Annales Mathematicae Silesianae 12 (1998), 123–130

Prace Naukowe Uniwersytetu Śląskiego nr 1751, Katowice

ON THE DIOPHANTINE EQUATION $x_1x_2\cdots x_n = h(n)(x_1 + x_2 + \cdots + x_n)$

JOZSEF BUKOR, PÉTER FILAKOVSZKY, AND JÁNOS T. TÓTH*

Abstract. We are concerned with the equation of the title, where n is a fixed positive integer, $h(n)$ is a given integer-valued arithmetic function and the unknowns take positive integral values $1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. We estimate the number m for which $x_i = 1$ $(i = 1, 2, \ldots, m)$ in every solution. Next we give an upper bound for the number of coordinates of a solution which can be greater than 1. Further we estimate the number of all solution of the equation, and the paper concludes with a list of open problems.

Introduction

In the present paper we are concerned with the equation

$$x_1x_2\cdots x_n = h(n)(x_1 + x_2 + \cdots + x_n), \tag{1}$$

where n is a fixed positive integer, $h(n)$ is a given integer-valued arithmetic function and the unknowns $1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ take positive integral values. Equation (1) was first considered by Schinzel [3] in the case $h(n) = 1$. He observed that for every n there exists a trivial solution $(1, \ldots, 1, 2, n)$. There was studied a similar diophantine equation

$$\prod_{i=0}^{k} x_i - \sum_{i=0}^{k} x_i = n. \tag{2}$$

Received on June 18, 1998.

1991 *Mathematics Subject Classification.* 11D72.

* This research was partially supported by the Slovak Academy of Sciences Grant 5123.

Schinzel conjectured (quoted in [1], p. 238) that there is a $k > 1$ such that, for every sufficiently large n, (2) is solvable in integers $x_i > 1$. From the Viola's results (see [4]) it follows that for any $k > 1$, the asymptotic density of the natural numbers n for which (2) is unsolvable is zero.
It is easy to check that for $h(n) = n$ the all solutions of (1) are
in the case $n = 2$: (3,6), (4,4) and
in the case $n = 3$: (1,4,15), (1,5,9), (1,6,7), (2,2,12), (2,3,5), (3,3,3).
For $n \geqslant 4$ we have trivial solutions

$$(1,\ldots,1,n+1,2n^2-n), \quad (1,\ldots,1,n+2,n^2), \quad (1,\ldots,1,2n-1,3n),$$

$$(1,\ldots,1,2,n,2n-1), \quad (1,\ldots,1,3,n,n), \quad (1,\ldots,1,2n,3n-2).$$

First, we estimate the number m, for which $x_i = 1$ $(i = 1, 2, \ldots, m)$ in every solution of (1). Next we give an upper bound for the number of coordinates of a solution which can be greater than 1. Further we estimate the number of solutions of (1) and the paper concludes with some open problems and with a table of the numbers of solutions of the equation (1) for several functions $h(n)$.

1. Estimates

PROPOSITION 1. *Let the n-tuple $(x_1, x_2, \ldots, x_n)$ be a solution of (1) and suppose $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. If an integer k, $(1 \leqslant k < n)$, satisfies the inequality*

$$2\big(2^k - (k+1)h(n)\big) > h(n)(n-k-1), \tag{3}$$

then $x_1 = x_2 = \cdots = x_{n-k} = 1$.

PROOF. To the contrary, suppose that $x_n \geqslant x_{n-1} \geqslant \cdots \geqslant x_{n-m} \geqslant 2$, where $m \geqslant k$ and $x_1 = x_2 = \cdots = x_{n-m-1} = 1$. Then we have

$$2^m x_n \leqslant x_1 \cdots x_n = h(n)(x_1 + \cdots + x_n) \leqslant h(n)\big(n - (m+1) + (m+1)x_n\big)$$

which implies

$$x_n\big(2^m - (m+1)h(n)\big) \leqslant h(n)(n-m-1). \tag{4}$$

From (3) it immediately follows that $2^k > h(n)$. Using this inequality it is easy to show by mathematical induction that for any $m \geqslant k$,

$$2\big(2^m - (m+1)h(n)\big) > h(n)(n-m-1),$$

which contradicts (4). $\square$

Note that the inequality (3) is valid for example for $k > \log_2 2h(n)n$. The solution $x_1 = \cdots = x_{n-k-1} = 1$, $x_{n-k} = \cdots = x_n = 2$, where $n = 2^{k+1} - (k+1)$, of the equation $x_1x_2\cdots x_n = x_1+x_2+\cdots+x_n$ (the case $h(n) = 1$) shows that the inequality (3) is essentially the best possible.

PROPOSITION 2. *Let $(x_1, x_2, \dots, x_n)$ be a solution of* (1). *Then either*

$$x_{n-1} \leqslant h(n) \quad \text{or} \quad x_n \leqslant h(n)\big(h(n)+n-1\big).$$

PROOF. Assume that $x_{n-1} > h(n)$ and write (1) in the form

$$x_n = \frac{h(n)(x_1+x_2+\cdots+x_{n-1})}{x_1x_2\cdots x_{n-1} - h(n)}. \tag{5}$$

First we shall establish the inequality

$$x_n \leqslant \frac{h(n)(n-2+x_{n-1})}{x_{n-1}-h(n)}. \tag{6}$$

In view of (5) the inequality (6) is equivalent to

$$(x_1+\cdots+x_{n-1})(x_{n-1}-h(n)) \leqslant (n-2+x_{n-1})(x_1x_2\cdots x_{n-1}-h(n)). \tag{7}$$

However, we have the obvious inequalities

$$x_1+\cdots+x_{n-1} \leqslant (n-2)\,x_1\cdots x_{n-2} + x_1\cdots x_{n-1}, \tag{8}$$

$$-h(n)(x_1+\cdots+x_{n-1}) \leqslant -h(n)(n-2+x_{n-1}). \tag{9}$$

Putting (8) and (9) together we get (7), hence also (6). It is not hard to check that for any x'_{n-1} satisfying $x'_{n-1} > h(n)$ we have

$$\frac{h(n)(n-2+x_{n-1})}{x_{n-1}-h(n)} < \frac{h(n)(n-2+x'_{n-1})}{x'_{n-1}-h(n)} \quad \text{if and only if} \quad x'_{n-1} < x_{n-1}.$$

Hence when we replace in the upper bound (6) the number x_{n-1} by the smallest admissible value $h(n)+1$ we still obtain an upper bound for x_n which then reads $x_n \leqslant h(n)(n-2+x_{n-1}) = h(n)\big(h(n)+n-1\big)$. □

PROPOSITION 3. *Let $(x_1, \dots, x_n)$ be a solution of* (1). *Then for any positive integer k either*

$$x_{n-k} \leqslant \sqrt[k]{k\cdot h(n)} \quad \text{or} \quad x_{n-k}^{k+1} - (k+1)h(n)x_{n-k} \leqslant h(n)\cdot n.$$

PROOF. Assume that $x_{n-k}^k > k \cdot h(n)$ and write (1) in the form

$$(10)\qquad x_{n-k} = \frac{h(n)\cdot(x_1+\cdots+x_{n-k-1}+x_{n-k+1}+\cdots+x_n)}{x_1\cdots x_{n-k-1}x_{n-k+1}\cdots x_n - h(n)}.$$

We have the obvious inequalities

$$(11)\qquad \begin{aligned} &x_1+\cdots+x_{n-k-1}+x_{n-k+1}+\cdots+x_n \\ &\leqslant (n-k-1)x_1\cdots x_{n-k-1}+x_1\cdots x_{n-k-1}(x_{n-k+1}+\cdots+x_n) \end{aligned}$$

$$(12)\quad x_1+\cdots+x_{n-k-1}+x_{n-k+1}+\cdots+x_n \geqslant n-k-1+x_{n-k+1}+\cdots+x_n.$$

Multiplying (11) by $x_{n-k+1}\cdots x_n$ and (12) by $-h(n)$ and adding the two inequalities we get

$$\begin{aligned} &(x_{n-k+1}\cdots x_n - h(n))(x_1+\cdots+x_{n-k-1}+x_{n-k+1}+\cdots+x_n) \\ &\leqslant (x_1\cdots x_{n-k-1}x_{n-k+1}\cdots x_n - h(n))(n-k-1+x_{n-k+1}\cdots+x_n). \end{aligned}$$

In view of (10) this inequality is equivalent to

$$(x_{n-k+1}\cdots x_n - h(n))x_{n-k} \leqslant h(n)(n-k-1+x_{n-k+1}+\cdots+x_n)$$

which implies

$$x_{n-k}^k x_n - h(n)x_{n-k} \leqslant h(n)(n+kx_n).$$

Taking into account that $x_{n-k}^k > kh(n)$ we can write x_{n-k} instead of x_n in the previous inequality and get

$$x_{n-k}^{k+1} - (k+1)h(n)x_{n-k} \leqslant h(n)\cdot n. \quad \square$$

2. Main theorem

In the proof of the main theorem we shall use the following auxiliary result. Let $V(n) = v(\alpha_1,\alpha_2,\ldots,\alpha_k)$ denote the number of factorizations of $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, $n = t_1t_2\cdots t_l$, where $l \geqslant 1$ and $2 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_l$.

LEMMA. $V(n) \leqslant n$.

PROOF. We claim $v(\alpha_1,\alpha_2,\ldots,\alpha_k) \leqslant 2^{\alpha_1}3^{\alpha_2}\cdots(k+1)^{\alpha_k}$, which implies that $V(n) \leqslant n$. We prove this by mathematical induction on $(\alpha_1+\alpha_2+\cdots+\alpha_k)$.

For simplicity, let $A := 2^{\alpha_1}3^{\alpha_2}\cdots(k+1)^{\alpha_k}$. Clearly, if p is a prime, then $V(p) = v(1) = 1 \leqslant 2^1$. In arbitrary factorization of $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ there is a factor divisible by p_k. For this reason

$$V(n) \leqslant \sum_{\substack{d|n \\ p_k|d}} V(\frac{n}{d}).$$

Hence

$$v(\alpha_1,\ldots,\alpha_k) \leqslant \sum v(\beta_1,\ldots,\beta_k),$$

where the sum runs over all $\beta_1,\beta_2,\ldots,\beta_k$ satisfying

$$\beta_1 \leqslant \alpha_1,\ldots,\beta_{k-1} \leqslant \alpha_{k-1}, \beta_k \leqslant \alpha_k - 1.$$

Hence with $0 \leqslant i_1 \leqslant \alpha_1$, $0 \leqslant i_2 \leqslant \alpha_2$, $\cdots$, $0 \leqslant i_{k-1} \leqslant \alpha_{k-1}$, $0 \leqslant i_k \leqslant \alpha_k - 1$, we have

$$\begin{aligned} v(\alpha_1,\alpha_2,\ldots,\alpha_k) &\leqslant \sum v(i_1,i_2,\ldots,i_k) \\ &\leqslant A\cdot\left[\prod_{j=2}^{k}(1+j^{-1}+j^{-2}+\cdots)\right]\cdot\left[(k+1)^{-1}+(k+1)^{-2}+\cdots\right] \\ &\leqslant A\cdot\left[\prod_{j=2}^{k}\frac{j}{j-1}\right]\cdot\frac{1}{k+1}\cdot\frac{k+1}{k} \quad\leqslant\quad A, \end{aligned}$$

where in the second line we have used the induction hypothesis. $\square$

REMARK. Professor A. Schinzel has informed us that this lemma is not, in fact, new. The same statement was proved by Mattics and Dodd (see [2]) in a slightly different way.

THEOREM. *For $f(n)$, the number of solution of the equation* (1), *we have*

$$f(n) \leqslant \frac{1}{2}h(n)n\big(h(n)n+1\big). \tag{13}$$

PROOF. Observe that from (1) it follows $x_1x_2\cdots x_n \leqslant h(n)nx_n$, that is, $x_1x_2\cdots x_{n-1} \leqslant h(n)n$. Hence, using the Lemma, we get

$$f(n) \leqslant \sum_{i=1}^{h(n)n} V(i) \leqslant \sum_{i=1}^{h(n)n} i = \frac{1}{2}h(n)n\big(h(n)n+1\big). \quad \square$$

REMARK. Professor A. Schinzel observed that the estimate (13) can be improved for large values of n by using the results in the papers: Oppenheim, J. London Math. Soc. **2** (1927), p. 130, Szekeres and Turan, Acta Litt. Scient. Szeged **6** (1933), pp. 143–154.

3. Open problems and numerical data

We propose several open problems related to the equation (1). Most of them are due to Professor P. Erdős.

Denote by $f(n)$ the number of solution of

$$x_1 x_2 \cdots x_n = n(x_1 + x_2 + \cdots + x_n), \qquad x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n.$$

1. Prove that $f(n) \to +\infty$ as $n \to +\infty$.

2. A number n is called a champion if $f(n) > f(m)$ for every $m < n$, it is called anti-champion if for every $m > n$ $f(m) > f(n)$. It is true that the anti-champions are always primes? Can we characterize the champions?

3. Is it true that the density of the integers n for which $f(n+1) > f(n)$ is $\frac{1}{2}$?

4. Are there infinitely many solutions of $f(n) = f(m)$, $n \neq m$?

5. Is it true that the density of the integers $f(n)$ is 0? In other words, is the density of the integers t for which there is an n with $f(n) = t$ equal to 0?

In the following table the number of solutions of (1) is given by computer program for $n \leqslant 40$ and several functions $h(n)$.

Number of solution of $x_1 \cdots x_n = h(n)(x_1 + \cdots + x_n)$						
n	$h(n) = 1$	$h(n) = 2$	$h(n) = 3$	$h(n) = n$	$h(n) = 2n$	$h(n) = n^2$
2	1	2	2	2	3	3
3	1	3	6	6	8	13
4	1	5	4	8	15	30
5	3	3	6	8	15	39
6	1	5	5	17	30	96
7	2	4	8	14	16	43
8	2	5	6	19	42	142
9	2	5	8	27	36	126
10	2	6	6	25	44	169
11	3	5	10	15	27	66

12	2	8	6	33	73	424
13	4	4	9	16	24	92
14	2	5	6	30	63	233
15	2	6	10	43	58	355
16	2	8	6	45	77	430
17	4	4	11	18	31	93
18	2	10	6	55	119	644
19	4	4	12	24	30	90
20	2	8	6	43	109	798
21	4	7	10	55	68	
22	2	7	7	43	62	
23	4	4	14	22	40	
24	1	10	9	92	166	
25	5	8	9	43	64	
26	4	7	7	35	83	
27	3	7	14	68	104	
28	3	9	8	69	140	
29	5	4	12	25	43	
30	2	11	6	107	174	
31	4	6	14	34	36	
32	3	10	8	80	173	
33	5	6	9	56	96	
34	2	7	9	48	90	
35	3	6	15	61	124	
36	2	13	8	130	252	
37	6	6	12	32	30	
38	3	6	11	45	101	
39	3	10	16	65	109	
40	4	12	6	119	220	

References

[1] P. Erdős, *Some unsolved problems*, Publ. Math. Inst. Hung. Acad. Sci., **6** (1961), 221–254.

[2] L. E. Mattics and F. W. Dodd, *A bound for the number of multiplicative partitions*, Amer. Math. Monthly, **93** (1986), 125–126.

[3] A. Schinzel, *Ungelöste Probleme*, Elem. Math., **11** (1956), 134–135.

[4] C. Viola, *On the diophantine equations* $\prod_0^k x_i - \sum_0^k x_i = n$ *and* $\sum_0^k \frac{1}{x_i} = \frac{a}{n}$, Acta Arithmetica, **XXII** (1973), 339–352.

Bukor, Tóth
Department of Mathematics
Faculty of Natural Sciences
Constantine the Philosopher University
Tr. A. Hlinku 1
949 74 Nitra
Slovakia

Filakovszky
Šoltésovej 21
940 59 Nové Zámky
Slovakia

e-mail:
jbukor@ukf.sk
jtoth@fpv.ukf.sk