THE 2ND CZECH AND POLISH CONFERENCE ON NUMBER THEORY
CIESZYN 1998
PROBLEM SESSION

The Problem Session, chaired by Andrzej Schinzel, took place on June 18, 1998. The following eight problems were proposed.

1.[1] **A. Schinzel.** Find an example of more than $4n$ consecutive integers all of which have a prime factor $\leqslant n$.

2.[2] **I. Korec.** For which m the following statement holds:

$$\begin{aligned} \forall x_0, y_0, z_0 \quad & (x_0^2 + y_0^2 \equiv z_0^2 \bmod m) \quad \Longrightarrow \\ \exists x, y, z \quad & (x^2 \equiv x_0^2 \bmod m \ \wedge\ y^2 \equiv y_0^2 \bmod m \ \wedge\ z^2 \equiv z_0^2 \bmod m \\ & \wedge\ x^2 + y^2 = z^2). \end{aligned}$$

3. I. Korec. Let G be the graph whose vertices are odd natural numbers, and two vertices are connected with an egde if and only if $|x^2 - y^2|$ is a square.

How many components has the graph G? The conjecture is that there are infinitely many components. And a lower bound for the number of components is 3 since 1 is isolated and 3 and 7 lie in distinct components.

4. K. Szymiczek. A. Schinzel proposed the following problem.[3] Show that the following statement is false: With p running through prime numbers,

$$\exists\, p_0 \quad \forall\, p > p_0 \quad \forall n \qquad p|2^n - 3 \Longleftrightarrow p|3^n - 2.$$

A much more difficult problem would be to find the $\gcd(2^n - 3, 3^n - 2)$. The results of computation with PARI/GP show that for all $n \leqslant 6600$ one has

$$\gcd(2^n - 3, 3^n - 2) = \begin{cases} 1 & \text{if} \quad n \equiv 0, 1, 2 \pmod 4 \\ 5 & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}$$

except for $n = 3783 = 3 \times 13 \times 97$. Here we have

$$\gcd(2^{3783} - 3, 3^{3783} - 2) = 26665 = 5 \times 5333.$$

This prompts the following question:

Do there exist infinitely many natural numbers n such that

$$\gcd(2^n - 3, 3^n - 2) = 1\ ?$$

Collected and edited by K. Szymiczek.

[1] David Jedelský of Ostrava University found that $n = 1741$ satisfies the requirement. The sequence of $4n + 1$ integers begins with an integer having 738 decimal digits. The result will be published elsewhere.

[2] For a solution see A. Schinzel's paper in this volume.

[3] For a solution of Schinzel's problem see the paper by G. Banaszak in this volume.

5. G. Banaszak. The following problem appeared during my joint work with W. Gajda and P. Krasoń at our K-theory Workshop in Zajączkowo (Poland) in May 1996.

Let F be a number field, F_v its completion at a prime ideal v and let $\mathbb{A}_F$ denote the adelé ring. The wild kernel $WK_n(F)$ is defined as the kernel of the natural map

$$K_n(F) \to \prod_v K_n(F_v)$$

(for details see G. Banaszak, W. Gajda, P. Krasoń, and P. Zelewski, A note on the Quillen-Lichtenbaum conjecture and the arithmetic of square rings, K-Theory (to appear in 1998)). There is another natural map

$$K_n(F) \to K_n(\mathbb{A}_F),$$

whose kernel is contained in $WK_n(F)$ because the first map factors through this map. Is it true that kernels of both maps are equal for all F?

6. W. Narkiewicz. Let $\Phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be a polynomial map, that is, $\Phi(x,y) = (f(x,y), g(x,y))$, where $f, g \in \mathbb{Z}[X,Y]$. It is known that the length of a cycle of Φ cannot exceed 800 (T. Pezda, Manuscripta Math. **83** (1994), 279–289) but the maximal known length of a cycle equals 6.

Determine the maximal cycle-length $M(\Phi)$ for Φ.

7. W. Narkiewicz. Lenstra showed under GRH the following. If K is a real number field, then there exist infinitely many primes $\mathfrak{p}$ in $\mathbb{Z}_K$ such that every nonzero residue class of $\mathbb{Z}_K$ modulo $\mathfrak{p}$ contains units of the ring $\mathbb{Z}_K$.

Without using GRH it has been proved (W. Narkiewicz, Archiv f. Math. **51** (1988), 238–241) that if K is real abelian and satisfies a certain technical condition, then the same holds with at most three exceptions.

The question arises whether one can obtain the same result for at least some classes of non-real abelian fields without using GRH. It would be also interesting to dispose of the three exceptional real abelian fields.

8. J. Tóth. For $A \subseteq \mathbb{N}$ put $R(A) = \{\frac{a}{b} : a, b \in A\}$. A is said to be a quotient base if $R(A) = \mathbb{Q}^+$. For $k \in \mathbb{N}$ let $f(k)$ be the least positive integer such that $\mathbb{N}$ can be decomposed into $f(k)$ pairwise disjoint sets $A_1, \ldots, A_{f(k)}$ in such a way that the union of k arbitrarily chosen from $A_1, \ldots, A_{f(k)}$ is not a quotient base. It is known that $f(1) = 2, f(2) = 4, 4 < f(3) \leqslant 8$.

Determine the exact value of $f(3)$. Nothing is known about the behavior of $f(n)$ for large n.