ON REGULAR MULTIVALUED COSINE FAMILIES

ANDRZEJ SMAJDOR

To the memory of Professor György Targonski

Abstract. Let K be a convex cone in a real normed space X. A one-parameter family $\{F_t : t \geq 0\}$ of set-valued functions $F_t : K \to n(K)$, where $n(K) := \{D : D \subset K, D \neq \emptyset\}$, is called *cosine* iff $F_{t+s} + F_{t-s} = 2F_t \circ F_s$, whenever $0 \leq s \leq t$ and F_0 is the identity map. A cosine family $\{F_t : t \geq 0\}$ is *regular* iff $\lim_{t\to 0+} F_t(x) = \{x\}$ for every x.

The growth and the continuity of regular cosine families are investigated.

Let X, Y, Z be nonempty sets and let $n(Y)$ denote the set of all nonempty subsets of Y. We recall that the superposition $G \circ F$ of set-valued functions $F : X \to n(Y)$ and $G : Y \to n(Z)$ is defined by the formula

$$(G \circ F)(x) := \bigcup\{G(y) : y \in F(x)\} \qquad \text{for} \quad x \in X.$$

A subset K of a real vector space X is called a *cone* if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be *convex* if it is a convex set.

A set-valued function $F : K \to n(Y)$, where K is a convex cone in X, is said to be *superadditive* iff $F(x) + F(y) \subset F(x+y)$ for $x, y \in K$.

Let K be a convex cone in X and let $\mathbb{Q}_+$ denote the set of all positive rational numbers. A set-valued function $F : K \to n(Y)$ is said to be $\mathbb{Q}_+$*-homogeneous* if $F(\lambda x) = \lambda F(x)$ for $\lambda \in \mathbb{Q}_+, x \in K$.

Now, we assume that X and Y are arbitrary real normed spaces. A set-valued function $F : K \to n(Y)$ is called *lower semicontinuous* at $x_0 \in$

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K iff for every open set V in Y such that $F(x_0) \cap V \neq \emptyset$ there exists a neighbourhood U of zero in X such that $F(x) \cap V \neq \emptyset$ for $x \in (x_0 + U) \cap K$. A set-valued function F is called lower semicontinuous iff it is lower semicontinuous at every point $x \in K$.

A set-valued function $F : K \to n(Y)$, is said to be *bounded* if for every bounded subset E of K the set $F(E) = \bigcup\{F(x) : x \in E\}$ is bounded in Y.

The following characterization of boundedness of $\mathbb{Q}_+$-homogeneous set-valued functions is easy to check.

LEMMA 1. *Let X and Y be two real normed spaces and let K be a convex cone in X. A $\mathbb{Q}_+$-homogeneous set-valued function $F : K \to n(Y)$ is bounded if and only if there exists a positive constant M such that*

$$\|F(x)\| := \sup\{\|y\| : y \in F(x)\} \le M\|x\| \qquad \text{for} \quad x \in K. \tag{1}$$

LEMMA 2. *Let X and Y be two real normed spaces and let K be a convex cone in X. Suppose that $F : K \to n(Y)$ is a $\mathbb{Q}_+$-homogeneous set-valued function. Then equality*

$$\lim_{x \to 0, x \in K} \|F(x)\| = 0.$$

holds if and only if there exists a positive constant M such that (1) holds.

The proof is similar as in the classical case (see [2] Theorem 2.4.1).

Under assumptions of Lemma 2 the functional

$$\|F\| = \sup_{x \in K, x \neq 0} \frac{\|F(x)\|}{\|x\|}$$

is finite for every $\mathbb{Q}_+$-homogeneous set-valued function $F : K \to n(Y)$ such that

$$\lim_{x \to 0, x \in K} \|F(x)\| = 0.$$

This functional will be called a *norm*.

COROLLARY 1. *Let X and Y be two real normed spaces and let K be a convex cone in X. Suppose that $F : K \to n(K)$ and $G : K \to n(Y)$ are bounded $\mathbb{Q}_+$-homogeneous set-valued functions. Then $G \circ F$ is bounded, $\mathbb{Q}_+$-homogeneous and inequality*

$$\|G \circ F\| \le \|G\| \|F\|$$

holds.

The set of all nonempty bounded subsets of a normed space Y will be denote by $B(Y)$.

LEMMA 3 (Theorem 3 in [7]). *Let X and Y be two real normed spaces and let K be a convex cone in X. Suppose that $(F_i : i \in I)$ is a family of superadditive lower semicontinuous in K and $\mathbb{Q}_+$-homogeneous set-valued functions $F_i : K \to n(Y)$. If $F(x) = \bigcup_{i\in I} F_i(x)$ and the set $B = \{x \in K : F(x) \in B(Y)\}$ is of the second category in K, then F is bounded and $B = K$.*

Lemma 3 and the same considerations as in the proof of Theorem 4 in [7] allow to derive the following lemma.

LEMMA 4. *Let X and Y be two real normed spaces and let K be a convex cone in X. Suppose that $(F_i : i \in I)$ is a family of superadditive lower semicontinuous in K and $\mathbb{Q}_+$-homogeneous set-valued functions $F_i : K \to n(Y)$. If K is of the second category in K and $\bigcup_{i\in I} F_i(x) \in B(Y)$ for $x \in K$, then there exists a constant $M \in (0, +\infty)$ such that*

$$\sup_{i\in I} \|F_i(x)\| \leq M\|x\| \; for \; x \in K.$$

REMARK 1. The assumption that the cone K is a set of the second category in K is essential and it can not be replaced by the completness of X.

In order to prove it we use an example from Chapter III, §3.7 of N. Bourbaki's book [1]. Let $X = \{x \in C(\mathbb{R},\mathbb{R}) : \lim_{t\to-\infty} x(t) = \lim_{t\to+\infty} x(t) = 0\}$, $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}\}$ for $x \in X$ and let $K = \{x \in X : \operatorname{supp} x \in c(\mathbb{R})\}$, where $c(\mathbb{R})$ is the set of all nonempty compact subsets of the set $\mathbb{R}$ of all real numbers. We can check that in this case $(X, \| \, \|)$ is a Banach space, K is a convex cone and set-valued functions $F_i : K \to n(\mathbb{R})$, $i = 1, 2, \ldots$ defined by formulas

$$F_i(x) = \{ix(i)\}$$

are additive, continuous and are $\mathbb{Q}_+$-homogeneous in K. Moreover, sets

$$\bigcup_{i\in\mathbb{N}} F_i(x) = \{ix(i) : i \in \mathbb{N}\}$$

are finite. So almost all assumptions of Lemma 4 hold except the "category" assumption. Let functions x_i, $i \in \mathbb{N}$ be defined as follows

$$x_i(t) = \begin{cases} 0, & \text{if } -\infty < t < i - \frac{1}{i}, \\ it + (1 - i^2), & \text{if } i - \frac{1}{i} \leq t \leq i, \\ -it + (1 + i^2), & \text{if } i < t \leq i + \frac{1}{i}, \\ 0, & \text{if } i + \frac{1}{i} < t. \end{cases}$$

We see that every x_i belongs to K, $F_i(x_i) = \{i\}$ and $\|x_i\| = 1$ for every $i \in \mathbb{N}$. Therefore the assertion of Lemma 4 does not hold.

REMARK 2. A convex cone K in Lemma 4 is of the second category in K if one of the three following cases holds true:

a) X is a Banach space and $\mathrm{int}K \neq \emptyset$,

b) X is a Banach space and K is closed,

c) X is a normed space and $\dim K = \dim (K - K) < +\infty$.

Cases a) and b) are obvious. In case c), let $n = \dim (K - K)$. Then there exist a basis $\{c_1 - d_1, ..., c_n - d_n\}$ of $\mathrm{lin}K = K - K$, such that the set $\{c_1, ...c_n, d_1, ...d_n\}$ is a subset of K. This subset is a spanning set of $K - K$, therefore it contains a basis $\{e_1, ..., e_n\} \subset K$ of $K - K$. The formula $\|x\| = \sum_{i=1}^{n} |\xi_i|$, for $x = \xi_1 e_1 + ... + \xi_n e_n$, defines a norm in $K - K$. It is easy to check that the ball $B(x_0, r_0)$ centered at $x_0 = \frac{1}{n}e_1 + ... + \frac{1}{n}e_n$ with the radius $r_0 = \frac{1}{n}$ is a subset of K. So the interior of K is nonempty.

Let T and S be two metric spaces and let $c(S)$ denote the set of all compact elements of $n(S)$. The Hausdorff distance derived from the metric in S is a metric in $c(S)$. A set-valued function $F : T \to c(S)$ is said to be continuous iff it is continuous as a single-valued function from T into the metric space $c(S)$.

Let Y be a normed space. We denote by $cc(Y)$ the family of all convex members of $c(Y)$. Observe that each linear set-valued function with closed values has to have convex ones.

LEMMA 5. *Let X and Y be two real normed spaces and let d be the Hausdorff distance derived from the norm in Y. Suppose that K is a convex cone in X with nonempty interior. Then there exists a positive constant M_0 such that for every linear continuous set-valued function $F : K \to c(Y)$ the inequality*

$$d(F(x), F(y)) \leq M_0 \|F\| \|x - y\|$$

holds.

PROOF. Let "$\sim$" denote the Rådström's equivalence relation between pairs of members of $cc(Y)$ defined by the formula

$$(A, B) \sim (C, D) \Leftrightarrow A + D = B + C.$$

For any pair (A, B), $[A, B]$ denotes its equivalence class. All equivalence classes form a real linear space $\mathcal{Z}$ with addition defined by the rule

$$[A, B] + [C, D] = [A + C, B + D],$$

and scalar multiplication

$$\lambda[A, B] = [\lambda A, \lambda B]$$

for $\lambda \geq 0$ and

$$\lambda[A, B] = [-\lambda B, -\lambda A]$$

for $\lambda < 0$.

The functional

$$\|[A, B]\| := d(A, B),$$

is a norm in $\mathcal{Z}$ (see [5]).

Now, let $F : K \to c(Y)$ be a linear continuous set-valued function. Then the function $f : K \to \mathcal{Z}$ given by

$$f(x) = [F(x), \{0\}],$$

is linear. Moreover, let $x_0 \in K$ and (x_n) be a sequence of elements of K such that $x_0 = \lim_{n\to\infty} x_n$. Then $F(x_0) = \lim_{n\to\infty} F(x_n)$ and

$$\lim_{n\to\infty} \|f(x_n) - f(x_0)\| = \lim_{n\to\infty} d(F(x_n), F(x_0)) = 0,$$

so f is continuous. The function f can be extended to a linear function $\hat{f} : X \to \mathcal{Z}$. This function is also continuous. Therefore

$$\lim_{x\to 0, x\in K} f(x) = \lim_{x\to 0} \hat{f}(x) = \hat{f}(0) = 0$$

and

$$\lim_{x\to 0, x\in K} d(F(x), \{0\}) = \lim_{x\to 0, x\in K} \|[F(x), \{0\}]\| = \lim_{x\to 0} \|f(x)\| = 0.$$

By Lemmas 1 and 2, F and f are bounded. Fix a $z \in \operatorname{int} K$. There exists an $\epsilon > 0$ such that $\frac{1}{\epsilon}z + S \subset K$, where S is the closed unit ball in X. If $v \in S$ and $u = \frac{1}{\epsilon}z + v$, then $u \in K$ and $\|u\| \leq \|\frac{1}{\epsilon}z\| + 1$, therefore

$$\begin{aligned}\|\hat{f}(v)\| &= \|\hat{f}(u) - \hat{f}(\frac{1}{\epsilon}z)\| \leq \|f(u)\| + \|f(\frac{1}{\epsilon}z)\| \\ &\leq \|f\|(\|u\| + \|\frac{1}{\epsilon}z\|) \leq \|f\|(1 + 2\|\frac{1}{\epsilon}z\|).\end{aligned}$$

Take $x, y \in K, x \neq y$. Since $\frac{x-y}{\|x-y\|} \in S$, we have

$$\|f(x) - f(y)\| = \|x - y\|\|\hat{f}(\frac{x-y}{\|x-y\|})\| \leq \|f\|M_0\|x - y\|,$$

where $M_0 := 1 + 2\|\frac{1}{\epsilon}z\|$. This implies that

$$d(F(x), F(y)) = \|[F(x), F(y)]\| = \|f(x) - f(y)\| \leq M_0\|F\|\|x - y\|.$$

This is a stronger version of Lemma 16 in [3](see also Lemma 7 in [4]). The application of the Rådström's equivalence relation allows to omit the assumption that X is a separable Banach space. This is an idea of dr Joanna Szczawińska.

LEMMA 6 (Lemma 1.9 in [6]). *Let X be a metric space with a metric ρ and let F be a set-valued function from X into X. If for a positive number M the inequality*

$$d(F(x), F(y)) \leq M\rho(x, y)$$

holds for every $x, y \in X$, then

$$d(F(A), F(B)) \leq Md(A, B)$$

for every nonempty subsets A, B of X, where d is the Hausdorff distance derived from the metric ρ.

Let $(K, +)$ be a semigroup. A one-parameter family $\{F_t : t \geq 0\}$ of set-valued functions $F_t : K \to n(K)$ is said to be a *cosine* family iff

$$F_0 = I,$$

where I denotes the identity map and

$$F_{t+s} + F_{t-s} = 2F_t \circ F_s, \tag{2}$$

whenever $0 \leq s \leq t$.

EXAMPLES:
1. $K = (-\infty, +\infty)$, $F_t(x) = x[\cos t, \cosh t]$.
2. $K = (-\infty, +\infty)$, $F_t(x) = x[\cos t, 1]$.
3. $K = [0, +\infty)$, $F_t(x) = x[1, \cosh t]$.

Let X be a real normed linear space. A cosine family $\{F_t : t \geq 0\}$ is *regular* iff

$$\lim_{t \to 0+} d(F_t(x), \{x\}) = 0,$$

where d is the Hausdorff distance derived from the norm in X.

THEOREM 1. *Let X be a real normed space, and let K be a convex cone in X of the second category in K. If $\{F_t : t \geq 0\}$ is a regular cosine family of continuous superadditive $\mathbb{Q}_+$-homogeneous set-valued functions $F_t : K \to c(K)$, then there exist two constants $M \geq 0$ and $\omega \geq 0$ such that*

$$||F_t|| \leq Me^{\omega t} \qquad \textit{for} \quad t \geq 0.$$

PROOF. The proof will be divided into three steps.

1^o *There exists an* η, $0 < \eta \leq 1$ *such that the function* $t \mapsto \|F_t\|$ *is bounded for* $0 \leq t \leq \eta$.

Suppose that it is false. Then there is a sequence (t_n) satisfying conditions: $t_n > 0$, $\lim_{n\to\infty} t_n = 0$ and $\|F_{t_n}\| \geq n$ for $n = 1, 2, \ldots$. From Lemma 4 it follows that for some $x \in K$ the sequence $(\|F_{t_n}(x)\|)$ is unbounded contrary to the regularity of the family $\{F_t : t \geq 0\}$. Thus there exist an η, $0 < \eta \leq 1$ and $L > 0$ such that

$$\|F_t\| \leq L \qquad \text{for} \quad t \in [0, \eta].$$

Since $\|F_0\| = \|I\| = 1$ we have $L \geq 1$.

2^o *Let* $m \geq 1$ *be an arbitrary constant. If* $s \geq 0$ *and* $\|F_s\| \leq m$, *then for every* $n = 1, 2, \ldots$ *we have* $\|F_{ns}\| \leq (3m)^n$.

The proof is by induction on n. The case $n = 1$ is trivial. If $n = 2$ we obtain

$$\|F_{2s}\| \leq \|F_{s-s}\| + 2\|F_s\|^2 \leq 2m^2 + 1 \leq (3m)^2.$$

Now, we suppose that $n \geq 3$ and

$$\|F_{ks}\| \leq (3m)^k \ \mathit{for}\ 1 \leq k \leq n.$$

By (2) we have

$$\begin{aligned}\|F_{(n+1)s}(x)\| &= d(F_{(n+1)s}(x) + F_{(n-1)s}(x), F_{(n-1)s}(x)) \\ &= d(2F_{ns} \circ F_s(x), F_{(n-1)s}(x))) \\ &\leq (2\|F_{ns}\|\ \|F_s\| + \|F_{(n-1)s}\|)\|x\|\end{aligned}$$

for every $x \in K$. Consequently,

$$\begin{aligned}\|(F_{(n+1)s}\| = \sup_{x\neq 0, x\in K} \frac{\|(F_{(n+1)s}(x)\|}{\|x\|} &\leq 2\|F_{ns}\|\ \|F_s\| + \|F_{(n-1)s}\| \\ \leq 2m(3m)^n + (3m)^{n-1} \leq (3m)^{n+1} &\leq (3m)^{n+1}.\end{aligned}$$

Hence the desired inequality is proved for $n = 1, 2, \ldots$.

3^o For each $t > 0$ there exists one and only one positive integer n such that $(n-1)\eta \leq t \leq n\eta$. Now if we take $s = t/n$ and use 2^o, we obtain

$$\|F_t\| = \|F_{ns}\| \leq (3L)^n = (3L)^{t/\eta}(3L)^{n-t/\eta} \leq 3L(3L)^{t/\eta}.$$

Let us define $M := 3L$ and $\omega = (1/\eta)\, ln(3L)$. Then we see that the assertion of the theorem holds.

A cosine family $\{F_t : t \geq 0\}$ is *continuous* iff the function $t \mapsto F_t(x)$ is continuous for every $x \in K$.

THEOREM 2. *Let X be a real Banach space and let K be a convex cone in X such that* $\mathrm{int}K \neq \emptyset$. *If $\{F_t : t \geq 0\}$ is a regular cosine family of continuous additive set-valued functions $F_t : K \to cc(K)$, then it is continuous.*

PROOF. The proof will be divided into eight steps.

1^o We assume that there exist $x_0 \in K$ and $t_0 \in [0, +\infty)$ such that the function $t \mapsto F_t(x_0)$ is discontinuous at the point t_0. Since the considered cosine family is regular, t_0 is positive.

2^o Let us define

$$L_n := \sup\{d(F_t(x_0), F_s(x_0)) : |t - t_0| \leq \frac{t_0}{8n}, |s - t_0| \leq \frac{t_0}{8n}, t \geq 0, s \geq 0\}.$$

for every positive integer n.

3^o *There exists $L > 0$ such that $L_n \geq L$ for every n.*

Obviously (L_n) is a non-negative and non-increasing sequence. Hence there exists a

$$\bar{L} = \lim_{n\to\infty} L_n \in \mathbb{R}.$$

We see that $L_n \geq \bar{L}$ for any n. Suppose that $\bar{L} = 0$. For every $\epsilon > 0$ there exists a positive integer n such that

$$d(F_t(x_0), F_s(x_0)) < \epsilon,$$

whenever $|t - t_0| \leq \frac{t_0}{8n}$, $|s - t_0| \leq \frac{t_0}{8n}$, $t > 0$ and $s > 0$. This implies that the function $t \mapsto F_t(x_0)$ is continuous at t_0 contrary to our assumption 1^o. Therefore $\bar{L} > 0$ and we take $L = \bar{L}$.

4^o For every positive integer n there exist two positive numbers s and t such that $|s - t_0| < \frac{t_0}{8n}$, $|t - t_0| < \frac{t_0}{8n}$ and $d(F_t(x_0), F_s(x_0)) \geq L_n - \frac{1}{n} > 0$. Hence $t \neq s$. Therefore there exist two sequences (t_n) and (s_n) such that $s_n > t_n > 0$, $|t_n - t_0| \leq \frac{t_0}{8n}$, $|s_n - t_0| \leq \frac{t_0}{8n}$ and $d(F_{t_n}(x_0), F_{s_n}(x_0)) \geq L_n - \frac{1}{n}$ for every n.

5^o $2t_n - s_n \in (0, +\infty)$ *for every n.*

It suffices to show that $t_n > s_n - t_n$ By 4^o clearly

$$s_n - t_n = (s_n - t_0) + (t_0 - t_n) \leq \frac{t_0}{4n}$$

and

$$t_n = t_0 - (t_0 - t_n) \geq \frac{8n-1}{8n} t_0 > \frac{t_0}{4n}.$$

6^o

$$d(F_{s_{4n}}(x_0), F_{2t_{4n}-s_{4n}}(x_0)) \leq L_n$$

for every $n = 1, 2, \ldots$.

From 4^o we have $|s_{4n} - t_0| \leq \frac{t_0}{32n} \leq \frac{t_0}{8n}$ and $|(2t_{4n} - s_{4n}) - t_0| = |2(t_{4n} - t_0) + (t_0 - s_{4n})| \leq 2|t_{4n} - t_0| + |t_0 - s_{4n}| \leq \frac{3t_0}{32n} \leq \frac{t_0}{8n}$. By 2^o we have the inequality.

7^o $\lim_{n\to\infty} L_n = 0$

We have

$$\begin{aligned} &2d(F_{t+s}(x_0), F_t(x_0)) \\ &\quad = d(2F_{t+s}(x_0) + 2F_{t-s}(x_0), 2F_t(x_0) + 2F_{t-s}(x_0)) \\ &\quad \leq d(2F_t \circ F_s(x_0) + F_{t+s}(x_0) + F_{t-s}(x_0), 2F_t(x_0) + 2F_{t-s}(x_0)) \\ &\quad \leq 2d(F_t \circ F_s(x_0), F_t(x_0)) + d(F_{t+s}(x_0), F_{t-s}(x_0)), \end{aligned}$$

whence

$$(3) \quad 2d(F_{t+s}(x_0), F_t(x_0)) \leq 2d(F_t \circ F_s(x_0), F_t(x_0)) + d(F_{t+s}(x_0), F_{t-s}(x_0))$$

According to Lemmas 5 and 6 the inequality

$$d(F_t[F_s(x_0)], F_t(x_0)) \leq M_0\|F_t\| d(F_s(x_0), \{x_0\})$$

holds for nonnegative t and s. Now we take $t = t_{4n}$, $s = s_{4n} - t_{4n}$ in (3). Then we obtain

$$\begin{aligned} &2d(F_{s_{4n}}(x_0), F_{t_{4n}}(x_0)) \\ &\quad = 2d(F_{t_{4n}+(s_{4n}-t_{4n})}(x_0), F_{t_{4n}}, (x_0)) \\ &\quad \leq 2d(F_{t_{4n}} \circ F_{s_{4n}-t_{4n}}(x_0), F_{t_{4n}}(x_0)) + d(F_{s_{4n}}(x_0), F_{2t_{4n}-s_{4n}}(x_0)) \\ &\quad \leq 2M_0\|F_{t_{4n}}\| d(F_{s_{4n}-t_{4n}}(x_0), \{x_0\}) + d(F_{s_{4n}}(x_0), F_{2t_{4n}-s_{4n}}(x_0)). \end{aligned}$$

Using 4^o and 6^o we have

$$2(L_{4n} - \frac{1}{4n}) \leq 2M_0\|F_{t_{4n}}\| d(F_{s_{4n}-t_{4n}}(x_0), \{x_0\}) + L_n.$$

Now, Theorem 1 implies that

$$2L_{4n} - L_n \leq 2M_0 M e^{\omega t_{4n}} d(F_{s_{4n}-t_{4n}}(x_0), \{x_0\}) + \frac{1}{2n}$$

for some $M \geq 0$ and $\omega \geq 0$ and for every n and we obtain the desired result.

Since 7^o contradicts 3^o we have proved our theorem.

REMARK 3. In the proof of Theorems 1 and 2 we have essentialy used ideas of M. Sova [8] for cosine operator functions.

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PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2
PL-30-084 KRAKÓW
POLAND