}$ is a singular set then, by Lemma 3.1, f belongs to W_0. Since any $f \in W_0$ is semiconjugate to irrational rotation of the circle (cf., e.g., [5]), f cannot be distributionally chaotic. If every $\tilde{\omega}$ either is cycle or solenoid then, by Lemma 3.2, f is not distributionally chaotic. □

PROPOSITION 3.6. *If $f \in C(\mathbb{S}, \mathbb{S})$ has a horseshoe (i.e., there are disjoint compact intervals U, V such that $f(U) \cap f(V) \supset U \cup V$) then f is distributionally chaotic.*

PROOF. Since $U \cup V$ contains a compact invariant set M such that $f|_M$ is semiconjugate to the standard shift τ on the space $X = \{0, 1\}^{\mathbb{N}}$ of sequences two symbols (cf., e.g., [1]), it suffices to show that τ is distributionally chaotic. But this is easy (cf., e.g., [6]). □

4. Proof of Theorem 2.2

Equivalence of conditions (i), (ii) and (iii) is proved in [1]. By Proposition 3.6 and Lemma 3.3, (ii) implies (iv), and by Proposition 3.5 (iv) implies (v), finally by Proposition 3.4 (v) implies (ii). □

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